Mostowski's collapsing function and the closed unbounded filter

by

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Abstract. We improve a result of Lévy by showing that if $P(s)$ is a $\Sigma$ predicate of set theory and if $P(s)$ holds then $P(b)$ holds for almost every countable approximation $b$ of $a$. Applications to model theory are discussed.

§ 1. Introduction. Our purpose here is to use Mostowski's collapsing function $\phi(s)$ to shed some light on an interesting Löwenheim-Skolem phenomenon in infinitary logic recently discovered by D. Kueker [4]. For each countable $s$, $\phi(s)$ can be thought of as a countable approximation $s'$ of $s$. Using the notion of "almost all" given by the closed unbounded filter we show that if $P$ is a $\Sigma$ predicate of set theory and $P(x, y)$ holds then $P(s', y')$ is true for almost all $s$. Several of Kueker's results follow as well as Lévy's result that if a $\Sigma$ predicate $P(x)$ has a solution $a$, then it has some hereditarily countable solution (namely $s'$ for almost all $s$).

For applications to model theory it is most natural for us to work in a universe of set theory which allows the existence of individuals. The universe of sets we have in mind can be described as follows. We are given a collection $M$ of individuals (atoms, urelements) which can be used to form sets.

$V_M(0) = 0$ the empty set,

$V_M(a+1) = \{ \text{the set of all subsets of } V_M(a) \cup M \}$,

$V_M(\lambda) = \bigcup V_M(a)$ for limit $\lambda$,

$V_M = \bigcup V_M(a)$.

The union in the last equation is taken over the class of all ordinals. A set on $M$ is, by definition, an element of $V_M$. The reader who feels un-

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comfortable with individuals can simply assume there are none, i.e., that $M = \emptyset$.

We use variables $p, q, r$ to range over $M$, $a, b, c$ to range over $V_M$, and $x, y, z$ to range over $M \cup V_M$.

We use $\epsilon$ for the membership relation on $V_M$, $R, S, T$ for predicates on $M$ and $P, Q$ for predicates on $M \cup V_M$.

A set $a$ is transitive if $s \in a \implies s \subseteq a$ for all $s, y$. Individuals $p$ are not considered transitive. For any $x$ there is a smallest transitive set $a$ such that $x \subseteq a$, called the transitive closure of $a$, $\text{TC}(a)$. If $x$ is an individual $p$ then $\text{TC}(x) = x$. If $x$ is a set then $\text{TC}(x) = x \cup \{ \{a\} \cup \cup ( \cup \{a\} \cup \ldots \}$ The support of a set $a$, $\text{Sp}(a)$, is the set of individuals in $\text{TC}(a)$. The pure sets are those sets with empty support. We use $[a]$ to denote the cardinality of $\text{TC}(a)$; $a$ is hereditarily countable if $|a| < \kappa$.

§ 2. The collapsing function. In Theorem 3 of Mostowski [8], it was shown that for any well founded relation $R$ there is a unique map $\sigma_R$ of $E$ onto a transitive set satisfying $\sigma_R(x) = \{ \sigma_R(y): y \in R x \}$ for $x$ in the field of $R$. This map $\sigma_R$ is called the collapsing function, or contraction function, for $R$. When $R = \epsilon \cap (\epsilon \times \epsilon)$ for some set $s$, we write $\sigma_s$ for $\sigma_R$. In this case, however, there is no reason to restrict the domain of $\sigma_s$ to $s$. In our context with urelements we define $x^\epsilon = \sigma_R(x)$ for any $x$ by 2.1.

2.1. Definition. For a fixed set $s$ we define, for every $s$, an approximation $x^\epsilon$ of $x$ by recursion on $\epsilon$ as follows:

\[ p^\epsilon = p, \]
\[ a^\epsilon = \{ a^\epsilon: x \in s \cap a \}. \]

The reader unfamiliar with this function might want to read the collapsing lemma in § 1.6 of Mostowski [9], though we will not need that lemma here. We need only the following.

2.2. Lemma. Given sets $a, s$ with $s$ countable we have the following facts:

(a) If $a$ is a set of individuals then $a^\epsilon = a \cap s$.
(b) The set $a^\epsilon$ is hereditarily countable.
(c) $\text{TC}(a) \subseteq s$ implies $a^\epsilon = a$.

(d) If $a$ is transitive so is $a^\epsilon$; hence if $a$ is an ordinal then $a^\epsilon$ is a countable ordinal.
(e) If $s \cap \text{TC}(a) = s^\epsilon \cap \text{TC}(a)$ for some other $s' \cap \text{TC}(a) = a^\epsilon$.

These facts are all easily verified; (b), (c) and (e) by induction on $\epsilon$.

§ 3. The closed unbounded filter. The following definition was given in Krener [4] and Jech [3] for the case where $A$ was an infinite cardinal but the results go through just as well in general.

3.1. Definition. Let $A$ be a transitive set and let $I$ be the set $P_{a_0}(A)$ of all countable subsets of $A$. The closed unbounded filter $D$ on $A$ consists of all $X \subseteq I$ such that for some $X_0 \subseteq X$,

(a) every $x \in X$ is a subset of some $x' \in X_0$,
(b) $X_0$ is closed under unions of countable chains.

3.2. Lemma. (Krener [4], Jech [3]). Given $A$ and $D$ as in 3.1, we have the following:

(a) $D$ is a countably complete proper filter.
(b) If $X_0 \in D$ for all $a \in A_0 \subseteq A$ then the diagonal

\[ Y = \{ s \in X_0: a \in A_0 \cap a \} \]

is in $D$.

(c) A subset $X \subseteq I$ is in $D$ iff player I has a strategy for the two person game $\Gamma_X$ given by the rules: I and II alternately choose elements of $A$, I wins if the set of their choices is in $X$, otherwise II wins (Krener [4]).

(A hint for the harder half of (c): Given a strategy $\alpha = \{ \alpha_0(a_0, \ldots, a_n) \}
\alpha < \alpha$ for $X$ let $X_0$ be set of $x \in X$ closed under the various $\alpha_n$ then 3.1 (a), (b) are clear for $X_0$.)

3.3. Definition. Let $Q$ be a predicate of sets and individuals. For given $a_0, \ldots, a_n$, in a transitive set $A$, we say that $Q(a_0, \ldots, a_n)$ holds almost everywhere if the set $\{ s \in P_{a_0}(A): Q(a_0', \ldots, a_n') \}$ is a member of the closed unbounded filter on $A$.

The following lemma is a simple extension of a remark in Krener [4], but since it is basic to our theorem, we sketch a proof.

3.4. Lemma. The notion of “almost everywhere” defined in 3.3 is independent of the particular transitive set $A$.

Proof. Let $Q$ be a place predicate to simplify notation, let $x \in A_0 \cap A_1, I_1 = P_{a_0}(A_1), D_1$ the closed unbounded filter on $A_1$ and $X_1 = \{ s \in I_1: Q(s^{a_0}) \}.

Assume $X_1 \in D_1$ and let us show $X_1 \in D_1$. By Lemma 3.2 (c), player I has a winning strategy $\sigma_I$ for the game $\Gamma_{X_1}$. By Lemma 3.2 (c) we can assume that this strategy only picks elements from $A_0 \cap A_1$ since $Q(a) \subseteq A_0 \cap A_1$.\]
and the value of \( x' \) depend only on \( s \cap TC(x) \). But then player I can use \( s' \) to get a winning strategy \( s_0 \) for \( H_X \). He simply ignores any move of \( \Pi \) outside \( A_1 \cup A_2 \), replacing it by \( x \) and uses \( s_0 \). Any \( s \in I \) which results from such a play will agree with an \( s' \in X_1 \), at least on \( TC(x) \), in which case \( x' = x'' \) by 2.2 (e). Thus I has a strategy for \( H_X \) and \( X_2 \in D_4 \) by 3.2 (e).

The results of the next section depend only on 3.2 and 3.4.

§ 4. The result. The language of set theory has a membership symbols \( \epsilon \), denoting \( \in \), an equality symbol \( = \), symbols for \( R, S, T, ... \) for any relations \( R, S, T \) on \( M \). The \( \Delta_0 \)-formulas, defined in Lévy [6], form the smallest class \( \Phi \) containing the atomic formulas closed under:

(i) If \( \varphi \) is in \( \Phi \) so is \( \neg \varphi \),

(ii) if \( \varphi, \psi \) are in \( \Phi \) so are \( (\varphi \land \psi), (\varphi \lor \psi) \),

(iii) if \( \varphi \) is in \( \Phi \) and \( u, v \) are any variables then \( Vu \epsilon \varphi \) and \( \exists u \epsilon \varphi \) are in \( \Phi \).

The \( \Sigma \)-formulas form the smallest class \( \Phi \) containing the \( \Delta_0 \)-formulas and closed under (ii), (iii) and (iv).

(iv) If \( \varphi \) is in \( \Phi \) and \( u \) is any variable then \( \exists u \epsilon \varphi \) is in \( \Phi \).

A predicate \( Q \) on \( M \) if \( \varphi \) is definable by a \( \Sigma \) formula of the above language; it is \( \Theta \) if it and its negation are both \( \Sigma \). The reader unfamiliar with \( \Sigma \) predicates should consult Lévy [6], p. 6 for basics and § 10 for examples and the theorem mentioned in the introduction.

4.1. Theorem. Let \( Q \) an \( n \)-ary \( \Sigma \) predicate. If \( Q(x_1, \ldots, x_n) \) holds, then \( Q(x_1', \ldots, x_n') \) holds a.e.

Since the closed unbounded filter is proper we have the following consequence of the theorem:

4.2. Corollary. Let \( P \) be a \( \Lambda \) predicate. For all \( x_1, \ldots, x_n \), \( P(x_1, \ldots, x_n) \) is true if \( P(x_1', \ldots, x_n') \) is true for almost all \( x \).

Extending some terminology of Knecr from classes of structures to arbitrary predicates we say that \( P \) is closed downward if for each \( x_1, \ldots, x_n \) with \( P(x_1, \ldots, x_n) \),

\[ P(x_1', \ldots, x_n') \] holds a.e.

Thus, the theorem states that all \( \Sigma \) predicates are closed downward. Note that "as is uncountable", whose negation is \( \Sigma \), is not closed downward.

To prove the theorem first note that every \( \Delta_0 \)-formula is equivalent to one where all negations occur in front of atomic formulas. Atomic predicates are trivially closed downward (if \( \varphi(x, y) \) then \( \varphi(x', y') \) whenever \( x \epsilon s \) and negated atomic predicates of individuals are also trivially closed downward. The other negated atomic formulas fall under 4.3.

4.3. Lemma. The predicates \( x \epsilon y \) and \( x \epsilon y \) are closed downward.

Proof. We prove \( \forall x \epsilon y \forall Q(x, y) \), where \( Q(x, y) \) is the conjunction of

\[ x \epsilon y \rightarrow x' \epsilon y' \text{ a.e.} \]

\[ x \epsilon y \rightarrow x' \epsilon y' \text{ a.e.} \]

\[ y \epsilon x \rightarrow y' \epsilon x' \text{ a.e.} \]

by a double induction over \( \epsilon \). Thus, given \( x_0, y_0 \) we prove \( Q(x_0, y_0) \) assuming

(1) \( \forall x \epsilon x_0 \forall y \forall Q(x, y) \) and

(2) \( \forall x \epsilon y \forall Q(x, y) \).

Case (i). Assume \( x_0 \neq y_0 \). If either \( x_0 \epsilon y_0 \) or \( y_0 \epsilon x_0 \) is an individual \( p \) then \( x_0' \neq y_0' \) as long as \( p \epsilon s \), so we may assume both are sets. If there is an \( x \epsilon x_0, y \epsilon y_0 \) then \( Q(x, y) \) by (1) so \( x' \epsilon y' \) a.e., whereas \( x' \epsilon y' \) a.e., so \( x_0' \neq y_0' \) a.e. If there is a \( y \epsilon y_0, y \neq x_0 \) then we use (2) similarly.

Case (ii). Assume \( x_0 \epsilon y_0 \). If \( y_0 \) is an individual then \( x_0' \epsilon y_0' \) for all \( s \) so we assume \( y_0 \) is a set. Now for each \( y \epsilon y_0 \) we have \( x_0 \neq y \) and \( Q(x_0, y) \) and hence the set \( X_\beta \in \mathcal{D} \), where

\[ X_\beta = \{ s : x_0' \neq y' \} \]

By 3.2, the diagonal

\[ Y = \{ s : s \epsilon X_\beta \} \]

is in \( \mathcal{D} \). Now let \( s \epsilon X \) be fixed. For every \( y \epsilon e \cap y_0, s \epsilon X_\beta \), i.e., \( x_0' \neq y' \), but \( y' \notin y'' \), \( s \epsilon e \cap y_0 \), so \( x_0' \notin y_0' \). Thus \( x_0' \notin y_0' \) a.e.

Case (iii). If \( y_0 \epsilon x_0 \), the proof is similar to (ii).

The following results, with earlier remarks and 4.3, completes the proof of Theorem 4.1. It is of interest in its own right, though, since there are predicates which are closed downward which are not \( \Sigma \) definable.

4.4. Proposition. The predicates closed downward are closed under conjunction, disjunction, bounded quantification and unbounded existential quantification.

Proof. Conjunction follows from one property for filters \( X, \mathcal{Y} \in \mathcal{D} \Rightarrow X \cap Y \in \mathcal{D} \) and disjunction from the other \( X \subseteq Y, X \in \mathcal{D} \Rightarrow Y \in \mathcal{D} \). Existential quantification is routine by induction, bounded existential follows from unbounded and Lemma 4.3. For bounded universal quantification, let \( P(x, \ldots, x_n, y) \) be closed downward and suppose

\[ \forall y \epsilon aP(x_1, \ldots, x_n, y) \].
We need to show that for almost all \( s \) we have
\[
V y \in \sigma' P(x'_1, \ldots, x'_n, y).
\]
For \( y \in a \) let
\[
X_y = \{ s : P(x'_1, \ldots, x'_n, y) \}
\]
(which is an element of \( D \) since \( D \) is closed downwards) and let \( Y \) be the diagonal
\[
Y = \{ s : s \in \bigcap \{ X_y \} \}
\]
which is an element of \( D \) by 3.2 (b). We see that
\[
s \in Y \iff P(x'_1, \ldots, x'_n, y') \text{ for all } y \in a \cap s
\]
so we have, for \( s \in Y \)
\[
V y \in \sigma' P(x'_1, \ldots, x'_n, y)
\]
since \( a' = (y' : y \in a \cap s) \).

If one is willing to talk about infinitary predicates over \( V_M \), then we see that the predicates closed downward are also closed under countable conjunctions and arbitrary disjunctions; the second is trivial, the first from the countable additivity of the closed unbounded filter.

§ 5. Applications to infinitary logic. Let \( L \) be a first order language with at least one countable \((?)\) number of symbols. We think of these symbols as individuals (elements of \( M \)), formulas are built up from them by set theoretic principles and so are in \( V_M \). We assume the reader is familiar with the infinitary languages \( L_\omega \omega \) and \( L_\omega \). If \( \varphi \in L_\omega \omega \), then \( \varphi' = \varphi \) for almost all \( s \) (for all \( s \geq \Sigma_2 \) by Lemma 2.2 (c)); if \( \varphi \in L_\omega \), then \( \varphi' \in L_\omega \) a.e., since \( L_\omega \) is a \( \Delta \) class and \( L_\omega = [\varphi \in L_\omega : |\varphi| < \aleph_0] \). To see more clearly what \( \varphi' \) means, for \( \varphi \in L_\omega \), define \( \varphi^{[s]} \) for all \( s \) and all \( \varphi \in L_\omega \) by recursion:
\[
\begin{align*}
\varphi^{[s]} &= \varphi & \text{if } \varphi \text{ is atomic,} \\
(\neg \varphi)^{[s]} &= \neg (\varphi^{[s]}) \\
(\exists y \varphi)^{[s]} &= \exists y (\varphi^{[s]}) \\
(\forall y \varphi)^{[s]} &= \forall y (\varphi^{[s]})
\end{align*}
\]
A simple inductive proof shows that \( \varphi' = \varphi^{[s]} \) for almost all \( s \); i.e., that our \( \varphi' \) is almost always equal to \( \varphi' \) as defined in Kueker [4].

(*) This requirement on \( L \) could be dropped by replacing \( L \) by \( L' \) at certain points. Since \( L \) is countable, \( L = L' \) a.e.

An \( L \)-structure \( \mathbb{A} \) is of the form \( \langle A, F \rangle \) where \( A \) is a set of individuals \((?)\) and \( F \) is a function which assigns to each symbol in \( L \) an interpretation of the appropriate kind. Then, for almost all \( s \), \( \mathbb{A} \) is just the substructure \( \mathbb{A}_s \) of \( \mathbb{A} \) with universe \( A_s = A \cap s \).

Theorem 4.1 and its corollary were inspired by two results in Kueker [4], 5.1 and 5.2 below.

5.1. Theorem (Kueker). Let \( \varphi \in L_\omega \) and let \( \mathbb{A} \) be an \( L \)-structure. Then
\[
\mathbb{A} \models \varphi \text{ if and only if}
\]
\[
\mathbb{A}' \models \varphi' \text{ a.e.}
\]

Proof. \( \varphi \) is a \( \Delta \) relation so the result follows from 4.3. ●

The proof shows that the result holds for logics stronger than \( L_\omega \). In the terminology of [2], the result goes through with \( L_\omega \) replaced by any absolute logic \( L' \). If \( L' \) is absolute and \( \varphi \in L' \) then \( \varphi' \in L'_\omega \) a.e. and \( \mathbb{A} \models \varphi \) iff \( \mathbb{A}' \models \varphi' \) a.e. If \( \varphi \in L'_\omega \) then \( \varphi = \varphi' \) a.e. In particular, these results for the logics \( L' = L_\omega \) and \( L'_\omega = L_\omega \) which allow some infinite alternations of quantifiers. This gives a very useful Löwenheim-Skolem result when applied to the theory of inductive definitions as in Moschovakis [7]; see his § 8.D.

5.2. Theorem (Kueker). Let \( \mathbb{A}, \mathbb{B} \) be \( L \)-structures.

(a) \( \mathbb{A} = \mathbb{B} \) iff \( \mathbb{A} \equiv \mathbb{B} \) a.e.
(b) \( \mathbb{A} \not= \mathbb{B} \) iff \( \mathbb{A} \not\equiv \mathbb{B} \) a.e.

Proof. The relation \( \mathbb{A} =_{\omega} \mathbb{B} \) means that \( \mathbb{A} \) and \( \mathbb{B} \) are models of the same sentences of \( L_\omega \). It is a \( \Delta \) relation (see [1]). For countable \( \mathbb{A}, \mathbb{B}, \mathbb{A} =_{\omega} \mathbb{B} \) iff \( \mathbb{A} \equiv \mathbb{B} \) ●

The reason for stating both (a) and (b) is that the closed unbounded filter is not an ultrafilter so (b) does not follow immediately from (a).

Let \( K \) be any class of \( L \)-structures closed under isomorphism and closed downward. Since Kueker [4], [5] give a number of interesting model theoretic results for such \( K \), it is useful to note the extremely easy with which such \( K \) can be identified, given Theorem 4.1, Proposition 4.4 and just a little familiarity with \( L \) predicates. For example, we have the following.

5.3. Proposition. Let \( K, K' \) be a class of \( L \) structures closed under isomorphism and closed downward. Then the classes of structures \( \mathbb{A} \) defined in the following are all closed downward.

(*) If one thinks of \( V_M \) as given in advance then the requirements \( L \subseteq M \) and \( A \subseteq M \) seem odd. If, on the other hand, one thinks of \( L \) as given and then forms \( P^M \) for some \( M \supseteq L \cup A \), then one sees that they are really not restrictions at all. It is just the usual mathematical practice; when working with some structure \( \mathbb{A} = \langle A, \ldots \rangle \) we ignore any structure on elements of \( A \) not given by \( \mathbb{A} \).
Section 5.3. One such comes from the class $K_4$ studied by Kueker and defined by $\mathcal{A} \in K_4$ if

there is a finite set $p_1 \ldots p_n \in A$ such that every $q \in A$ is definable in $(\mathcal{A}, p_1 \ldots p_n)$ by a formula $\varphi(x)$ of $L_{\text{adm}}$.

5.4. Proposition. The class $K_4$ is $\Delta$ definable.

Proof. Simply writing out the above condition gives a $\Sigma$ definition of $K_4$. It is not quite so obvious how to write $\mathcal{A} \not\in K_4$ as a $\Sigma$ condition. To do it we use the following observation of Nadel [10]: If an element $q$ of a structure $\mathcal{B}$ is definable by some formula $\varphi(x)$ of $L_{\text{adm}}$, then it is definable by a formula $\psi \in L_{\text{adm}}^{(g)}$, where $(\mathcal{B})^{(g)}$ is the smallest admissible set with $\mathcal{B}$ an element and $L_{\text{adm}}^{(g)} = L_{\text{adm}} \cap L_{\text{adm}}^{(g)}$. We can now write $\mathcal{A} \not\in K_4$ iff

$\exists x \ (x$ is admissible $\land \mathcal{A} \ not \ definable$ on $(\mathcal{A}, p_1 \ldots p_n) \ by \ any \ formula \ $ $\varphi(x)$.

The part inside the parentheses is easily seen to be $\Delta$. [1]

From 5.4 and 4.2 we obtain Kueker's result that $\mathcal{A} \in K_4$ iff $\mathcal{B} \in K_4$, a.e., a result which Kueker puts to good use in [5].

References


