Relativization of cylindric algebras (1)

by

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Abstract. The classes of cylindric algebras, and of cylindric relativized algebras, are described, and it is shown that the latter can be characterized by equational identities for dimension $\leqslant 2$, but not for higher dimensions. Some other results for higher dimensions are mentioned without proof.

It is well and widely known that if

$$\mathfrak{D} = \langle B, +, \cdot, -, 0, 1\rangle$$

is any Boolean algebra (BA) and a is any (3) element of B, then the function h_a on B such that $h_a(x) = a\cdot x$ for all $x \in B$, is a homomorphism of $\mathfrak{D}$ onto another BA

$$\mathfrak{D}_a = \langle B_a, +, \cdot, -_a, 0, a\rangle ,$$

called the *relativization of* $\mathfrak{D}$ *with respect to* a, where $B_a = \{x \in B|\ x \leqslant a\}$ and $-_a x = a - x$ for all $x \in B_a$. In this paper we shall study the process of relativization in the case of cylindric algebras (CA's), which are a kind of multi-dimensional BA's; we shall be concerned especially with CA's of dimension 2 and 3.

It turns out that the process of relativization extends in a natural way from BA's to CA's, but that in general an algebra obtained by relativizing a CA of dimension greater than 1 is not itself a CA; we call a structure obtained in this way a *cylindric relativized algebra* (Cr). In § 2 below we shall characterize the class of 2-dimensional Cr's by means of equational identities, and in § 3 we shall show that such a characteri-

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(2) Some of the results to be reported below were obtained by Resek in 1968 while working at the University of Warsaw, some were obtained by Henkin in 1969 while working at the Mathematical Institute and at All Souls College, Oxford University. Both authors have had the pleasure and mathematical stimulation of contact with Andrzej Mostowski.

(3) If BA's are defined to require that $0 \neq 1$, then a must be chosen so that $a \neq 0$.

zation is impossible for the class of Cr's of dimension 3 or higher. Some results concerning CA's and subalgebras of Cr's for dimensions greater than 2 are briefly mentioned in the concluding § 4. In § 1 we present the basic definitions and known results which are used in the sequel.

§ 1. The concept of BA's is abstracted from the notion of a Boolean set algebra

$$\mathfrak{F} = \langle F, \cup, \cap, \sim, \emptyset, V \rangle,$$

where V is an arbitrary set, $\sim$ is the operation of complementation with respect to V, and F is a non-empty family of subsets of V closed under $\cup, \cap, \sim$. In case V is the Cartesian power of some other set U, say $V = {}^{\alpha}U$ where α is any ordinal number ([4]), then we can consider certain richer structures called *cylindric set algebras*, obtained as follows. For $\varkappa, \lambda < \alpha$ we distinguish the *diagonal sets*

$$D_{\varkappa\lambda} = \{w \in {}^{\alpha}U \mid w_{\varkappa} = w_{\lambda}\},$$

and we consider the operations $C_{\varkappa}$ of *cylindrification* such that, for any $X \subseteq {}^{\alpha}U$,

$C_{\varkappa}X = \{y \in {}^{\alpha}U \mid$ for some $x \in X$, $y_{\lambda} = x_{\lambda}$ for every $\lambda \neq \varkappa$, $\lambda < \alpha\}$ ([5]).
If G is any family of subsets of ${}^{\alpha}U$ such that $D_{\varkappa\lambda} \in G$ for all $\varkappa, \lambda < \alpha$ and G is closed under each of the operations $\cup, \cap, \sim, C_{\varkappa}$, then the structure

$$\mathfrak{G} = \langle G, \cup, \cap, \sim, \emptyset, {}^{\alpha}U, C_{\varkappa}, D_{\varkappa\lambda} \rangle_{\varkappa,\lambda<\alpha}$$

is called an *α-dimensional cylindric set algebra* (CSA_{α}).

The notion of a cylindric algebra is obtained by abstraction from the notion of a cylindric set algebra by selecting certain equations which hold identically in every cylindric set algebra and using these as axioms to define the class of CA's. Specifically, by an *α-dimensional cylindric algebra*, CA_{α}, we mean any structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_{\varkappa}, d_{\varkappa\lambda} \rangle_{\varkappa,\lambda<\alpha}$$

such that

(C_0) the structure $\langle A, +, \cdot, -, 0, 1 \rangle$ is a BA, each $d_{\varkappa\lambda} \in A$ and each $c_{\varkappa}$ is a one-place operation on A; and for every $x, y \in A$ and $\varkappa, \lambda, \mu < \alpha$:

(C_1) $c_{\varkappa}0 = 0$;

(C_2) $x \leqslant c_{\varkappa}x$ (i.e., $x \cdot c_{\varkappa}x = x$);

(C_3) $c_{\varkappa}(x \cdot c_{\varkappa}y) = c_{\varkappa}x \cdot c_{\varkappa}y$;

(C_4) $c_{\varkappa}c_{\lambda}x = c_{\lambda}c_{\varkappa}x$;

(C_5) $d_{\varkappa\varkappa} = 1$;

(C_6) if $\varkappa \neq \lambda, \mu$ then $d_{\lambda\mu} = c_{\varkappa}(d_{\lambda\varkappa} \cdot d_{\varkappa\mu})$;

(C_7) if $\varkappa \neq \lambda$ then $c_{\varkappa}(d_{\varkappa\lambda} \cdot x) \cdot c_{\varkappa}(d_{\varkappa\lambda} \cdot -x) = 0$ ([6]).

Among the laws, derivable from these axioms, which hold in every CA_{α}, are the following:

(1.1)
(i) $c_{\varkappa}c_{\varkappa}x = c_{\varkappa}x$;
(ii) $c_{\varkappa}(x+y) = c_{\varkappa}x + c_{\varkappa}y$;
(iii) $c_{\varkappa}(-c_{\varkappa}x) = -c_{\varkappa}x$;
(iv) $d_{\varkappa\lambda} = d_{\lambda\varkappa}$;
(v) $c_{\varkappa}d_{\varkappa\lambda} = 1$;
(vi) $c_{\varkappa}d_{\lambda\mu} = d_{\lambda\mu}$ if $\varkappa \neq \lambda, \mu$;
(vii) $c_{\varkappa}(d_{\varkappa\lambda} \cdot -x) = -c_{\varkappa}(d_{\varkappa\lambda} \cdot x)$ if $\varkappa \neq \lambda$.

From (i), (ii), (iii), and (C_3) we see that the set

$$c_{\varkappa}^{*}A = \{c_{\varkappa}x \mid x \in A\}$$

is closed under $+, \cdot, -$, and hence by (C_1) the structure

$$c_{\varkappa}^{*}\mathfrak{A} = \langle c_{\varkappa}^{*}A, +, \cdot, -, 0, 1 \rangle \tag{1.2}$$

is a BA, the BA *of $\varkappa$-cylinders of* $\mathfrak{A}$.

For any element a of a $\mathrm{CA}_{\alpha}\mathfrak{A}$, let $d_{\varkappa\lambda}^{a} = a \cdot d_{\varkappa\lambda}$ and let $c_{\varkappa}^{a}$ be the operation such that $c_{\varkappa}^{a}x = a \cdot c_{\varkappa}x$ for all $x \in A$. Then the structure

$$\mathfrak{A}_a = \langle A_a, +, \cdot, -_a, 0, a, c_{\varkappa}^{a}, d_{\varkappa\lambda}^{a} \rangle_{\varkappa,\lambda<\alpha}$$

is called a *cylindric relativized algebra of dimension α*, Cr_{α}. Note that the mapping h_a of A onto A_a, such that $h_a x = a \cdot x$ for all $x \in A$, is *not* in general a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_a$, since in general we will not have $h_a(c_{\varkappa}x) = c_{\varkappa}^{a}(h_a x)$ (although this will be true in case $c_{\varkappa}a = a$, by (C_3)). In consequence we cannot be sure that all of the equational identities (C_1)-(C_7) holding in the $\mathrm{CA}_{\alpha}\mathfrak{A}$, will also hold in the $\mathrm{Cr}_{\alpha}\mathfrak{A}_a$. It turns out that while every Cr_{α} is a CA_{α} for $\alpha = 0,1$, it is easy to construct a Cr_{α} which is not a CA_{α} for every $\alpha > 1$; for example, in the CSA_{α} $\mathfrak{A}$ of all subsets of ${}^{\alpha}U$, where U is a set with more than one element, if we put $a = -d_{01}$ then the $\mathrm{Cr}_{\alpha}\mathfrak{A}_a$ will fail to satisfy (C_4) and (C_6). However, we have the following result.

(1.3) Proposition. *In any* Cr_{α}, *all of the laws* (C_0)-(C_3), (C_5), (C_7), *and* (1.1) (i)-(iv) *hold identically. If* $\mathfrak{A}$ *is a* CA_{α} *and* $\mathfrak{A}_a$ *is a* Cr_{α} *obtained from* $\mathfrak{A}$ *by relativizing to the element* a, *then* $\mathfrak{A}_a$ *will also satisfy* (C_4) *and*

([4]) We assume ordinals defined so that each ordinal number coincides with the set of its predecessors.

([5]) If we visualize ${}^{\alpha}U$ as an α-dimensional Cartesian space over U, then each $D_{\varkappa\lambda}$ is a diagonal hyperplane of the space, and for any point set X of the space, $C_{\varkappa}X$ is the cylinder generated by translating X parallel to the $\varkappa$th coordinate axis.

([6]) This definition and much of the elementary theory of CA's, in particular all results stated below in this section without proof, are to be found in [2].

(C_6) — *and hence will be a* CA_α — *iff the following conditions hold for all* $\varkappa, \lambda < \alpha$ *and* $x \in A_a$:

(i) $c_\varkappa(c_\lambda x \cdot a) \cdot a \leqslant c_\lambda(c_\varkappa x \cdot a)$;

(ii) $a \leqslant c_\varkappa(d_{\varkappa\lambda} \cdot a)$.

The principal result of § 2, Theorem 2.8, states that for dimension $\alpha = 2$, the identities (C_0)-(C_3), (C_5), (C_7), and (1.1) (iv) characterize the class of Cr_α' s. Given any structure $\mathfrak{D}$ satisfying these identities, we shall construct a CA_2 $\mathfrak{A}$, and an element a of $\mathfrak{A}$, such that $\mathfrak{D} = \mathfrak{A}_a$. This construction will proceed by first forming certain BA's associated with $\mathfrak{D}$, and then combining these in a suitable way by forming direct products and Boolean products.

The well-known construction of the *direct product* $\underset{i \in I}{\boldsymbol{P}} \mathfrak{C}_i$ of a given family $\{\mathfrak{C}_i\}_{i \in I}$ of similar structures [7] produces a structure $\mathfrak{C}$, and canonical homomorphisms p_i (called projections) of $\mathfrak{C}$ onto $\mathfrak{C}_i$ for each i, which can be characterized up to isomorphism by the property that, given any structure $\mathfrak{C}'$ with homomorphisms p'_i onto $\mathfrak{C}_i$ for each i, there exists a unique homomorphism h of $\mathfrak{C}'$ onto $\mathfrak{C}$ such that $p'_i = p_i \circ h$ for each $i \in I$. Since the formation of a direct product preserves equational identities, we can be sure that the direct product of BA's is itself a BA. In case each $\mathfrak{C}_i$ is a Boolean set algebra of subsets of some set V_i, where $V_i \cap V_j = \emptyset$ when $i \neq j$, the direct product $\mathfrak{C}$ of the algebras $\mathfrak{C}_i$ is isomorphic to a Boolean set algebra of subsets of $\bigcup_{i \in I} V_i$, the elements of $\mathfrak{C}$ consisting of all sets $\bigcup_{i \in I} X_i$ such that $X_i \in \mathfrak{C}_i$ for each $i \in I$.

The construction of a *Boolean product* $\mathfrak{D}$ of given BA's $\{\mathfrak{D}_i\}_{i \in I}$, while related to the general algebraic notion of free product, is special to the theory of BA's [8]. The Boolean product $\mathfrak{D}$ is a BA, equipped with canonical homomorphisms q_i of $\mathfrak{D}_i$ into $\mathfrak{D}$ for each i, and can be characterized up to isomorphism by the property that given any BA $\mathfrak{D}'$ with homomorphisms q'_i of $\mathfrak{D}_i$ into $\mathfrak{D}'$ for each i, there exists a unique homomorphism g of $\mathfrak{D}$ into $\mathfrak{D}'$ such that $q'_i = g \circ q_i$ for each $i \in I$. It is not hard to show that if $\mathfrak{D}$ is the Boolean product of the BA's $\{\mathfrak{D}_i\}_{i \in I}$ and if q_i is the canonical homomorphism of $\mathfrak{D}_i$ into $\mathfrak{D}$, then q_i is one-one, and $\mathfrak{D}$ is generated by the union of the subalgebras $q_i^*(\mathfrak{D}_i)$.

In case each $\mathfrak{D}_i$ is a Boolean set algebra of subsets of some set V_i, the Boolean product $\mathfrak{D}$ of the algebras $\mathfrak{D}_i$ is isomorphic to a Boolean set algebra of subsets of the Cartesian product $\underset{i \in I}{\boldsymbol{P}} V_i$; the elements of this field consist of all finite unions of sets $\underset{i \in I}{\boldsymbol{P}} X_i$ such that $X_i \in \mathfrak{D}_i$ for each

(7) See [2], Sec. 0.3.

(8) See [4], § 13. Within the theory of duality for BA's, the constructions of direct products and Boolean products are dual notions.

$i \in I$ and (in case I is infinite) $X_i = V_i$ for all but finitely many $i \in I$. The canonical isomorphism q_i in this case is defined so that, for each element X of $\mathfrak{D}_i$, $q_i(X) = \underset{j \in I}{\boldsymbol{P}} Y_j$ where $Y_i = X$ and $Y_j = V_j$ for each $j \in I \sim \{i\}$.

Since every BA is isomorphic to a Boolean set algebra by the Stone representation theorem, the above example actually provides a way to construct the Boolean product of an arbitrary family $\{\mathfrak{D}_i\}_{i \in I}$ of BA's. When a Boolean product $\mathfrak{D}$ is obtained in this way as a Boolean set algebra of subsets of some set $\underset{i \in I}{\boldsymbol{P}} V_i$, we can introduce cylindrifications c'_j on $\mathfrak{D}$ for each $j \in I$ in the same way as the operations c_j were defined for CSA's, specifying that for any element z of $\mathfrak{D}$,

$$c'_j z = \{v \in \underset{i \in I}{\boldsymbol{P}} V_i \mid \text{for some } u \in z,\ v_i = u_i \text{ for all } i \in I \sim \{j\}\}.$$

It is easy to infer that $\mathfrak{D}$ will be closed under each c'_j, from the fact that if $X_i \in \mathfrak{D}_i$ for every $i \in I$, then $c'_j(\underset{i \in I}{\boldsymbol{P}} X_i) = \underset{i \in I}{\boldsymbol{P}} Y_i$ where $Y_j = V_j$ or $Y_j = \emptyset$ according as $X_j \neq \emptyset$ or $X_j = \emptyset$, and $Y_i = X_i$ for all $i \in I \sim \{j\}$. The operations $\{c'_j\}_{j \in I}$ defined in this way satisfy all of the laws (C_1)-(C_4) that hold in CA's.

In the general case the connection between the canonical isomorphisms q_i of the BA's $\mathfrak{D}_i$ into the Boolean product $\mathfrak{D}$, and the cylindrifications c'_j on $\mathfrak{D}$ described above, is not very simple. However, for our purposes we shall only need to deal with this situation in the case where $I = \{0, 1\}$. In this special case the definitions for q_0, q_1 and c'_0, c'_1 show that q_j is an isomorphism of $\mathfrak{D}_j$ *onto* the BA $c'^*_{1-j}\mathfrak{D}$ for each $j = 0, 1$. Also, in this case, we have $c'_0 c'_1 z = 1$ and $c'_1 c'_0 z = 1$ whenever z is an element of $\mathfrak{D}$ with $z \neq 0$. Writing c_j for c'_{1-j} for convenience, we express these results by the following.

(1.4) Proposition. *Let* $\mathfrak{D}_0, \mathfrak{D}_1$ *be arbitrary* BA*'s, and let* $\mathfrak{D}$ *be their Boolean product with canonical isomorphisms* q_i *of* $\mathfrak{D}_i$ *into* $\mathfrak{D}$ *for* $i = 0, 1$. *Then there exist operations* c_0, c_1 *on* $\mathfrak{D}$ *satisfying the laws* (C_1)-(C_4), *such that* q_i *is an isomorphism of* $\mathfrak{D}_i$ *onto the* BA $c_i^*\mathfrak{D}$ *formed from* $\mathfrak{D}$ *as in* (1.2). *Furthermore, for any element* z *of* $\mathfrak{D}$, *if* $z \neq 0$ *then* $c_0 c_1 z = c_1 c_0 z = 1$. *We call the system* $\langle\mathfrak{D}, c_0, c_1\rangle$ *the cylindric product of the* BA*'s* $\mathfrak{D}_0$ *and* $\mathfrak{D}_1$.

§ 2. Throughout this section we shall use the word *structure* to refer to any system $\mathfrak{A}$ of the same similarity type as CA_2's and Cr_2's, i.e., to systems

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$$

where A is a set, $+$ and $\cdot$ are binary operations on A, $-$, c_0, c_1 are unary operations on A, and $0, 1, d_{00}, d_{01}, d_{10}, d_{11} \in A$.

(2.1) Definition. We let Ω be the *class of all structures* $\mathfrak{A}$ satisfying the equational identities (C_0)-(C_3), (C_5), (C_7), and (1.1) (iv).

Our aim is to show that $\Omega = \mathrm{Cr}_2$. Since we already know that $\mathrm{Cr}_2 \subseteq \Omega$ by (1.3), we concentrate on demonstrating the opposite inclusion. We shall need several elementary facts about structures in Ω, which we collect in the following.

(2.2) LEMMA. *Let $\mathfrak{A}$ be any structure in Ω. Then the following hold (for any $x, y \in A$ and any $\varkappa = 0, 1$):*

(i) $c_\varkappa c_\varkappa y = c_\varkappa y$,

(ii) $c_\varkappa(x+y) = c_\varkappa x + c_\varkappa y$,

(iii) $c_\varkappa(-c_\varkappa x) = -c_\varkappa x$,

(iv) $c_\varkappa x \leqslant c_\varkappa y$ *whenever* $x \leqslant y$,

(v) $c_\varkappa(-x\cdot d_{01}) = c_\varkappa d_{01}\cdot - c_\varkappa(x\cdot d_{01})$,

(vi) $c_\varkappa(x\cdot d_{01})\cdot d_{01} = x\cdot d_{01}$,

(vii) *the structure* $c_\varkappa^*\mathfrak{A}$, *formed as in* (1.2), *is a* BA,

(viii) *if* $x \in c_\varkappa^* A$ *then* $c_\varkappa(x\cdot y) = x\cdot c_\varkappa y$.

Proof. Parts (i)-(iii) are copied from (1.1); their proofs in the theory of CA's depend only on (C_0)-(C_3), which hold for $\mathfrak{A} \in \Omega$ by (2.1). Part (iv) is an immediate consequence of (ii), since from $x \leqslant y$ follows $x+y = y$.

To obtain (v) we first use (ii) to get

$$c_\varkappa(x\cdot d_{01}) + c_\varkappa(-x\cdot d_{01}) = c_\varkappa d_{01},$$

then by (C_7) get

$$c_\varkappa(x\cdot d_{01})\cdot c_\varkappa(-x\cdot d_{01}) = 0;$$

(v) then follows from the theory of BA's.

To obtain (vi) we first use (C_2) to get $x\cdot d_{01} \leqslant c_\varkappa(x\cdot d_{01})$, from which we see that $x\cdot d_{01} \leqslant c_\varkappa(x\cdot d_{01})\cdot d_{01}$.

The opposite inequality may be obtained by computing

$$\begin{aligned} c_\varkappa[c_\varkappa(x\cdot d_{01})\cdot d_{01}\cdot -(x\cdot d_{01})] &= c_\varkappa(x\cdot d_{01})\cdot c_\varkappa(d_{01}\cdot -(x\cdot d_{01})) && \text{by } (C_3) \\ &= c_\varkappa(x\cdot d_{01})\cdot c_\varkappa d_{01}\cdot - c_\varkappa(x\cdot d_{01}) && \text{by (v)} \\ &= 0, \end{aligned}$$

whence

$$c_\varkappa(x\cdot d_{01})\cdot d_{01}\cdot -(x\cdot d_{01}) = 0 \quad \text{by } (C_2),$$

and so $c_\varkappa(x\cdot d_{01})\cdot d_{01} \leqslant x\cdot d_{01}$, completing the proof of (vi).

Turning to part (vii), we see that $c_\varkappa^* A$ is closed under $+$ by (ii) and under $-$ by (iii) which, with (C_1), is enough to assure (vii). Finally, as to (viii), if $x \in c_\varkappa^* A$ then $x = c_\varkappa x$ by (i), hence $c_\varkappa(x\cdot y) = x\cdot c_\varkappa y$ by (C_3).

With (2.2) proved, let us return to the task of showing that $\Omega \subseteq \mathrm{Cr}_2$. To do this we choose any $\mathfrak{D} \in \Omega$ and must produce an $\mathfrak{A} \in \mathrm{CA}_2$, and an element a of $\mathfrak{A}$, such that $\mathfrak{D} = \mathfrak{A}_a$. To construct this $\mathfrak{A}$ we shall form various direct products and cylindric products of BA's, but in addition we need a new type of product for structures in Ω, which we now describe.

(2.3) DEFINITION. Let $\mathfrak{B}, \mathfrak{B}' \in \Omega$, where

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1},$$

$$\mathfrak{B}' = \langle B', +', \cdot', -', 0', 1', c'_\varkappa, d'_{\varkappa\lambda}\rangle_{\varkappa,\lambda=1,0}.$$

Suppose that for $\varkappa = 0, 1$ we have a one-one map $h_\varkappa\colon c_\varkappa^* B \to c'^*_\varkappa B'$, and let $b_\varkappa = h_\varkappa(1)$. Let

$$\langle A, +^A, \cdot^A, -^A, 0^A, 1^A, d^A_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$$

be the direct product

$$\langle B, +, \cdot, -, 0, 1, d_{\varkappa\lambda}\rangle \times \langle B', +', \cdot', -', 0', 1', d'_{\varkappa\lambda}\rangle,$$

and define operations c_0^A, c_1^A on A as follows: For $\varkappa = 0, 1$ and any $\langle x, y\rangle \in A$ set

$$c_\varkappa^A\langle x, y\rangle = \langle c_\varkappa x, c'_\varkappa y\rangle +^A \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y), h_\varkappa c_\varkappa x\rangle.$$

Then the structure

$$\mathfrak{A} = \langle A, +^A, \cdot^A, -^A, 0^A, 1^A, c_\varkappa^A, d^A_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$$

is called the *superposition product of $\mathfrak{B}$ and $\mathfrak{B}'$ (with respect to h_0 and h_1)*. In case h_0 and h_1 are fixed in advance we denote this $\mathfrak{B} \otimes \mathfrak{B}'$.

(2.4) LEMMA. *Let $\mathfrak{B}$, $\mathfrak{B}'$, h_0, h_1, b_0, b_1, be as in Definition* (2.3), *and let $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{B}'$ as defined there. Assume that for $\varkappa = 0, 1$, the map $h_\varkappa$ is an isomorphism of the* BA *$c_\varkappa^*\mathfrak{B}$ onto the* BA *$(c'^*_\varkappa\mathfrak{B})_{b_\varkappa}$ obtained by relativizing $c'^*_\varkappa\mathfrak{B}'$ to $b_\varkappa$. Assume, furthermore, that for $\varkappa = 0, 1$, we have $h_\varkappa(c_\varkappa d_{01})\cdot' d'_{01} = 0'$. Then $\mathfrak{A} \in \Omega$. If, furthermore, we have $c'_\varkappa c'_\lambda y = 1'$ whenever $\varkappa \neq \lambda$ and $y \neq 0'$ in $\mathfrak{B}'$, then also $c_\varkappa^A c_\lambda^A z = 1^A$ whenever $z \neq 0^A$ in $\mathfrak{A}$.*

Proof. It is clear from (2.3) that $\mathfrak{A}$ satisfies (C_0), since a direct product of BA's is a BA. It is also clear that $\mathfrak{A}$ satisfies (1.1) (iv), since $\mathfrak{B}$ and $\mathfrak{B}'$ do by (2.1), and $d^A_{\varkappa\lambda} = \langle d_{\varkappa\lambda}, d'_{\varkappa\lambda}\rangle$ for each $\varkappa, \lambda = 0, 1$. Thus, to show $\mathfrak{A} \in \Omega$ it remains to show that $\mathfrak{A}$ satisfies (C_1)-(C_3), (C_5), (C_7). Of these, (C_1), (C_2), and (C_5) are very simple and straight-forward computations, so we confine our efforts to proving (C_3) and (C_7).

Before demonstrating (C_3), let us show that (2.2) (ii), the additive law, holds for $c_\varkappa^A$ in $\mathfrak{A}$. For any $\varkappa = 0, 1$ and $\langle x, y\rangle, \langle u, v\rangle \in A$ we have:

$$\begin{aligned} c_\varkappa^A[\langle x, y\rangle +^A \langle u, v\rangle] &= c_\varkappa^A\langle x+u, y+'v\rangle \quad (\text{by definition of } +^A), \\ &= \langle c_\varkappa(x+u), c'_\varkappa(y+'v)\rangle +^A \langle h_\varkappa^{-1}(b_\varkappa\cdot' c'_\varkappa(y+'v)), h_\varkappa c_\varkappa(x+u)\rangle \\ &\qquad (\text{by definition of } c_\varkappa^A), \end{aligned}$$

$$= \langle c_\varkappa x + c_\varkappa u,\ c'_\varkappa y +' c'_\varkappa v\rangle +^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y +' b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa(c_\varkappa x + c_\varkappa u)\rangle$$

(by 2.2 (ii) for $\mathfrak{B}$, $\mathfrak{B}'$),

$$= \langle c_\varkappa x,\ c'_\varkappa y\rangle +^{\mathcal{A}} \langle c_\varkappa u,\ c'_\varkappa v\rangle +^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y),\ h_\varkappa c_\varkappa x\rangle$$

$$+^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa c_\varkappa u\rangle$$ (by the fact that $h_\varkappa$ is an isomorphism and by definition of $+^{\mathcal{A}}$),

$$= c_\varkappa^{\mathcal{A}}\langle x, y\rangle +^{\mathcal{A}} c_\varkappa^{\mathcal{A}}\langle u, v\rangle$$ (by (C_0) for $\mathfrak{A}$ and by definition of $c_\varkappa^{\mathcal{A}}$).

Thus (2.2) (ii) holds in $\mathfrak{A}$, and we now use it to show that (C_3) holds in $\mathfrak{A}$.

(C_3) For $\varkappa = 0, 1$ and $\langle x, y\rangle, \langle u, v\rangle \in A$ we have:

$$c_\varkappa^{\mathcal{A}}[\langle x, y\rangle \cdot^{\mathcal{A}} c_\varkappa^{\mathcal{A}}\langle u, v\rangle]$$

$$= c_\varkappa^{\mathcal{A}}[\langle x, y\rangle \cdot^{\mathcal{A}} \langle c_\varkappa u,\ c'_\varkappa v\rangle +^{\mathcal{A}} \langle x, y\rangle \cdot^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa c_\varkappa u\rangle]$$

$$= c_\varkappa^{\mathcal{A}}\langle x \cdot c_\varkappa u,\ y \cdot' c'_\varkappa v\rangle +^{\mathcal{A}} c_\varkappa^{\mathcal{A}}\langle x \cdot h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ y \cdot' h_\varkappa c_\varkappa u\rangle$$

(by definition of $\cdot^{\mathcal{A}}$ and by (2.2) (ii) for $\mathfrak{A}$),

$$= \langle c_\varkappa(x \cdot c_\varkappa u),\ c'_\varkappa(y \cdot' c'_\varkappa v)\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(y \cdot' c'_\varkappa v)\big),\ h_\varkappa c_\varkappa(x \cdot c_\varkappa u)\rangle$$
$$+^{\mathcal{A}} \langle c_\varkappa\big(x \cdot h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v)\big),\ c'_\varkappa(y \cdot' h_\varkappa c_\varkappa u)\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(y \cdot' h_\varkappa c_\varkappa u)\big),\ h_\varkappa c_\varkappa\big(x \cdot h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v)\big)\rangle$$

(by definition of $c_\varkappa^{\mathcal{A}}$)

$$= \langle c_\varkappa x \cdot c_\varkappa u,\ c'_\varkappa y \cdot' c'_\varkappa v\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y) \cdot h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa c_\varkappa x \cdot h_\varkappa c_\varkappa u\rangle$$
$$+^{\mathcal{A}} \langle c_\varkappa x \cdot h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ c'_\varkappa y \cdot' h_\varkappa c_\varkappa u\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y) \cdot c_\varkappa u,\ h_\varkappa c_\varkappa x \cdot' c'_\varkappa v\rangle$$

(by (C_3) for $\mathfrak{B}$, $\mathfrak{B}'$, the facts that $h_\varkappa$ is an isomorphism with range $\subseteq c'^{*}_\varkappa \mathfrak{B}'$ and that $b_\varkappa = h_\varkappa(1)$, and by (2.2) (viii),

$$= \langle c_\varkappa x,\ c'_\varkappa y\rangle \cdot^{\mathcal{A}} \langle c_\varkappa u,\ c'_\varkappa v\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa y),\ h_\varkappa c_\varkappa x\rangle \cdot^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa c_\varkappa u\rangle$$
$$+^{\mathcal{A}} \langle c_\varkappa x,\ c'_\varkappa y\rangle \cdot^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa v),\ h_\varkappa c_\varkappa u\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}(b_\varkappa \cdot' c_\varkappa y),\ h_\varkappa c_\varkappa x\rangle \cdot^{\mathcal{A}} \langle c_\varkappa u,\ c'_\varkappa v\rangle$$

(by definition of $\cdot^{\mathcal{A}}$),

$$= c_\varkappa^{\mathcal{A}}\langle x, y\rangle \cdot^{\mathcal{A}} c_\varkappa^{\mathcal{A}}\langle u, v\rangle$$ (by (C_0) for $\mathfrak{A}$ and by definition of $c_\varkappa^{\mathcal{A}}$).

With (C_3) proved, let us now consider (C_7).

(C_7) In order to demonstrate that this condition holds for $\mathfrak{A}$ we first show that for $\varkappa = 0, 1$ and $\langle u, v\rangle \in A$ we have:

$$(*)\qquad c_\varkappa(d_{01} \cdot u) \cdot h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' v)\big) = 0 \quad\text{and}\quad h_\varkappa c_\varkappa(d_{01} \cdot u) \cdot' c'_\varkappa(d'_{01} \cdot' v) = 0'.$$

To obtain $(*)$ we begin with the hypothesis $h_\varkappa(c_\varkappa d_{01}) \cdot' d'_{01} = 0'$ of (2.4). Multiplying through by v, applying $c'_\varkappa$, and using (C_1) gives

$$c'_\varkappa\big(h_\varkappa(c_\varkappa d_{01}) \cdot' d'_{01} \cdot' v\big) = 0'.$$

Since range $(h_\varkappa) \subseteq c'^{*}_\varkappa B'$ we can use (2.2) (viii) to get

$$h_\varkappa(c_\varkappa d_{01}) \cdot' c'_\varkappa(d'_{01} \cdot' v) = 0'.$$

Now multiplying by $b_\varkappa$ and recalling that $h_\varkappa$ an isomorphism, we get

$$c_\varkappa d_{01} \cdot h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' v)\big) = 0.$$

But $d_{01} \leqslant c_\varkappa d_{01}$ by (C_2), so

$$(d_{01} \cdot u) \cdot h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' v)\big) = 0.$$

Finally apply $c_\varkappa$ and use (2.2) (viii) again to get the first equation of $(*)$. The second equation is derived from the first by applying the isomorphism $h_\varkappa$ and recalling that $b_\varkappa = h_\varkappa(1)$.

Now let us use $(*)$ to derive (C_7). For any $\varkappa = 0, 1$ and $\langle x, y\rangle \in A$ we have:

$$c_\varkappa^{\mathcal{A}}(d_{01}^{\mathcal{A}} \cdot^{\mathcal{A}} \langle x, y\rangle) \cdot^{\mathcal{A}} c_\varkappa^{\mathcal{A}}(d_{01}^{\mathcal{A}} \cdot^{\mathcal{A}} -^{\mathcal{A}} \langle x, y\rangle)$$

$$= c_\varkappa^{\mathcal{A}}\langle d_{01} \cdot x,\ d'_{01} \cdot' y\rangle \cdot^{\mathcal{A}} c_\varkappa^{\mathcal{A}}\langle d_{01} \cdot -x,\ d'_{01} \cdot' -' y\rangle$$

$$= \big[\langle c_\varkappa(d_{01} \cdot x),\ c'_\varkappa(d'_{01} \cdot' y)\rangle +^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' y)\big),\ h_\varkappa c_\varkappa(d_{01} \cdot x)\rangle\big]$$
$$\cdot^{\mathcal{A}}\big[\langle c_\varkappa(d_{01} \cdot -x),\ c'_\varkappa(d'_{01} \cdot' -' y)\rangle +^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' -' y)\big),\ h_\varkappa c_\varkappa(d_{01} \cdot -x)\rangle\big]$$

$$= \langle c_\varkappa(d_{01} \cdot x) \cdot c_\varkappa(d_{01} \cdot -x),\ c'_\varkappa(d'_{01} \cdot' y) \cdot' c'_\varkappa(d'_{01} \cdot' -' y)\rangle$$
$$+^{\mathcal{A}} \langle c_\varkappa(d_{01} \cdot x) \cdot h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' -' y)\big),\ c'_\varkappa(d'_{01} \cdot' y) \cdot' h_\varkappa c_\varkappa(d_{01} \cdot -x)\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' y)\big) \cdot c_\varkappa(d_{01} \cdot -x),\ h_\varkappa c_\varkappa(d_{01} \cdot x) \cdot' c'_\varkappa(d'_{01} \cdot' -' y)\rangle$$
$$+^{\mathcal{A}} \langle h_\varkappa^{-1}\big(b_\varkappa \cdot' c'_\varkappa(d'_{01} \cdot' y) \cdot' c'_\varkappa(d'_{01} \cdot' -' y)\big),\ h_\varkappa\big(c_\varkappa(d_{01} \cdot x) \cdot c_\varkappa(d_{01} \cdot -x)\big)\rangle$$

$= \langle 0, 0'\rangle$ by (C_7) for $\mathfrak{B}$ and $\mathfrak{B}'$, and by equations $(*)$,

$= 0^{\mathcal{A}}$.

The proof above that (C_7) holds in $\mathfrak{A}$ completes our proof that $\mathfrak{A} \in \Omega$, and we turn to the second part of Lemma (2.4). Assume, therefore, that for $\varkappa \neq \lambda$ and any $y \in B'$ we have $c'_\varkappa c'_\lambda y = 1'$, and consider any $\langle x, y\rangle \in A$ with $\langle x, y\rangle \neq 0^A$. Then:

$$(**)\qquad c^A_\varkappa c^A_\lambda \langle x, y\rangle$$
$$= c^A_\varkappa[\langle c_\lambda x, c'_\lambda y\rangle +^A \langle h_\lambda^{-1}(b_\lambda \cdot' c'_\lambda y), h_\lambda c_\lambda x\rangle]$$
$$= [\langle c_\varkappa c_\lambda x, c'_\varkappa c'_\lambda y\rangle +^A \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa c'_\lambda y), h_\varkappa c_\varkappa c_\lambda x\rangle]$$
$$+^A[\langle c_\varkappa h_\lambda^{-1}(b_\lambda \cdot' c'_\lambda y), c'_\varkappa h_\lambda c_\lambda x\rangle +^A \langle h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa h_\lambda c_\lambda x), h_\varkappa c_\varkappa h_\lambda^{-1}(b_\lambda \cdot' c_\lambda y)\rangle].$$

Now from $\langle x, y\rangle \neq 0^A$ we have either $x \neq 0$ or $y \neq 0'$. In case $y \neq 0'$, then by hypothesis $c'_\varkappa c'_\lambda y = 1'$, and hence $h_\varkappa^{-1}(b_\varkappa \cdot' c'_\varkappa c'_\lambda y) = h_\varkappa^{-1} b_\varkappa = h_\varkappa^{-1} h_\varkappa(1) = 1$, and thus (**) gives $c^A_\varkappa c^A_\lambda \langle x, y\rangle = 1^A$ by definition of $\cdot^A$ and by (C_0) for $\mathfrak{A}$.

On the other hand, suppose that $x \neq 0$. Then $c_\lambda x \neq 0$ by (C_2) for $\mathfrak{B}$, and hence $h_\lambda c_\lambda x \neq 0'$ because h_λ is an isomorphism. But $h_\lambda c_\lambda x = c'_\lambda h_\lambda c_\lambda x$ by (1.1) (i) for $\mathfrak{B}'$, since the range of h_λ is $c'^*_\lambda B'$. Thus $c'_\varkappa h_\lambda c_\lambda x = 1'$ by the hypothesis; and from this we have also $h_\varkappa^{-1}(b_\varkappa \cdot c'_\varkappa h_\lambda c_\lambda x) = h_\varkappa^{-1}(b_\varkappa) = 1$. Using these in (**) gives $c^A_\varkappa c^A_\lambda \langle x, y\rangle = 1^A$ again.

This completes the proof of Lemma (2.4).

Having defined superposition products in (2.3) and derived their basic properties in (2.4), let us see how we can use them. Recall that we wish to show that $\Omega \subseteq \mathrm{Cr}_2$, by considering any $\mathfrak{B} \in \Omega$ and showing that $\mathfrak{B} = \mathfrak{A}_a$ for some CA_2 $\mathfrak{A}$ and element a of $\mathfrak{A}$. The way in which we shall find $\mathfrak{A}$ and a is suggested in the following lemma.

(2.5) LEMMA. *Suppose that $\mathfrak{B}, \mathfrak{B}' \in \Omega$. Assume that for each $\varkappa = 0, 1$ we have an isomorphism $h_\varkappa$ of $c^*_\varkappa \mathfrak{B}$ onto $(c'^*_\varkappa \mathfrak{B}')_{b_\varkappa}$, where $b_\varkappa = h_\varkappa(1)$. Let $\mathfrak{A}$ be the superposition product $\mathfrak{B} \otimes \mathfrak{B}'$ (with respect to h_0, h_1), and suppose that $\mathfrak{A} \in \mathrm{CA}_2$. Then $\mathfrak{B}$ is isomorphic to the $\mathrm{Cr}_2\mathfrak{A}_a$, where $a = \langle 1, 0'\rangle \in A$.*

Proof. Let $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{A}$ be as in (2.3), and let j be the map of B into A such that $j(x) = \langle x, 0'\rangle$ for all $x \in B$. Clearly j is a one-one map of B onto $A_a = \{a \cdot^A z |\ z \in A\}$.

It is obvious that $j(0) = 0^A$, $j(1) = a$, j carries $+$ and $\cdot$ into $+^A$ and $\cdot^A$ respectively, and for any $x \in B$

$$j(-x) = \langle -x, 0'\rangle = \langle 1, 0'\rangle \cdot^A \langle -x, 1'\rangle$$
$$= a \cdot^A -^A \langle x, 0'\rangle = a \cdot^A -^A j(x)$$
$$= a -^A j(x).$$

Thus it remains only to show that j carries $d_{\varkappa\lambda}$ to $(d^A_{\varkappa\lambda})^a$ and $c_\varkappa$ into $(c^A_\varkappa)^a$. Let us compute

$$j(d_{\varkappa\lambda}) = \langle d_{\varkappa\lambda}, 0'\rangle = \langle 1, 0'\rangle \cdot^A \langle d_{\varkappa\lambda}, d'_{\varkappa\lambda}\rangle$$
$$= a \cdot^A d^A_{\varkappa\lambda} = (d^A_{\varkappa\lambda})^a.$$

And for any $x \in B$,

$$j(c_\varkappa x) = \langle c_\varkappa x, 0'\rangle = \langle 1, 0'\rangle \cdot^A \langle c_\varkappa x, h_\varkappa c_\varkappa x\rangle$$
$$= a \cdot^A[\langle c_\varkappa x, 0'\rangle +^A \langle 0, h_\varkappa c_\varkappa x\rangle]$$
$$= a \cdot^A c^A_\varkappa \langle x, 0'\rangle = (c^A_\varkappa)^a j(x).$$

This completes the proof of (2.5).

In the light of Lemma (2.5) and the remarks immediately preceding it, our aim will now be to show how, starting with any $\mathfrak{B} \in \Omega$, we can find another $\mathfrak{B}' \in \Omega$, with isomorphisms $h_\varkappa$ of $c^*_\varkappa \mathfrak{B}$ onto $(c'^*_\varkappa \mathfrak{B}')_{b_\varkappa}$ for $\varkappa = 0, 1$, such that $\mathfrak{B} \otimes \mathfrak{B}' \in \mathrm{CA}_2$. We shall construct $\mathfrak{B}'$ in two stages.

(2.6) LEMMA. *Let $\mathfrak{B}$ be any structure in Ω,*

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}.$$

Then we can find another structure $\mathfrak{D}' \in \Omega$,

$$\mathfrak{D}' = \langle D, +^D, \cdot^D, -^D, 0^D, 1^D, c^D_\varkappa, d^D_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$$

with the following properties:

(i) $d^D_{01} = 0^D$;

(ii) $c^D_0 c^D_1 x = 1^D$ *and* $c^D_1 c^D_0 x = 1^D$ *whenever* $x \in D$ *and* $x \neq 0^D$;

(iii) *there are isomorphisms $f_\varkappa$ of $c^*_\varkappa \mathfrak{B}$ onto $(c^{D*}_\varkappa \mathfrak{D}')_{e_\varkappa}$ for each $\varkappa = 0, 1$, where $e_\varkappa = f_\varkappa(1)$;*

(iv) *there is an isomorphism m of $c^{D'}_0\mathfrak{D}'$ onto $c^{D'}_1\mathfrak{D}'$ such that $mf_0(c_0 d_{01}) = f_1(c_1 d_{01})$.*

Proof. For $\varkappa = 0, 1$ let $\mathfrak{C}_\varkappa$ be the BA $c^*_\varkappa\mathfrak{B}$. Set $a_\varkappa = -c_\varkappa d_{01}$, so that $a_\varkappa$ is an element of $\mathfrak{C}_\varkappa$ by (1.1) (iii), and form the BA $\mathfrak{C}'_\varkappa = (\mathfrak{C}_\varkappa)_{a_\varkappa}$ by relativization. Now form further BA's $\mathfrak{D}_0$ and $\mathfrak{D}_1$ by taking direct products,

$$\mathfrak{D}_\varkappa = \mathfrak{C}_\varkappa \times \mathfrak{C}'_{1-\varkappa} \quad \text{for} \quad \varkappa = 0, 1.$$

Let $\mathfrak{D} = \langle D, +^D, \cdot^D, -^D, 0^D, 1^D, c^D_0, c^D_1\rangle$ be the *cylindric product* of $\mathfrak{D}_0$ and $\mathfrak{D}_1$, as in (1.4), where $q_\varkappa$ is the canonical isomorphism of $\mathfrak{D}_\varkappa$ onto $c^{D*}_\varkappa\mathfrak{D}$. Finally, set $d^D_{00} = d^D_{11} = 1^D$ and $d^D_{01} = d^D_{10} = 0^D$, and put

$$\mathfrak{D}' = \langle D, +^D, \cdot^D, -^D, 0^D, 1^D, c^D_\varkappa, d^D_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}.$$

Clearly $\mathfrak{D}'$ satisfies (C_0)-(C_4) by (1.4). Also, $\mathfrak{D}'$ satisfies (C_5) and (C_7) by choice of $d^D_{\varkappa\lambda}$. Thus $\mathfrak{D}' \in \Omega$, and clearly (2.6) (i) and (ii) hold — the former by definition of d_{01}, the latter by (1.4).

Now for $\varkappa = 0, 1$ define $f_\varkappa$ from $c^*_\varkappa\mathfrak{B}$ $(= \mathfrak{C}_\varkappa)$ to $c^{D*}_\varkappa\mathfrak{D}'$, as follows. For any $x \in B$ form the element $\langle c_\varkappa x, 0\rangle \in \mathfrak{D}_\varkappa = \mathfrak{C}_\varkappa \times \mathfrak{C}'_{1-\varkappa}$, and set $f_\varkappa(c_\varkappa x) = q_\varkappa\langle c_\varkappa x, 0\rangle \in c^{D*}_\varkappa\mathfrak{D}'$. Set $e_\varkappa = f_\varkappa(1) = q_\varkappa\langle 1, 0\rangle$. Then clearly $f_\varkappa$ is an isomorphism of $\mathfrak{C}_\varkappa$ onto $(c^{D*}_\varkappa\mathfrak{D}')_{e_\varkappa}$, so that (2.6) (iii) is satisfied.

Finally, let us define a map m of $c_0^D{*}\mathfrak{D}'$ to $c_1^D{*}\mathfrak{D}'$, as follows. Consider any element $z \in c_0^D{*}\mathfrak{D}'$, say $z = q_0\langle x, y\rangle$ with $x \in \mathfrak{C}_0 = c_0^*\mathfrak{B}$ and $y \in \mathfrak{C}_1' = (c_1^*\mathfrak{B})_{a_1} = (c_1^*\mathfrak{B})_{-c_1d_{01}}$. Then we set

(1) $$m(z) = mq_0\langle x, y\rangle = q_1\langle c_1(x\cdot d_{01})+y\,,\, x\cdot - c_0d_{01}\rangle\,.$$

To establish (2.6) (iv) we wish to show that m is an isomorphism of the BA $c_0^D{*}\mathfrak{D}'$ onto $c_1^D{*}\mathfrak{D}'$. Since (1), the definition of m, leads directly to the conclusion

(2) $$m[q_0\langle x, y\rangle +^D q_0\langle x', y'\rangle] = mq_0\langle x, y\rangle +^D mq_0\langle x', y'\rangle$$

for all $q_0\langle x, y\rangle, q_0\langle x', y'\rangle \in c_0^D{*}\mathfrak{D}'$, using (2.2) (ii) for $\mathfrak{B}$, let us show that m preserves negation.

Recalling that

(3) $$-^D q_0\langle x, y\rangle = q_0\langle -x, -y\cdot - c_1d_{01}\rangle$$

for $\langle x, y\rangle \in \mathfrak{D}_0 = \mathfrak{C}_0\times\mathfrak{C}_1'$, we easily get

(4) $$mq_0\langle x, y\rangle +^D m(-^D q_0\langle x, y\rangle) = q_1\langle 1, -c_0d_{01}\rangle = 1^D$$

by (2), definition of $+^D$, (3), and (1), since $y \leqslant -c_1d_{01}$. Now

$$mq_0\langle x, y\rangle \cdot^D m(-^D q_0\langle x, y\rangle)$$
$$= q_1\langle [c_1(x\cdot d_{01})+y]\cdot[c_1(-x\cdot d_{01})+-y\cdot - c_1d_{01}], 0\rangle$$

by (1) and (3), since q_1 is a homomorphism,

$$= q_1\langle (c_1(x\cdot d_{01})\cdot -y\cdot - c_1d_{01}) + (y\cdot c_1(-x\cdot d_{01})), 0\rangle \quad (\text{by } (C_7))\,.$$

But $c_1(x\cdot d_{01}) \leqslant c_1d_{01}$ by (2.2) (iv), so $c_1(x\cdot d_{01})\cdot - c_1d_{01} = 0$; and similarly $c_1(-x\cdot d_{01}) \leqslant c_1d_{01}$, so $y\cdot c_1(-x\cdot d_{01}) = 0$ since $y \leqslant -c_1d_{01}$. Hence

$$mq_0\langle x, y\rangle \cdot^D m(-^D q_0\langle x, y\rangle) = q_1(0, 0) = 0^D,$$

which, together with (4), shows that

(5) $$m(-^D q_0\langle x, y\rangle) = -^D mq_0\langle x, y\rangle\,.$$

From (2) and (5) we see that m is a homomorphism, so let us now show that it is one-one. Suppose, then, that $q_0\langle x, y\rangle$ and $q_0\langle x', y'\rangle$ are elements of $c_0^D{*}\mathfrak{D}'$ such that

(6) $$mq_0\langle x, y\rangle = mq_0\langle x', y'\rangle\,.$$

By (1) and the fact that q_1 is an isomorphism we can infer from (6) that

(7a) $$c_1(x\cdot d_{01})+y = c_1(x'\cdot d_{01})+y',$$

and

(7b) $$x\cdot - c_0d_{01} = x'\cdot - c_0d_{01}\,.$$

Since $c_1(x\cdot d_{01}) \leqslant c_1d_{01}$ by (2.2) (iv) while $y \leqslant -c_1d_{01}$, and similarly $c_1(x'\cdot d_{01}) \leqslant c_1d_{01}$, $y' \leqslant -c_1d_{01}$, we obtain from (7a) that

(8a) $$c_1(x\cdot d_{01}) = c_1(x'\cdot d_{01})$$

and

(8b) $$y = y'.$$

From (8a), multiplying through by d_{01} and using (2.2) (vi), we get $x\cdot d_{01} = x'\cdot d_{01}$ which, upon applying c_0 to both sides and using (2.2) (viii), gives $x\cdot c_0d_{01} = x'\cdot c_0d_{01}$. When this is combined with (7b) we get, finally, $x = x'$. Since this equation, as well as (8b), have been derived from (6), we have completed the proof that m is one-one, and so by (2) and (5) we know that

(9) m is an isomorphism of $c_0^D{*}\mathfrak{D}'$ into $c_1^D{*}\mathfrak{D}'$.

Let us now show that the range of m is the whole of $c_1^D{*}\mathfrak{D}'$. Consider, therefore, an arbitrary element $q_1\langle u, v\rangle \in c_1^D{*}\mathfrak{D}'$, where $u \in \mathfrak{C}_1 = c_1^*\mathfrak{B}$ and $v \in \mathfrak{C}_0'$ (so that $v \in c_0^*\mathfrak{B}$ and $v \leqslant -c_0d_{01}$). Put

(10) $$x = c_0(u\cdot d_{01})+v\,, \qquad y = u\cdot - c_1d_{01}\,.$$

Using (1) we compute

$$mq_0\langle x, y\rangle = q_1\Big\langle c_1\big((c_0(u\cdot d_{01})+v)\cdot d_{01}\big)+u\cdot - c_1d_{01}, \big(c_0(u\cdot d_{01})+v\big)\cdot - c_0d_{01}\Big\rangle,$$

so that

(11) $$mq_0\langle x, y\rangle = q_1\langle c_1(u\cdot d_{01})+c_1(v\cdot d_{01})+u\cdot - c_1d_{01}, c_0(u\cdot d_{01})\cdot - c_0d_{01} + {} + v\cdot - c_0d_{01}\rangle\,,$$

by (2.2) (ii) and (2.2) (vi). But $c_1(u\cdot d_{01}) = u\cdot c_1d_{01}$ by (2.2) (viii), $v\cdot d_{01} \leqslant v\cdot c_0d_{01}$ by (C_2) and so $v\cdot d_{01} = 0$ (since $v \leqslant -c_0d_{01}$), and $c_0(u\cdot d_{01})\cdot - c_0d_{01} \leqslant c_0d_{01}\cdot - c_0d_{01} = 0$ by (2.2) (iv). Thus (11) yields $mq_0\langle x, y\rangle = q_1\langle u, v\rangle$. Since $q_1\langle u, v\rangle$ was an arbitrary element of $c_1^D{*}\mathfrak{D}'$, we can now conclude from (9) that

(12) m is an isomorphism of $c_0^D{*}\mathfrak{D}'$ onto $c_1^D{*}\mathfrak{D}'$.

Thus, the following computation completes the proof of (2.6) (iv):

$$mf_0(c_0d_{01}) = mq_0\langle c_0d_{01}, 0\rangle \quad (\text{by definition of } f_0)\,,$$
$$= q_1\langle c_1((c_0d_{01})\cdot d_{01})+0, (c_0d_{01})\cdot - c_0d_{01}\rangle \quad (\text{by definition of } m)\,,$$
$$= q_1\langle c_1d_{01}, 0\rangle \quad (\text{by (2.2) (vi)})$$
$$= f_1(c_1d_{01})\,.$$

This finishes the proof of Lemma (2.6).

We are now ready, starting from any $\mathfrak{B} \in \Omega$, to form a structure $\mathfrak{B}'$ of the kind described in Lemma (2.5).

(2.7) Lemma. *Let $\mathfrak{B}$ be any structure in Ω,*

$$\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, c_\varkappa, d_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}.$$

Then we can find another structure $\mathfrak{B}' \in \Omega$,

$$\mathfrak{B}' = \langle B', +', \cdot', -', 0', 1', c'_\varkappa, d'_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$$

with the following properties:

(i) $c'_0c'_1x = 1'$ *and* $c'_1c'_0x = 1'$ *whenever* $x \in B'$ *and* $x \neq 0'$;

(ii) *there is an isomorphism $h_\varkappa$ of the* BA$c^*_\varkappa\mathfrak{B}$ *onto* $(c'^*_\varkappa\mathfrak{B}')_{b_\varkappa}$, *where* $b_\varkappa = h_\varkappa(1)$, *for each* $\varkappa = 0, 1$;

(iii) $c'_\varkappa d'_{01} = -'h_\varkappa(c_\varkappa d_{01})$ *for* $\varkappa = 0, 1$;

(iv) *the superposition product* $\mathfrak{B} \otimes \mathfrak{B}'$ *(with respect to h_0, h_1) is a* CA$_2$.

Proof. Starting with the given structure $\mathfrak{B}$, we first find a structure $\mathfrak{D}'$, mappings f_0, f_1, m, and elements e_0, e_1, satisfying the conditions of Lemma (2.6).

For $\varkappa = 0, 1$ let $i_\varkappa = -^D f_\varkappa(c_\varkappa d_{01}) \in D$, let $\mathfrak{I}$ be the BA$(c_0^{D*}\mathfrak{D}')_{i_0}$, and write

$$\mathfrak{I} = \langle I, +^I, \cdot^I, -^I, 0^I, 1^I\rangle;$$

in particular, we have $I = \{c_0^Dx \in D \mid c_0^Dx \leqslant i_0\}$, $1^I = i_0$, and $-^Iy = i_0 \cdot^D -^D y$ for each $y \in I$.

Now for $\varkappa, \lambda = 0, 1$ define $d^I_{\varkappa\lambda} = 1^I$ and $c^I_\varkappa y = y$ for any $y \in I$. It is then a trivial matter to verify that

(1) the structure $\mathfrak{I}' = \langle I, +^I, \cdot^I, -^I, 0^I, 1^I, c^I_\varkappa, d^I_{\varkappa\lambda}\rangle_{\varkappa,\lambda=0,1}$ is an element of Ω.

Next we need certain mappings j_0 and j_1 of I into D. In fact, we can take j_0 to be the identity map on I (recalling that $I \subseteq D$). As for j_1, for any $y \in I$ we define $j_1(y) = m(y)$. Since $j_1(1^I) = m(i_0) = m(-^Df_0(c_0d_{01}))$ $= -^Df_1(c_1d_{01}) = i_1$, we easily see that

(2) For $\varkappa = 0, 1$, $j_\varkappa$ is an isomorphism of the BA$c^{I*}_\varkappa\mathfrak{I}$ onto $(c^{D*}_\varkappa\mathfrak{D}')_{i_\varkappa}$.

Furthermore, by (2.6) (i) we have, of course,

(3) $$j_\varkappa(c^I_\varkappa d^I_{01}) \cdot d^D_{01} = 0^D \quad \text{for} \quad \varkappa = 0, 1.$$

By combining (1), (2), and (3) we are able to apply Lemma (2.4). Letting $\mathfrak{B}'$ be the superposition product $\mathfrak{I}' \otimes \mathfrak{D}'$ (with respect to j_0 and j_1) we thus see that $\mathfrak{B}' \in \Omega$. Also (2.7) (i) holds by (2.4), in view of (2.6) (ii).

To obtain (2.7) (ii), we define $h_\varkappa$ from $c^*_\varkappa B$ to $c^*_\varkappa B'$, for each $\varkappa = 0, 1$, by setting

(4) $$h_\varkappa(z) = c'_\varkappa(0^I, f_\varkappa z) \quad \text{for all } z \in c^*_\varkappa B.$$

Using the definition of superposition products (2.3) we get

$$h_\varkappa(z) = \langle c^I_\varkappa 0^I, c^D_\varkappa f_\varkappa z\rangle +' \langle j^{-1}_\varkappa(i_\varkappa \cdot^D c^D_\varkappa f_\varkappa z), j_\varkappa c^I_\varkappa 0^I\rangle$$

and so, by (2.6) (iii),

(5) $$h_\varkappa(z) = \langle j^{-1}_\varkappa(i_\varkappa \cdot f_\varkappa z), f_\varkappa z\rangle \quad \text{for all } z \in c^*_\varkappa B.$$

From this, using (2.6) (iii) and (2) above, we obtain (2.7) (ii).

Next, using (2.3) and the definitions of $\mathfrak{I}'$ and $\mathfrak{D}'$, we compute

$$\begin{aligned} c'_\varkappa(d'_{01}) &= c'_\varkappa(d^I_{01}, d^D_{01}) = c'_\varkappa(1^I, 0^D) \\ &= \langle c^I_\varkappa 1^I, c^D_\varkappa 0^D\rangle +' \langle j^{-1}_\varkappa(i_\varkappa \cdot c^D_\varkappa 0^D), j_\varkappa c_\varkappa 1^I\rangle \\ &= \langle 1^I, j_\varkappa 1^I\rangle \\ &= -'\langle 0^I, -^D i_\varkappa\rangle \\ &= -'h_\varkappa(c_\varkappa d_{01}) \quad \text{(by (4) and definition of } i_\varkappa), \end{aligned}$$

which proves (2.7) (iii).

Now let $\mathfrak{A}$ be the superposition product $\mathfrak{B} \otimes \mathfrak{B}'$ (with respect to h_0, h_1). Since $h_\varkappa(c_\varkappa d_{01}) \cdot' d'_{01} = 0'$ by (2.7) (iii), and using (2.7) (ii), we can apply (2.4) to obtain $\mathfrak{A} \in \Omega$. Again by (2.4), this time with (2.7) (i), we obtain $c^A_\varkappa c^A_\lambda z = c^A_\lambda c^A_\varkappa z = 1^A$ whenever $z \neq 0^A$, so that by (C$_1$) for $\mathfrak{A}$ we have $c^A_\varkappa c^A_\lambda z = c^A_\lambda c^A_\varkappa z$ for *all* $z \in A$; thus (C$_4$) holds for $\mathfrak{A}$. Finally, by (2.3) we compute

$$\begin{aligned} c^A_\varkappa(d^A_{01}) &= c^A_\varkappa\langle d_{01}, d'_{01}\rangle \\ &= \langle c_\varkappa d_{01}, c'_\varkappa d'_{01}\rangle +^A \langle h^{-1}_\varkappa(b_\varkappa \cdot' c'_\varkappa d'_{01}), h_\varkappa c_\varkappa d_{01}\rangle \\ &= \langle 1, 1'\rangle \quad \text{(by (2.7) (iii))} \\ &= 1^A, \end{aligned}$$

so that (C$_6$) holds for $\mathfrak{A}$.

Putting together the facts that $\mathfrak{A} \in \Omega$ and that (C$_4$), (C$_6$) hold for $\mathfrak{A}$, we obtain $\mathfrak{A} \in$ CA$_2$ by definition of Ω and CA$_2$.

All the lemmas are now at hand for our equational characterization of the class Cr$_2$ of 2-dimensional cylindric relativized algebras.

(2.8) Theorem [9]. *For any structure $\mathfrak{A}$ we have $\mathfrak{A} \in$ Cr$_2$ iff $\mathfrak{A} \in \Omega$.*

[9] This theorem was first shown by Henkin in 1969 by a method involving the embedding of a given structure of Ω into an atomistic structure of Ω. The present proof involving the notion of superposition products is new, and is also due to Henkin.

Proof. By (1.3) we have $\mathrm{Cr}_2 \subseteq \Omega$. To obtain the opposite inclusion we take any $\mathfrak{B} \in \Omega$ and, by (2.7) (ii), (2.7) (iv), and (2.5), we conclude that $\mathfrak{B}$ is isomorphic to some Cr_2, $\mathfrak{A}_a$. But then, by the "exchange principle" of the general theory of structures, $\mathfrak{B}$ is itself a Cr_2.

§ 3. In the preceding section we presented a set of equational identities (2.1) which characterize the class Cr_2 of 2-dimensional cylindric relativized algebras. In the present section we shall show that no such characterization is possible for Cr_3 (or for any Cr_α with $\alpha > 2$) [10].

(3.1) THEOREM. *There is a structure $\mathfrak{B} \in \mathrm{Cr}_3$, and a subalgebra $\mathfrak{D}$ of $\mathfrak{B}$, such that $\mathfrak{D} \notin \mathrm{Cr}_3$.*

Proof. In § 1 we introduced the notion of an α-dimensional cylindric set algebra (CSA_α), from which the notion of a cylindric algebra is abstracted. To construct the $\mathfrak{B}$ of (3.1) we begin with an $\mathfrak{A} \in \mathrm{CSA}_3$, chosen as follows. We take 1^A to be the set ${}^3\omega$ of all ordered triples $\langle x_0, x_1, x_2\rangle$ of natural numbers, we let A be the set of all subsets of 1^A, and we take

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, \emptyset, 1^A, c_\varkappa^A, d_{\varkappa\lambda}^A\rangle_{\varkappa,\lambda<3}.$$

Next we choose an element $a \in A$ by specifying that for any $\langle x_0, x_1, x_2\rangle \in {}^3\omega$,

(1) $\langle x_0, x_1, x_2\rangle \in a$ iff:

(i) Either $x_0 = x_1+1$ or $x_1 = x_0+1$, and

(ii) Either x_0 is even and $x_2 = 0$, or else x_0 is odd and $x_2 = 1$.

And then we take $\mathfrak{B} = \mathfrak{A}_a$, the relativization of $\mathfrak{A}$ to a. Thus, writing

$$\mathfrak{B} = \langle B, \cup, \cap, {}_a\!\sim, \emptyset, a, c_\varkappa^B, d_{\varkappa\lambda}^B\rangle_{\varkappa,\lambda<3},$$

we see that B is the set of all subsets of a; that $c_\varkappa^B Y = a \cap c_\varkappa^A Y$ for all $Y \in B$ (so $\langle x_0, x_1, x_2\rangle \in c_1^B Y$, e.g., iff $\langle x_0, x_1, x_2\rangle \in a$ and $\langle x_0, y, x_2\rangle \in Y$ for some $y \in \omega$); and that $d_{\varkappa\lambda}^B = a \cap d_{\varkappa\lambda}^A$, so that by (1) we get

(2)
$$d_{00}^B = d_{11}^B = d_{22}^B = a, \quad d_{01}^B = d_{12}^B = \emptyset, \quad \text{and}$$
$$d_{02}^B = \{\langle 0,1,0\rangle, \langle 1,0,1\rangle, \langle 1,2,1\rangle\}.$$

Now let D be the set of all those elements X of B such that either X or ${}_a\!\sim X$ is finite. It is well known (and obvious) that D is closed under $\cup, \cap, {}_a\!\sim$, and that it contains $\emptyset$ and a among its elements; and from (2) we see also that each $d_{\varkappa\lambda}^B \in D$. Let us check now that D is also closed under

[10] The results of this section (but for Cr_4 instead of Cr_3) were obtained by Resek in 1968. Her proof was simplified in 1969 by Henkin, and was then seen to yield the sharper result for Cr_3. It is this proof that is given here for (3.1).

each $c_\varkappa^B$. Indeed, for any $Y \in B$ and $\varkappa < 3$ we have [11] $c_\varkappa^B Y = \bigcup_{x \in Y} c_\varkappa^B\{x\}$, and for any $x \in a$ we see from (1) that $c_\varkappa^B\{x\}$ has at most two elements. Thus if Y is finite then so is $c_\varkappa^B Y$, while if ${}_a\!\sim Y$ is finite then so is ${}_a\!\sim c_\varkappa^B Y$ because $Y \subseteq c_\varkappa^B Y$; so D is closed under $c_\varkappa^B$. These observations show that:

(3) The structure $\mathfrak{D} = \langle D, \cup, \cap, {}_a\!\sim, \emptyset, a, c_\varkappa^B, d_{\varkappa\lambda}^B\rangle_{\varkappa,\lambda<3}$ is a subalgebra of $\mathfrak{B}$.

Thus, to complete the proof of (3.1) we need to show that $\mathfrak{D} \notin \mathrm{Cr}_3$. We do this by contradiction.

(4) *Assume* that $\mathfrak{D} \in \mathrm{Cr}_3$, so that for some CA_3 $\mathfrak{E}$ having a as an element, $\mathfrak{D}$ is the relativization of $\mathfrak{E}$ to a, i.e., $\mathfrak{D} = \mathfrak{E}_a$.

If we write, as usual,

$$\mathfrak{E} = \langle E, +^E, \cdot^E, -^E, 0^E, 1^E, c_\varkappa^E, d_{\varkappa\lambda}^E\rangle_{\varkappa,\lambda<3},$$

then from (1) we see that $D = \{v \in E \mid v \leqslant^E a\}$, and that for any $Y \in D$ and $\varkappa < 3$ we have $c_\varkappa^B Y = a \cdot^E c_\varkappa^E Y$.

Now we are going to derive from (4) the conclusion that

(5) For any $x = \langle x_0, x_1, x_2\rangle \in a$ we have $\{x\} \leqslant^E c_0^E c_1^E\{\langle 0,1,0\rangle\}$ iff $x_2 = 0$.

To see this observe that by (1), if $\langle 2n+2, y_1, 0\rangle \in a$ then also $\langle 2n+2, 2n+1, 0\rangle$ and $\langle 2n, 2n+1, 0\rangle$ are in a, and by definition of c_1^B and c_0^B we get:

$$\{\langle 2n+2, y_1, 0\rangle\} \subseteq c_1^B\{\langle 2n+2, 2n+1, 0\rangle\}$$
$$\subseteq c_1^B c_0^B\{\langle 2n, 2n+1, 0\rangle\}.$$

Since, for any $X, Z \in D$, we have $X \subseteq Z$ iff $X \leqslant^E Z$, and $c_\varkappa^B X \leqslant^E c_\varkappa^E X$, by definition of $\mathfrak{E}$, and since (2.2) (iv) holds in $\mathfrak{E}$, we thus obtain

$$\{\langle 2n+2, y_1, 0\rangle\} \leqslant^E c_1^E c_0^E\{\langle 2n, 2n+1, 0\rangle\}.$$

Using (C_4) for $\mathfrak{E}$, and (1), we can then employ induction to conclude:

(6) Whenever $\langle x_0, x_1, 0\rangle \in a$ we have

$$\{\langle x_0, x_1, 0\rangle\} \leqslant^E c_1^E c_0^E\{\langle 0,1,0\rangle\}.$$

[11] The additive law for $c_\varkappa$ given as (1.1) (ii) is only a special case of the completely additive law $c_\varkappa \sum_{i\in I} x_i = \sum_{i\in I} c_\varkappa x_i$ which holds in any CA_α whenever the sum on the left exists. This law is derivable from (C_0)-(C_3), and hence holds also in every Cr_α.

And by similar reasoning we obtain:

(7) Whenever $\langle y_0, y_1, 1\rangle \in a$ we have

$$\{\langle 1, 2, 1\rangle\} \leqslant^E c_1^E c_0^E \{\langle y_0, y_1, 1\rangle\}.$$

Let us *assume*, now, that there is some $\langle y_0, y_1, 1\rangle \in a$ such that

(8) $$\{\langle y_0, y_1, 1\rangle\} \leqslant^E c_1^E c_0^E \{\langle 0, 1, 0\rangle\}.$$

Applying c_0^E and then c_1^E to both sides of this inequality (as we may by (2.2) (iv)), and using (C_4) and (1.1) (i) for $\mathfrak{E}$, we get

$$c_1^E c_0^E \{\langle y_0, y_1, 1\rangle\} \leqslant^E c_1^E c_0^E \{\langle 0, 1, 0\rangle\}$$

and hence, by (7),

$$\{\langle 1, 2, 1\rangle\} \leqslant^E c_1^E c_0^E \{\langle 0, 1, 0\rangle\}.$$

But $\{\langle 1, 2, 1\rangle\} \leqslant^E d_{02}^E \leqslant d_{02}^E$, and similarly $\{\langle 0, 1, 0\rangle\} \leqslant^E d_{02}^E$. Thus

$$\begin{aligned}
\{\langle 1, 2, 1\rangle\} &\leqslant^E c_1^E c_0^E [d_{02}^E \cdot^E \{\langle 0, 1, 0\rangle\}] \cdot^E d_{02}^E \\
&= c_0^E [d_{02}^E \cdot^E c_1^E \{\langle 0, 1, 0\rangle\}] \cdot^E d_{02}^E \quad \text{(by } (C_4) \text{ and (1.1) (vi))}, \\
&= d_{02}^E \cdot^E c_1^E \{\langle 0, 1, 0\rangle\} \quad \text{(by (2.2) (vi))}, \\
&\leqslant^E c_1^E \{\langle 0, 1, 0\rangle\}.
\end{aligned}$$

But $\{\langle 1, 2, 1\rangle\} \subseteq a$, so by (4) we get

$$\{\langle 1, 2, 1\rangle\} \subseteq c_1^E \{\langle 0, 1, 0\rangle\}.$$

But this is impossible, since by definition of c_1^E each triple in $c_1^E\{\langle 0, 1, 0\rangle\}$ has the form $\langle 0, z, 0\rangle$. This contradiction arose from assuming that (8) holds for some $\langle y_0, y_1, 1\rangle \in a$.

It follows, by (1), that whenever $\langle y_0, y_1, y_2\rangle \in a$ and

$$\{\langle y_0, y_1, y_2\rangle\} \leqslant^E c_1^E c_0^E \{\langle 0, 1, 0\rangle\},$$

we must have $y_2 = 0$. This fact, together with (6), completes the demonstration that (5) holds — assuming (4).

Of course $a \cdot^E c_0^E c_1^E \{\langle 0, 1, 0\rangle\}$ is an element Y of $\mathfrak{D}$, according to (4), and so by definition of D we must have either Y or ${}_a{\sim}Y$ finite. On the other hand from (5) we infer that Y consists of all those $\langle x_0, x_1, x_2\rangle \in a$ such that $x_2 = 0$, whence by (1) both Y and ${}_a{\sim}Y$ are infinite.

Thus a contradiction has been obtained from the assumption (4), and this shows that $\mathfrak{D} \notin \mathrm{Cr}_3$. This fact, together with (3), completes the proof of Theorem (3.1).

Since any equational class is closed under formation of substructures, (3.1) leads directly to the following.

(3.2) COROLLARY. *There is no set of equational identities that characterizes the class of structures* Cr_3.

It is a simple matter to modify the proof of (3.1) so as to establish a counterpart of this theorem for any dimension $a > 3$; and hence its Corollary (3.2) may be similarly extended.

Although, for $a \geqslant 3$, we see that there are subalgebras of Cr_a's which are not themselves Cr_a's, it is an open question whether every homomorphic image of a Cr_a is itself a Cr_a.

§ 4. In this section we mention several results without presenting proofs.

Let SCr_a be the class of all subalgebras of Cr_a's. In the preceding section we have seen that for $a \geqslant 3$ we can find a structure $\mathfrak{A} \in \mathrm{SCr}_a$ such that $\mathfrak{A} \notin \mathrm{Cr}_a$, and hence we have concluded that Cr_a is not an equational class for $a \geqslant 3$ (although it is for $a < 3$). What about the class SCr_a — is it equational?

(4.1) THEOREM [12]. *For each* $a \geqslant 3$ *an infinite set* Γ_a *of equational identities is known which characterizes the class of structures* SCr_a. *This class cannot be characterized by any finite set of such identities.*

In addition to the equations defining Ω, Γ_a contains (1.1) (vi) and the equation

$$c_\varkappa(d_{\lambda\varkappa} \cdot d_{\varkappa\mu}) \leqslant d_{\lambda\mu} \quad \text{for} \quad \varkappa \neq \lambda, \mu,$$

a weakened form of (C_6). The remaining equations of Γ_a are grouped into various infinite bundles, of which one example is:

$$c_\varkappa c_\lambda c_\varkappa \ldots c_\lambda c_\varkappa (x \cdot d_{\varkappa\mu}) \cdot d_{\varkappa\mu} \leqslant c_\lambda x$$

whenever $\varkappa, \lambda, \mu$ are distinct. By a general theorem of Jonsson and Tarski on Boolean algebras with operators [13], it follows that any structure satisfying the equational identities of Γ_a can be extended to a complete and atomistic structure satisfying those same identities. The proof of (4.1) is obtained by showing that any complete atomistic structure satisfying Γ_a is a Cr_a, and then applying the Jonsson-Tarski result. This method of proof gives the following.

[12] This theorem was proved by Resek in 1968; a full description of the set Γ_a, and a proof of the theorem, will be included in her doctoral dissertation. An indirect proof of the equational character of SCr_a was given by Pigozzi in 1969, who showed that the class SCr_a is closed under formation of homomorphisms; since closure under direct products was known, and closure under formation of subalgebras is obvious, this allows the application of Garrett Birkhoff's characterization of equational classes. (See [2], pp. 11 and 266). Pigozzi's proof does not furnish the set Γ_a explicitly, nor deal with the question of finite sets of equations.

[13] See [3], Theorem 2.15.

(4.2) COROLLARY. *Every complete atomistic structure of* SCr_α *is a* Cr_α.

It is natural to conjecture that (4.2) can be strengthened, namely, that every complete structure in SCr_α, whether atomistic or not, is a Cr_α. However, this remains an open question at present. In this connection recall that the structure $\mathfrak{D}$ of § 3, which is in SCr_3 but not in Cr_3, is incomplete, and that this incompleteness plays a central rôle in showing that $\mathfrak{D} \notin \mathrm{Cr}_3$.

We now turn our attention to the operation relativization on the cylindric set algebras (CSA's) introduced in § 1. Although every BA is isomorphic to a Boolean set algebra, there can be no similar result for CA's in general since for finite α a CSA_α is always simple, and hence a direct product of two such algebras, while obviously a CA_α, is not isomorphic to any CSA_α. A CA_α which is isomorphic to a subdirect product of CSA_α's is called *representable*. Equivalently, an $\mathfrak{A} \in \mathrm{CA}_\alpha$ is representable if, and only if, for every non-zero element x of $\mathfrak{A}$ there is a homomorphism h of $\mathfrak{A}$ to some CSA_α such that $h(x) \neq 0$.

For every $\alpha \geqslant 2$ there are non-representable CA_α's, and the relativization of CSA_α's is one of the principal ways of constructing these [14]. Another method of constructing non-representable CA_α's that has been employed is a method of "adjoining elements" which is closely related to the formation of superposition products (introduced in 2.3 above). However, all non-representable CA_α's that can be constructed by adjoining elements to a CSA_α turn out to be isomorphic to relativized CSA_α's.

The last observation obviously raises the question whether every CA_α may be isomorphic to a relativized CSA_α [15]. But the answer is negative. There are equational identities which hold in every CA_α which is a relativized CSA_α, but which fail in certain CA_α's obtained from CSA_α's by a process of "twisting cylindrifications." These identities are related to certain operations called *substitutions* which can be defined in every CA_α.

If $\varkappa \neq \lambda$, and $\varkappa, \lambda < \alpha$, we define the operation $S^\varkappa_\lambda$ on any $\mathrm{CA}_\alpha \mathfrak{A}$ by the rule $S^\varkappa_\lambda x = c_\varkappa(d_{\varkappa\lambda} \cdot x)$ for all elements x of $\mathfrak{A}$. This operation is called a substitution because if $\mathfrak{A}$ is a CSA_α whose elements are subsets of ${}^\alpha U$, then for every $\omega \in {}^\alpha U$ we have $\omega \in S^\varkappa_\lambda(c_\lambda x)$ iff $\omega' \in c_\lambda x$, where ω' is obtained from ω by substituting the $\varkappa$th coordinate for the λth coordinate. The identities to which we have referred in the preceding paragraph can be expressed in terms of these substitution operations as follows:

For all distinct $\varkappa_0, \ldots, \varkappa_{n-1}$, $\lambda < \alpha$ and all $x \in A$,

$$S^\lambda_{\varkappa_0} S^{\varkappa_0}_{\varkappa_1} \ldots S^{\varkappa_{n-2}}_{\varkappa_{n-1}} S^{\varkappa_{n-1}}_\lambda (c_\lambda x) = S^\lambda_{\varkappa_1} S^{\varkappa_1}_{\varkappa_2} \ldots S^{\varkappa_{n-1}}_{\varkappa_0} S^{\varkappa_0}_\lambda (c_\lambda x) .$$

[14] See [1] for a description of such a construction in a metalogical, rather than a set-theoretical, context.

[15] In particular, every representable CSA_α is known to be isomorphic to a relativized CSA_α. This is an unpublished result of Henkin's.

These identities are called *MGR* — the "merry-go-round" identities.

Although it is not known whether MGR characterizes the class of CA_α's that are isomorphic to relativized CSA_α's, we have the following.

(4.3) THEOREM. *An atomistic* CA_α *is isomorphic to a relativized* CSA_α *iff it satisfies* MGR. *Furthermore, any* CA_α *is isomorphic to a subalgebra of a relativized* CSA_α *iff it satisfies* MGR [16].

[16] The proof of this theorem of Resek's, found in 1969, will be included in her dissertation.

References

[1] L. Henkin, *Relativization with respect to formulas and its use in proofs of independence*, Composito Math. 20 (1968), pp. 88–106.

[2] — J. D. Monk, and A. Tarski, *Cylindric Algebras*, Part I, Amsterdam 1971.

[3] B. Jonsson and A. Tarski, *Boolean algebras with operators*, Part I, American J. Math. 73 (1951), pp. 891–939.

[4] R. Sikorski, *Boolean Algebras*, New York 1969.

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