$K$ is the least model-class containing $\{[\sigma_n]\}_{n\in\omega}$ as element). Then $K \subseteq M_1 \cap \cap M_2$ and $N \subseteq K$. We show that $\neg \mathrm{Mcl}_{E1}(K)$. Suppose the contrary. Then $K$ contains a selector for $\{[\sigma_n]\}_{n\in\omega}$, i.e. there are $\tau_n \in [\sigma_n]$ such that $\{\tau_n\}_{n\in\omega} \in K$. The system $\{\tau_n\}_{n\in\omega}$ is representable as a subset $\tau$ of $\mathbf{L}$ (as an exact functor see [7] 1408) and is in $M_1 \cap M_2$. Consequently, $\tau \leqslant_D^{\square} \sigma$ and $\tau \leqslant_D^{\square} \pi$ and hence there is an $n$ such that $\tau \in N_n$. But $\{[\sigma_n]\}_{n\in\omega} \leqslant_C \tau$ and hence $K \subseteq N_n$ which contradicts to $N \subseteq K$.

4.5. Remark. (1) The set $\{[\sigma_n]\}_{n\in\omega}$ can be represented as a sequence of disjoint sets of subsets of $\omega$ and hence one has a countable disjointed system of sets of reals without a selector.

(2) $K$ is the least model-class with $N$ as a subclass and we have $N \neq K$ by 4.3. Is $K = M_1 \cap M_2$? Can one obtain $K$ as a "symmetric submodel of a support extension of $\mathbf{L}$"?

(3) An analysis shows that the assumption "$\mathbf{V}$ has a set support over $\mathbf{L}$" in 4.4 can be weakened to the Gödel's form of the axiom of choice (E2 of [7]).

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# Elementary interpretations of negationless arithmetic

by

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**Abstract.** Some systems of negationless arithmetic (in the spirit, if not the form of Griss' negationless mathematics) are introduced and their relation to intuitionism are considered.

## § 1. The negationless mathematics of G. F. C. Griss

**1.1.** Few would disagree with Griss' criterion that all the well-formed parts of a meaningful formula should also be meaningful. Unfortunately there is plenty of room for disagreement on the meaning of "meaningful". Although it may well be impossible to determine what Griss had in mind, two aspects of this interpretation are (more or less) evident: namely that for a formula to have meaning it is necessary that it have a constructive interpretation and that it be satisfiable.

If one accepts such a condition on the notion of meaningful and still adheres to the principle that all the well-formed parts of a meaningful formula should also be meaningful, then one finds that the propositional connectives "or", "if ... then" and "it is not the case that" cause a lot of problems. For example the sentence

$$\mathbf{0} = \mathbf{0} \vee \mathbf{0} = \mathbf{1}$$

could not possibly have any meaning for Griss, since if it had then so would $\mathbf{0} = \mathbf{1}$ and the latter could not have any meaning for him since it is not satisfiable.

The connective "$\rightarrow$" is even more problematic. Already the impredicative aspect of the intuitionistic interpretation of "$\rightarrow$" leaves much to be desired, and if to the intuitionistic interpretation one adds Griss' criterion, the formulae such as

$$\mathbf{0} = \mathbf{1} \rightarrow \mathbf{1} = \mathbf{1} \quad \text{and} \quad \mathbf{0} = \mathbf{0} \rightarrow \mathbf{0} = \mathbf{1}$$

are banished from mathematics. What is worst still is that a sentence of the form $\forall x A x$ may have meaning and yet $A\mathbf{n}$ may be meaningless for certain numerals $\mathbf{n}$, for example let $Ax \equiv (x = \mathbf{0} \rightarrow x = \mathbf{0})$.

Intuitionistically, the negation of $\neg A$ is identified with $A \to \mathbf{0} = \mathbf{1}$, so negation has all the problems of "$\to$" (and a few more).

**1.2. Other propositional connectives.** In view of the above complications the connectives "$\vee$", "$\to$" and "$\neg$" are not usually included in the formalizations of negationless mathematics (name usually given to the mathematics proposed by G. F. C. Griss 1946). In their place other connectives are added (see for example: Nelson 1966, Minichiello 1968). One favourite connective is quantified implication: $Ax \to_x Bx$, which could be read (by a classical logician) as $\emptyset \neq \{n\colon A\mathbf{n}\} \subseteq \{n\colon B\mathbf{n}\}$.

On the whole the new connectives are quite complicated and the formalizations of negationless mathematics (or even arithmetic) that have been given so far are rather cumbersome.

**1.3. Some of the aims for negationless mathematics.** Null implications, i.e. implications of the form $A \to B$ where $A$ is false in all instances, and null predicates are so ingrained in modern mathematics that Griss' attempts to develop (constructive) mathematics without their use appears to be doomed right from the start. In addition the available formalizations are not aesthetically pleasing so that the programme of Griss will probably not attract much attention.

Yet negationless mathematics could be useful in the foundations of mathematics. Of course no one doubts that null implications and null predicates are harmless, what is desired is a proof that they are indeed harmless. Hilbert did not want to be expelled from the paradise created by Cantor, we should at least be able to prove that there is no need to live in the purgatory created by Griss.

**1.4. The disparity between $\vee$ and $\exists$.** Both $A_0$ and $A_1$ are well-formed parts of $A_0 \vee A_1$, so according to the criterion of Griss in order that $A_0 \vee A_1$ be (negationless) meaningful it is necessary that both $A_0$ and $A_1$ be satisfiable. On the other hand for the sentence $\exists x Ax$ to be a meaningful statement about the natural numbers it is not required that $A\mathbf{n}$ be satisfiable for all natural numbers $n$, but only for at least one.

One could modify Griss' criterion so that it applies to subformulae ($At$ is a subformula of $\exists x Ax$ for all terms $t$), but then the quantifier $\exists$ would be just about useless; in any case, it is doubtful that Griss had in mind such a stringent non-nullity criterion.

In order to maintain the standard interpretation of an existential statement as a large disjunction it is best to relax the conditions on disjunctions so that only one of the disjuncts of a disjunction need be satisfiable. Or in other words, $A_0 \vee A_1$ will be allowed in our formalization of negationless arithmetic and its negationless interpretation is, loosely speaking, the same as that of $\exists i (i < \mathbf{2} \wedge A_i)$.



**1.5. The connective $\neg$.** From an intuitionistic viewpoint the negation of $A$, $\neg A$, is often interpreted as $A \to \mathbf{0} = \mathbf{1}$. However such a complex notion is not needed for the negation of equations between numerical terms. It is consistent with the constructive interpretation of the natural numbers to have, in addition to the equality predicate "$=$" an apartness relation "$\#$".

In the formalization given in § 2 we shall not use unlimited negation, we shall restrict ourselves to the apartness relation.

**1.6. The connective $\to$.** From time immemorial the connective $\to$ has been riddled with problems and misconceptions. Part of the problem may be connected with the insistence that the connective "if ... then" be somehow related to the verb "implies". If one is willing to give up any such connection then "$\to$" causes no problem. A beautiful example of this is the classical truth value interpretation of "$A \to B$" as "$\neg A \vee B$". Under the classical interpretation, iterated conditionals, for example: $((A \to B) \to C)$, have an easily understood interpretation (e.g. $(A \wedge \neg B) \vee C$). On the other hand, if one uses the intuitionistic interpretation for $\to$, then even simple looking statements such as:

$$(\mathbf{0} = \mathbf{0} \to \mathbf{1} = \mathbf{1}) \to \mathbf{2} = \mathbf{2}$$

involve not only constructions dealing with natural numbers, but also constructions on constructions on constructions.

Now, although we believe that eventually a negationless interpretation for the conditional must be found, we also believe that we should determine how much can be done if we simply eliminate conditional sentences from the formalism. Of course from a classical viewpoint nothing has been lost since we propose to keep $\forall$, $\exists$, and $\vee$ and negations of atomic formulae. Unfortunately if we use the classical interpretation then we loose constructivity.

If we were considering only the predicate calculus then the normalization theorem for the intuitionistic predicate calculus, **IPC**, and its associated subformula property can be used to show that **IPC** is a conservative extension of the system obtained by eliminating the rules corresponding to $\to$.

However the situation for arithmetic is different.

## § 2. The system NA for negationless arithmetic

**2.1.** The system **NA** will be a system of natural deduction formalized in the style of Prawitz 1965 and most of the notational conventions will be taken from there. Deductions ($\Pi, \Pi_0, \ldots$) will be in tree form and as is usual in systems of natural deduction we shall distinguish between *variables* $(x, y, \ldots)$ which can only occur bound in formulae and *parameters* $(a_0, a_1, \ldots)$ which can only occur free in formulae. We shall also have an individual constant $\mathbf{0}$ (zero).

**2.2.** In addition to the function symbols: $s$ (successor) and $p$ (predecessor) we shall also include a function symbol for each primitive recursive function. More specifically, we shall use the indexing of the primitive recursive functions given in Feferman 1962 and to each natural number $b$ such that $\mathrm{In}(b)$ we shall introduce the $(b)_1$-ary function symbol $f_b$ (see also 4.1).

**2.3.** The *language* of **NA** is to contain symbols for conjunction $(\wedge)$, disjunction $(\vee)$, universal quantification $(\forall)$ and existential quantification $(\exists)$.

The only (primitive) relation symbols that will be used are

$$= \text{(for equality)}, \qquad \# \text{(for apartness)}.$$

Atomic formulae, formulae, numerals etc. are then defined as usual.

**2.4. Axioms of NA.**

i. $t = t$ for any term $t$.

ii. $t \# st$ for any term $t$.

iii. $\mathbf{0} \# st$ for any term $t$.

iv. $ft_1 \ldots t_k = t_{k+1}$ for any equation "$ft_1 \ldots t_k = t_{k+1}$" used in the primitive recursive definition of a primitive recursive function.

**2.5. Logical rules of inference.**

$(\wedge)$ $$\frac{A, B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$(\vee)$ $$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad \frac{A \vee B, \begin{array}{c} A \\ \vdots \\ C \end{array}, \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}$$

$(\forall)$ $$\frac{Aa}{\forall x Ax}{}^{*} \qquad \frac{\forall x Ax}{At}$$

$(\exists)$ $$\frac{At}{\exists x Ax} \qquad \frac{\exists x Ax, \begin{array}{c} Aa^{*} \\ \vdots \\ C \end{array}}{C}$$



[In the rules * it is assumed that the parameter $a$ (the proper parameter of the inference) does not occur in any formula, other than $Aa$, on which $Aa$ depends; for definition of dependency see Prawitz 1965].

**2.6. Rules on inference for $=$ and $\#$.**

$$\frac{t_1 = t_2,\ t_3 = t_2}{t_3 = t_1} \qquad \frac{t_1 = t_2}{st_1 = st_2}$$

$$\frac{t_i = t_i'}{f_b t_1 \ldots t_i \ldots t_n = f_b t_1 \ldots t_i' \ldots t_n} \qquad \frac{st_1 = st_2}{t_1 = t_2}$$

$$\frac{t_1 \# t_2}{st_1 \# st_2} \qquad \frac{t \# \mathbf{0}}{spt = t}$$

$$\frac{t_1 \# t_2,\ t_1 = t_3}{t_2 \# t_3}.$$

**2.7. Rule of induction** (IND).

$$\frac{A\mathbf{0}, \begin{array}{c} Aa \\ \vdots \\ Asa \end{array}}{At}$$

where $a$ is a parameter not occurring in assumptions on which $Asa$ depends except for formulae of the form $Aa$.

**2.8. Deductions, derivations.** Deductions are supposed to be arranged in tree form and we shall assume that when a deduction is given it is specified at each node of the deduction which rule of inference it is being applied. In addition we shall assume that when a deduction is given it is stated at each of the top nodes of the deduction at which inference the formula is discharged.

By a *derivation* we understand a deduction without any undischarged assumption formulae (other than axioms).

## § 3. NA as a subsystem of HA

**3.1.** For simplicity we shall assume that intuitionistic arithmetic **HA**, has been developed as a system of natural deduction using $\rightarrow$ and all the symbols of **NA**. Since all the axioms and rules of **NA** are derivable in **HA** we then obtain that **NA** is a subsystem of **HA**.

In section § 5 we prove that **HA** is not a conservative extension of **NA**. In fact we shall exhibit a prenex sentence only involving $\forall$, $\exists$ and *positive* atomic formulae which is provable in **HA** but which is not provable in **NA**.

We conclude this section with some of the sentences that can be proved in **NA**. We shall use the notation "$[b]$" of Feferman 1962 for the primitive recursive function whose index is $b$.

**3.2.** Lemma.

i. $\mathbf{NA} \vdash \mathbf{n} = \mathbf{n}$.

ii. $\mathbf{NA} \vdash \mathbf{n} \# \mathbf{m}$ *if* $n \neq m$.

iii. $\mathbf{NA} \vdash f_b \mathbf{n}_1 \ldots \mathbf{n}_k = \mathbf{n}_{k+1}$ *if* $[b](n_1, \ldots, n_k) = n_{k+1}$.

iv. $\mathbf{NA} \vdash f_b \mathbf{n}_1 \ldots \mathbf{n}_k \# \mathbf{n}_{k+1}$ *if* $[b](n_1, \ldots, n_k) \neq n_{k+1}$.

v. $\mathbf{NA} \vdash t_1 = t_2$ *if* $t_1, t_2$ *are closed terms equal under the canonical interpretation.*

vi. $\mathbf{NA} \vdash t_1 \# t_2$ *if* $t_1, t_2$ *are closed terms unequal under the canonical interpretation.*

vii. $\mathbf{NA} \vdash \forall x (x \# t_1 \vee x \# t_2)$ *if* $t_1, t_2$ *are closed terms unequal under the canonical interpretation.*

Proofs. Of i-vi by informal induction. vii uses also the rule of induction.

**3.3.** Definition. If $A, B$ are formulae, then by $\mathbf{NA} + A \vdash B$ we understand that there is a *deduction* of $B$ whose undischarged assumption formulae occurrences are all of the same form as $A$.

The following lemma is not interesting from a negationless viewpoint, it is included in order to show that there are some vestigial remains of null-implications in **NA**.

**3.4.** Lemma.

i. $\mathbf{NA} + \mathbf{0} = \mathbf{1} \vdash \mathbf{n} = \mathbf{0}$.

ii. $\mathbf{NA} + \mathbf{0} = \mathbf{1} \vdash \mathbf{n} \# \mathbf{0}$.

iii. $\mathbf{NA} + \mathbf{0} = \mathbf{1} \vdash a_0 = a_1$.

iv. $\mathbf{NA} + \mathbf{0} = \mathbf{1} \vdash a_0 \# a_1$.

v. $\mathbf{NA} + \mathbf{0} = \mathbf{1} \vdash A$, $A$ *any formula.*

vi. $\mathbf{NA} + \mathbf{0} \# \mathbf{0} \vdash \mathbf{0} = \mathbf{1}$.

vii. $\mathbf{NA} + \mathbf{0} \# \mathbf{0} \vdash A$, $A$ *any formula.*

## § 4. Realizability and NA

**4.1.** We shall use Kleene's 1945 notion of realizability except that instead of using Gödel numbers of partial recursive functions we shall use indices of primitive recursive functions. $\mathrm{In}^m(b)$ iff $b$ is an index for determining a function $\varphi$ of $(b)_1$ arguments from any function $\psi$ of $m$ arguments by adjoining instances of primitive recursive schemata to the true numerical equations for $\psi$. The $n$-ary function defined in that case from the function $\psi$ will be denoted by $[b]^\psi$.



**4.2.** Definition. $\theta$ is the *universal recursive function* for the primitive recursive functions; that is, for all $n$

$$\theta(b, n) = [b](n).$$

**4.3.** Definition. Given a formula $E$ with parameters $\vec{a} = a_0, \ldots, a_k$ we define when a natural number $b$ *prim-realizes* $E$ for a given assignment $\vec{n} = n_0, \ldots, n_k$ to the parameters $\vec{a}$, and express it symbolically

$$b \, R_{\vec{n}} \, E$$

as follows:

i. $E$ is an atomic formula and $E\mathbf{n}_0 \ldots \mathbf{n}_k$ is true under the canonical interpretation.

ii. $E = E_0 \wedge E_1$ and $(b)_0 R_{\vec{n}} E_0$, $(b)_1 R_{\vec{n}} E_1$.

iii. $E = E_0 \vee E_1$ and either $(b)_0 = 0$, $(b)_1 R_{\vec{n}} E_0$ or $(b)_0 = 1$, $(b)_1 R_{\vec{n}} E_1$.

iv. $E = \exists x A x$ and $(b)_1 R_{\vec{n},(b)_0} A a_{k+1}$.

v. $E = \forall x A x$ and $\mathrm{In}^2(b)$ and for all $m$, $[b]^\theta(m) R_{\vec{n},m} A a_{k+1}$.

**4.4.** We shall assume that the syntax of **NA** has been arithmetized and we shall follow the custom of identifying formulae, deductions, etc. with their Gödel numbers.

**4.5.** Definition. $\Pi$ is a *proper deduction* iff

i. the proper parameter of an application of $\forall$-introduction occurs in $\Pi$ only in formulae occurrences above the consequence of the application of the rule,

ii. the proper parameter of an application of $\exists$-elimination in $\Pi$ occurs only in formulae occurrences above the minor premisses,

iii. every proper parameter in $\Pi$ is a proper parameter of exactly one application of the $(\forall I)$ rule or the $(\exists E)$ rule.

It can be shown without to much difficulty that every deduction in **NA** can be transformed into a proper deduction (see Prawitz 1965, § 3).

**4.6.** Given a deduction $\Pi$ it is clear that we can primitive recursively determine the parameters that occur in undischarged assumptions of $\Pi$. We shall let Param $(\Pi)$ be the enumeration, in increasing order, of the parameters that occur in undischarged assumptions of $\Pi$. Similarly, Assum $(\Pi)$ is an enumeration of the formulae occurrences of the undischarged assumption formulae of $\Pi$.

**4.7.** Theorem. *If a sentence $A$ is provable in* **NA** *then $A$ is prim-realizable.*

Proof. We shall prove by induction of the length of $\Pi$ that to every proper deduction $\Pi$ we can associate a natural number $e$ such that

$$[e]^\theta(\vec{n}, \vec{b}, \vec{m}) \, R_{\vec{n}\vec{m}} \, A$$

whenever

$\vec{n}$ is an assignment to Param ($\Pi$),

$\vec{b}$ is a sequence of numbers which prim-realize the formulae in Assum ($\Pi$),

$\vec{m}$ is an assignment to the parameters of $A$ that do not occur in Param ($\Pi$),

(such a number $e$ is said to realize the deduction $\Pi$).

Basis step. $\Pi$ consists of exactly one formula. Then $A$ must either be the assumption formula of $\Pi$ or else $A$ is an axiom (and hence an atomic formula). In either case it is clear how to determine the required natural number $e$.

Induction steps.

Case 1. The last rule of inference was $(\wedge I)$ so that $\Pi$ must have been of the form

$$\frac{\Pi_0, \Pi_1}{A_0 \wedge A_1}.$$

According to the induction hypothesis there are natural numbers $e_0, e_1$ which realize $\Pi_0, \Pi_1$ respectively. Let $\vec{n}$ be an assignment to Param ($\Pi$), $\vec{b}$ a sequence of numbers which realize the formulae in Assum ($\Pi$), $\vec{m}$ an assignment to the parameters of $A_0 \wedge A_1$. The assignments $\vec{n}, \vec{b}, \vec{m}$ can be primitive recursively split into $\vec{n}_0, \vec{n}_1, \vec{b}_0, \vec{b}_1, \vec{m}_0, \vec{m}_1$ so that $\vec{n}_0$ is an assignment to the Param ($\Pi_0$) and so on. We thus need a natural number $e$ such that

$$[e]^\theta(\vec{n}, \vec{b}, \vec{m}) = \langle [e_0]^\theta(\vec{n}_0, \vec{b}_0, \vec{m}_0), [e_1]^\theta(\vec{n}_1, \vec{b}_1, \vec{m}_1)\rangle.$$

Such a number can clearly be found.

Cases 2-6. The last rule of inference applied in $\Pi$ is either $(\wedge E)$, $(\vee I)$, $(\forall I)$, $(\forall E)$, $(\exists I)$. Similar to case 1.

Case 7. The last rule of inference applied in $\Pi$ is $(\vee E)$. Thus $\Pi$ must be of the form

$$\frac{\begin{matrix} & A_0 & A_1 \\ \ddots\vdots\,.\cdot & \vdots & \vdots \\ A_0 \vee A_1 & C & C \end{matrix}}{C} = \frac{\Pi_0, \Pi_1, \Pi_2}{C}.$$

Let $\vec{n}, \vec{b}, \vec{m}$ be assignments to Param ($\Pi$), Assum ($\Pi$) and parameters of $C$ that do not occur in Param ($\Pi$). Let $P, P_0, P_1$ be the set of parameters that occur in $A_0 \vee A_1$, $A_0$, $A_1$ resp. that occur neither in Param ($\Pi$) nor in $C$. Let $\vec{n}_0, \vec{n}_1, \vec{n}_2$ be assignments to Param ($\Pi_0$), Param ($\Pi_1$), Param ($\Pi_2$) which agree with $\vec{n}$ on Param ($\Pi$) and which assign 0 to the parameters in $P, P_0, P_1$ respectively. Then let $\vec{m}_0$ be the assignment which agrees with $\vec{m}$ outside of $P$ and which assigns 0 to the parameters in $P$.



According to the induction hypothesis there are natural numbers $e_0, e_1, e_2$ which realize the deductions $\Pi_0, \Pi_1, \Pi_2$. Let $e$ be a natural number such that:

$$[e]^\theta(\vec{n}, \vec{b}, \vec{m}) = \mathrm{sg}\big(([e_0]^\theta(\vec{n}_0, \vec{b}, \vec{m}_0))_0\big)\cdot[e_1]^\theta\big(\vec{n}_1, \vec{b}^* \langle([e_0]^\theta(\vec{n}_0, \vec{b}, \vec{m}_0))_1\rangle, \vec{m}\big) + \\ + \overline{\mathrm{sg}}\big(([e_0]^\theta(\vec{n}_0, \vec{b}, \vec{m}_0))_0\big)\cdot[e_2]^\theta\big(\vec{n}_2, \vec{b}^* \langle([e_0]^\theta(\vec{n}_0, \vec{b}, \vec{m}_0))_1\rangle, \vec{m}\big).$$

Such a number $e$ can be found and $e$ realizes $\Pi$.

Case 8. The last rule of inference applied in $\Pi$ is $(\exists E)$. Similar to case 7.

Cases 9-15. The last rule of inference applied in $\Pi$ is one of the rules of inference for $=$ or $\#$. In this case the conclusion is an atomic formula so the determination of $e$ is straightforward.

Case 16. The last rule of inference applied was (IND). Thus $\Pi$ must be of the form

$$\frac{\dfrac{\Sigma_0}{A0} \qquad \dfrac{\begin{matrix}[Aa]\\ \Sigma_1\end{matrix}}{Asa}}{At} = \frac{\Pi_0, \Pi_1}{At}.$$

According to the induction hypothesis there are natural numbers $e_0, e_1$ which realize the deductions $\Pi_0, \Pi_1$ respectively. Let $\vec{n}, \vec{b}, \vec{m}$ be such that

$\vec{n}$ is an assignment to Param($\Pi$),

$\vec{b}$ is a sequence of numbers which prim-realizes the formulae in Assum($\Pi$),

$\vec{m}$ is an assignment to the parameters of $At$ that do not occur in Param($\Pi$).

First of all note that there is a natural number $e_3$ such that

$[e_3]^\theta(\vec{n}, \vec{m})$ = the value of the term $t$ under the assignment $\vec{n}^*\vec{m}$ to the parameters of $t$. Next we observe that we can obtain a natural number $e_4$ such that

$$[e_4]^\theta(\vec{n}, \vec{b}, \vec{m}, 0) = [e_0]^\theta(\vec{n}, \vec{b}, \vec{m}),$$
$$[e_4]^\theta(\vec{n}, \vec{b}, \vec{m}, k+1) = [e_1]^\theta(\vec{n}, \vec{b}^*\langle[e_4]^\theta(\vec{n}, \vec{b}, \vec{m}, k)\rangle, \vec{m}).$$

Finally let $e$ be such that

$$[e]^\theta(\vec{n}, \vec{b}, \vec{m}) = [e_4]^\theta(\vec{n}, \vec{b}, \vec{m}, [e_3]^\theta(\vec{n}, \vec{m})).$$

**4.8.** Remark. The use of $\theta$ can be avoided if we assume that **NA** has only finitely many function symbols.

## § 5. HA as an extension of NA

**5.1.** Theorem. *There is a prenex sentence $A$ of the language of* **NA** *such that* **HA** $\vdash A$ *and yet $A$ is not derivable in* **NA**.

Proof. Let $T(x, y, z)$ be Kleene's $T$ predicate and let $f_T$ be the function symbol corresponding to the representing function of $T$. Then let $k$ be the Gödel number of a provably recursive function which is not primitive recursive in $\theta$. Then **HA** $\vdash \forall x \exists y [f_T(\mathbf{k}, x, y) = \mathbf{0}]$. On the other hand $\forall x \exists y [f_T(\mathbf{k}, x, y) = \mathbf{0}]$ could not be provable in **NA** because suppose it were; then there would be a function $\varphi$ primitive recursive in $\theta$ such that for all $n$, $T(k, n, \varphi(n))$, contradicting the choice of $k$.

## § 6. Extensions of NA

**6.1.** It is unlikely that anyone would claim that the sentence $\forall x \exists y [f_T(\mathbf{k}, x, y) = \mathbf{0}]$ fails to be meaningful from a negationless viewpoint. Thus **NA** is incomplete in the sense that rather simple negationless correct sentences in the language of **NA** are not provable in **NA**.

One way to try to eliminate the weakness of **NA** is to introduce more logical connectives in particular one could introduce connectives that approximate, in a negationless fashion, the conditional. However, there are other alternatives.

**NA** is supposed to be a theory about the natural numbers. In particular the quantifiers are intended to range over the natural numbers. Or put in another way, the quantifiers $\forall$ and $\exists$ may be viewed as infinite conjunctions and disjunctions resp. Naturally, arbitrary infinite conjunctions and disjunctions may be objectionable to some constructivist, however, if one restricts oneself to conjunctions $\bigwedge_n A_n$ and disjunctions $\bigvee_n A_n$ where the $A_n$, $n = 0, 1, 2, \ldots$ are given by some predetermined type of rule (for example, a primitive recursive function) then such objections can be minimized. There might still be objections on the grounds that a sentence $\bigwedge_n A_n$ (specially if written $A_0 \wedge A_1 \wedge A_2 \wedge \ldots$) requires that an infinite set be constructed (and completed). However such fears are groundless because $\bigwedge_n A_n$ can be considered to be another name for the rule that generates the $A_0, \ldots, A_n, \ldots$ Or put in more formalistic terms: The infinitary formulae can be constructed as well-founded trees given by lawlike functions and all we ever set down is the law (of the lawlike function).

There are some further advantages to using the infinitary conjunctions and disjunctions; for example one can avoid the use of parameters. Also $\exists$, $\forall$, $\vee$ and $\wedge$ are definable in terms of the infinitary conjunction and disjunction. Let $\mathbf{NA}_\infty$ be the system so obtained. That is,



$\mathbf{NA}_\infty$ has as axioms all true atomic sentences (recall that the distinctness relation $\#$ produces atomic formulae). The rules of inference of $\mathbf{NA}_\infty$ are the rules for $=$ and $\#$ given in 2.6 together with the following rules for $\wedge$, $\vee$.

$$(\wedge I)\ \frac{\{A_n\colon n = 0, 1, \ldots\}}{\bigwedge_n A_n} \qquad (\wedge E)\ \frac{\bigwedge_n A_n}{A_m}$$

$$(\vee I)\ \frac{A_m}{\bigvee_n A_n} \qquad (\vee E)\ \frac{\bigvee_n A_n \quad \begin{matrix} A_0 & A_1 & A_2 & \\ \vdots & \vdots & \vdots & \\ C, & C, & C, & \ldots \end{matrix}}{C}$$

and the structural rule of repetition

$$(\mathrm{Rep})\ \frac{A}{A}.$$

**6.2.** Derivations in $\mathbf{NA}_\infty$ are well-founded trees of sentences. The top most occurrences being either assumption sentences or axioms. Using a suitable arithmetization, the derivation in $\mathbf{NA}_\infty$ can be coded into number theoretical functions. Let $\mathbf{NA}_{\infty,\mathcal{F}}$ be the system obtained by requiring that the derivations (as functions) be in the class $\mathcal{F}$ of functions.

**6.3.** We shall assume that $\mathcal{F} \supseteq \boldsymbol{PR}$ (the class of primitive recursive functions).

**6.4.** Given a sentence $A$ of **HA** let $A^*$ be the sentence obtained by replacing subformulae of the forms: $\exists x Bx$, $\forall x Bx$, $B \vee C$, $B \wedge C$ by $\bigvee_n B\mathbf{n}$, $\bigwedge_n B\mathbf{n}$, $B \vee C \vee C \vee C \vee \ldots$ and $B \wedge C \wedge C \wedge \ldots$ respectively. Clearly, if $A$ is a sentence of **NA** then $A^*$ is a sentence of $\mathbf{NA}_\infty$.

**6.5.** Lemma. *If the sentence $A$ is provable in* **NA** *then $A^*$ is provable in* $\mathbf{NA}_{\infty,\mathcal{F}}$.

Proof. By induction on the length of the proof of $A$ in **NA**. The only case worth mentioning is that of (IND). Given an application of (IND)

$$\frac{\dfrac{\Sigma'}{A0} \qquad \dfrac{\dfrac{[Aa]}{\Sigma(a)}}{Asa}}{\forall x Ax}$$

on then transforms it to an application of $(\bigwedge I)$ as follows

$$\dfrac{\dfrac{\Sigma'}{A0}, \quad \dfrac{\begin{matrix}\dfrac{\Sigma'}{[A0]} \\ \Sigma(0)\end{matrix}}{A1}, \quad \dfrac{\begin{matrix}\dfrac{\Sigma'}{[A0]} \\ \Sigma(0) \\ [A1] \\ \Sigma(1)\end{matrix}}{A2}, \ldots}{\bigwedge_n An}$$

**6.6.** Let $\mathbf{HA}_{\infty,\mathcal{F}}$ be the corresponding infinitary extension of Heyting's arithmetic (i.e. using the connectives $\bigwedge, \bigvee, \to$). Then one immediately obtains that for any sentence $A$ of **HA** if $\mathbf{HA} \vdash A$, then $\mathbf{HA}_{\infty,\mathcal{F}} \vdash A^*$. In addition we have the following conservative extension result.

**6.7.** Lemma. *If $A$ is a sentence of* $\mathbf{NA}_\infty$, *then $A$ is provable in* $\mathbf{HA}_{\infty,\mathcal{F}}$ *iff $A$ is provable in* $\mathbf{NA}_{\infty,\mathcal{F}}$.

Proof. An immediate consequence of the subformula property of the normal derivations in $\mathbf{HA}_{\infty,\mathcal{F}}$ and the fact that every derivation in $\mathbf{HA}_{\infty,\mathcal{F}}$ can be transformed into a normal derivation [cf. Prawitz, 1971].

**6.8.** Given any sentence $A$ of $\mathbf{NA}_\infty$ let $\sim A$ be the sentence obtained as follows:

if $A \equiv (\tau = \sigma)$ then $\sim A \equiv (\tau \neq \sigma)$,

if $A \equiv (\tau \neq \sigma)$ then $\sim A \equiv (\tau = \sigma)$,

if $A \equiv \bigwedge_n B_n$ then $\sim A \equiv \bigvee_n \sim B_n$,

if $A \equiv \bigvee_n B_n$ then $\sim A \equiv \bigwedge_n \sim B_n$.

Let $\mathfrak{A}$ be the collection of *all* number theoretic functions. Then a simple (non-constructive) induction on the rank of $A$ gives the following lemma.

**6.9.** Lemma. *For every sentence $A$ of* $\mathbf{NA}_\infty$ *either $A$ or $\sim A$ is provable in* $\mathbf{NA}_{\infty,\mathfrak{A}}$.

## § 7. Implicative extensions of NA

**7.1.** One may reasonably claim, in view of the results above, that any worthwhile discussion of negationless arithmetic must include some kind of implication which can be iterated. The favourite candidate appears to be quantified implication: $Ax \to_x Bx$.

Quantified implication, the use of variables and Griss' criterion on the well-formed points have some strange consequences. Note first that



the negationless interpretation of $Ax \to_x Bx$ is basically that of $\forall x(Ax \to Bx) \wedge \exists x Ax$. Suppose now that $x$ does not occur free in $A$, then $A \to_x Bx$ leads to $A \wedge \forall x Bx$. In particular $y \neq z \to_x B(y, z, x)$ leads to $y \neq z \wedge \forall x B(y, z, x)$.

Instead of quantified implication $\to_x$ one may introduce an infinitary connective $\Rightarrow$ such that whenever $\langle A_n \rangle_n$, $\langle B_n \rangle_n$ are infinite sequences of formulae (given in the reader's favourite constructive way) then $\langle A_n \rangle_n \Rightarrow \langle B_n \rangle_n$ is also a formula. Instead of $\langle A_n \rangle_n \Rightarrow \langle B_n \rangle_n$ one may write: $A_n \Rightarrow_n B_n$. We propose that the classical interpretation of $A_n \Rightarrow_n B_n$ be $\bigwedge_n (A_n \to B_n) \wedge \bigvee_n A_n$. Whether $A_n \Rightarrow_n B_n$ can be justified negationless remains to be seen; one small thing in its favour is that $A(x) \to_x B(x)$ can be interpreted as $A(\mathbf{n}) \Rightarrow_n B(\mathbf{n})$.

**7.2.** The rules of inference associated with $\Rightarrow$ are

$$(\Rightarrow I) \qquad \dfrac{\begin{matrix} A_0 & A_1 & & A_n & \\ \vdots & \vdots & & \vdots & \\ A_m, & B_0, & B_1, \ldots, & B_n, & \ldots \end{matrix}}{A_n \Rightarrow_n B_n}$$

and

$$(\Rightarrow E) \qquad \dfrac{A_n \Rightarrow_n B_n, A_m}{B_m}.$$

The derivations should be restricted to some suitable class of constructive functions. In any case for reasonable classes of constructive functions one can prove a normalization theorem (i.e. eliminability of maximal segments in the segments in the sense of Prawitz 1971). From the normalization theorem one can show that unsatisfiable subformulae can be removed from closed derivations. More specifically given a formula $A$ let $A^*$ be the formula obtained from $A$ by deleting all unsatisfiable atomic formulae (some convention must be agreed upon for dealing with infinite disjunctions (conjunctions) which may become finite when the unsatisfiable disjuncts (conjuncts) are eliminated). Then if $A$ is derivable, so is $A^*$ and $A^*$ has a derivation in which only formulae of the form $B^*$ are used.

## § 8. The functional interpretation

**8.1.** A calculus which only involves equations between numerical terms is certainly negationless acceptable. Thus one might be tempted to believe that Gödel's Dialectica interpretation gives a negationless interpretation for intuitionistic arithmetic.

There is no doubt that if $A^G$ is the Gödel interpretation of $A$, then $A^G$ is negationless acceptable (at least for $A$ which are provable in **HA**) even if $A$ is not. The trouble is that $A$ itself may be negationless acceptable but the negationless interpretation of $A$ may be different from that $A^G$. Thus the Dialectica interpretation does not succeed in showing that every negationless acceptable statement which is provable using non-acceptable formulae has a derivation which only uses negationless acceptable formulae.

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# On nonmonotone inductive definability

by

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**Abstract.** The paper studies the class of relations on a set $A$ which are defined inductively by nonmonotone operators in some collection $\mathfrak{F}$ satisfying certain minimal structural conditions. There are several concrete applications, including the construction of some interesting admissible sets.

The purpose of this paper is to apply the methods of EIAS (2) to the study of nonmonotone inductions.

In the first three sections we have attempted to codify the most basic properties of nonmonotone induction. These are general versions of tricks and methods which have been well known to the researchers in this field for some time. Many of them were formulated in similar abstract forms independently by P. Aczel, see Aczel [1973].

After the basics, we apply the theory of *Spector classes* of Chapter 9 of EIAS in Sections 4, 5 to obtain a characterization of the class of inductive relations relative to a "typical, nonmonotone class of operators." This is Theorem 15, the main result of the paper.

In Section 6 we consider in some detail the important examples of inductive definability in the higher order language over a structure — i.e. $\Sigma_k^m$- and $\Pi_k^m$-inductive definability, $m = 0$ and $k \geqslant 2$ or $m, k \geqslant 1$. The significant but "atypical" case of $\Pi_1^0$-induction is discussed briefly in Section 8.

Finally, in Section 7 we apply the theory of *companions of Spector classes* of Chapter 9 of EIAS to characterize various nonmonotone inductions in terms of admissible sets with related, interesting properties. The main result here is Theorem 21. There are also some applications

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(2) By EIAS we will refer to *Elementary Induction on Abstract Structures*, Amsterdam 1973.