Relativization of cylindric algebras (*)

by

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Abstract. The classes of cylindric algebras, and of cylindric relativized algebras, are described, and it is shown that the latter can be characterized by equational identities for dimension ≤ 2, but not for higher dimensions. Some other results for higher dimensions are mentioned without proof.

It is well and widely known that if

\[ D = \langle B, +, \cdot, -a, 0, 1 \rangle \]

is any Boolean algebra (BA) and \( a \) is any (*) element of \( B \), then the function \( h_a \) on \( B \) such that

\[ h_a(x) = a \cdot x \text{ for all } x \in B, \]

is a homomorphism of \( D \) onto another BA

\[ D_a = \langle B_a, +, \cdot, -a, 0, a \rangle, \]

called the relativization of \( D \) with respect to \( a \), where \( B_a = \{ x \in B \mid x \leq a \} \) and \( -a = a \cdot x \) for all \( x \in B_a \). In this paper we shall study the process of relativization in the case of cylindric algebras (CA's), which are a kind of multi-dimensional BA's; we shall be concerned especially with CA's of dimension 2 and 3.

It turns out that the process of relativization extends in a natural way from BA's to CA's, but that in general an algebra obtained by relativizing a CA of dimension greater than 1 is not itself a CA; we call a structure obtained in this way a cylindric relativized algebra (Cr). In § 2 below we shall characterize the class of 2-dimensional Cr's by means of equational identities, and in § 3 we shall show that such a characteri-

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(**) Some of the results to be reported below were obtained by Resek in 1968 while working at the University of Warsaw, some were obtained by Henkin in 1969 while working at the Mathematical Institute and at All Souls College, Oxford University. Both authors have had the pleasure and mathematical stimulation of contact with Andrzej Mostowski.

(*) If BA's are defined to require that \( 0 \neq 1 \), then \( a \) must be chosen so that \( a \neq 0 \).
zation is impossible for the class of CA’s of dimension 3 or higher. Some results concerning CA’s and subalgebras of CA’s for dimensions greater than 2 are briefly mentioned in the concluding § 4. In § 1 we present the basic definitions and known results which are used in the sequel.

§ 1. The concept of BA’s is abstracted from the notion of a Boolean set algebra

\[ \mathcal{B} = \langle \mathcal{F}, \cup, \cap, \sim, \Theta, V \rangle, \]

where \( V \) is an arbitrary set, \( \sim \) is the operation of complementation with respect to \( V \), and \( \mathcal{F} \) is a non-empty family of subsets of \( V \) closed under \( \cup, \cap, \sim \). In case \( V \) is the Cartesian power of another set \( U \), say \( V = U^a \) where \( a \) is any ordinal number \(^{(*)}\), then we can consider certain richer structures called cylindric set algebras, obtained as follows. For \( \kappa, \lambda \in \alpha \) we distinguish the diagonal sets

\[ D_{\alpha} = \{ \sigma \in U^\alpha \} \]

and we consider the operations \( C_{\alpha} \) of cylindrification such that, for any

\[ \mathcal{X} \subseteq U^\alpha, \]

\[ C_{\alpha}(\mathcal{X}) = \{ y \in U \} \]

for some \( x \in \mathcal{X}, y_1 = x_2 \) for every \( \lambda \neq \kappa, \lambda \in \alpha \). \(^{(*)}\)

If \( G \) is an arbitrary set of subsets of \( U^\alpha \) such that \( D_{\alpha} \in G \) for all \( \alpha, \beta \in \alpha \) and \( \beta \) is closed under each of the operations \( \cup, \cap, \sim, C_{\alpha} \), then the structure

\[ \mathcal{B} = \langle G, \cup, \cap, \sim, \Theta, U^\alpha, D_{\alpha} \backslash \lambda \subseteq \alpha \rangle \]

is called an \( \alpha \)-dimensional cylindric set algebra (CSA\( \alpha \)).

The notion of a cylindric algebra is obtained by abstraction from the notion of a cylindric set algebra by selecting certain equations which hold identically in every cylindric set algebra and using these as axioms to define the class of CSA’s. Specifically, by an \( \alpha \)-dimensional cylindric algebra, CSA\( \alpha \), we mean any structure

\[ \mathcal{B} = \langle A, \cup, \cap, \sim, 0, 1, a, d_{\alpha} \backslash \lambda \subseteq \alpha \rangle \]

such that

\( (C_0) \) \( a, b \in A \) and each \( a \) is a one-place operation on \( A \); and for every \( x, y \in A \) and \( x, \mu, \nu < \alpha \):

\( (C_1) \) \( 0 = 0 \);

\( (C_2) \) \( a \leq a \) (i.e., \( a \cdot a = a \));

\( (C_3) \) \( a \circ a = a \).

\( (\ast) \) We assume ordinals defined so that each ordinal number coincides with the set of its predecessors.

\( (\ast) \) If we visualize \( U \) as an \( \alpha \)-dimensional Cartesian space over \( U \), then each \( D_{\alpha} \) is a diagonal hyperplane of the space, and for any point set \( X \) of the space, \( C_{\alpha}(X) \) is the cylinder generated by translating \( X \) parallel to the \( \alpha \)th coordinate axis.

Among the laws, derivable from these axioms, which hold in every CSA\( \alpha \), are the following:

\[ (1.1) \]

\( (i) \) \( a \circ a = a \);

\( (ii) \) \( a \circ (b \circ c) = (a \circ b) \circ c \);

\( (iii) \) \( a \circ (b \circ c) = (a \circ b) \circ c \);

\( (iv) \) \( d_{\alpha} = a \);

\( (v) \) \( a \circ d_{\alpha} = a \);

\( (vi) \) \( a \circ d_{\alpha} = a \) if \( \alpha \neq \lambda, \mu \);

\( (vii) \) \( a \circ d_{\alpha} = a \) if \( \alpha \neq \lambda \).

From (i), (ii), (iii), and (C\( \alpha \)) we see that the set

\[ a \circ \alpha = \{ a | x \in A \} \]

is closed under \( \cup, \cap, \sim, \) and hence by (C\( \alpha \)) the structure

\[ a \circ \alpha = \{ a | x \in A \} \]

is a BA, the BA of \( \alpha \)-cylinders of \( a \).

For any element \( a \) of a CSA\( \mathcal{A} \), let \( \alpha \circ \alpha \) be the operation such that \( \alpha \circ \alpha = a \circ a \) for all \( x \in A \). Then the structure

\[ a \circ \alpha = \{ a | x \in A \} \]

is a cylindric relativized algebra of dimension \( a \), \( a \circ \alpha \). Note that the mapping \( h_a \) of \( A \) onto \( A \), such that \( h_a(x) = a \cdot x \) for all \( x \in A \), is not in general a homomorphism of \( \mathcal{A} \) onto \( a \circ \alpha \), since in general we will not have \( h_a(c \circ a) = c \circ h_a(a) \) (although this will be true in case \( c \circ a = a \) by (C\( \alpha \))).

In consequence we cannot be sure that all of the equations (C\( \alpha \)) holding in the CSA\( \mathcal{A} \) will also hold in the CSA\( \alpha \). It turns out that while each CSA\( \mathcal{A} \), for \( \alpha = 0 \), it is easy to construct a CSA\( \alpha \) which is not a CSA\( \mathcal{A} \) for every \( \alpha > 1 \); for example, in the CSA\( \alpha \), \( \mathcal{A} \) of all subsets of \( U \), where \( U \) is a set with more than one element, if we put \( a = \alpha \), then the CSA\( \alpha \) will fail to satisfy (C\( \alpha \)) and (C\( \alpha \)). However, we have the following result.

\[ (1.3) \] Proposition. In any CSA\( \mathcal{A} \), all of the laws (C\( \alpha \)) hold and (C\( \alpha \)) hold identically. If \( a \circ \alpha \) is a CSA\( \alpha \) obtained from \( a \circ \alpha \) by relativizing to the element \( a_0 \), then \( a \circ \alpha \) will also satisfy (C\( \alpha \)) and
(C_1) — and hence will be a CA. — if the following conditions hold for all 
_1 < a < _2 and _1 < _2:
(i) _1 < _1 + _1 < _2 + _2;
(ii) _2 < _2 + _1 < _1 + _2.

The principal result of § 2, Theorem 2.8, states that for dimension 
_1 = 2, the identities (C_1) - (C_6), (C_7), (C_8), and (1.1) (iv) characterize
the class of CAs. Given any structure D satisfying these identities, we shall con-
struct a CA _2, and an element _1, such that D = _1. This con-
struction will proceed by first forming certain BA's associated with D,
and then combining these in a suitable way by forming direct products
and Boolean products.

The well-known construction of the direct product P_1 C_1 of a given
family (B_1)_i of similar structures (i) produces a structure C_1, and
canonical homomorphisms p_i (called projections) of C onto C_1 for each i,
which can be characterized up to isomorphism by the property that,
given any structure C with homomorphisms p_i onto C_1 for each i, there
exists a unique homomorphism _1 of C onto C such that p_i = _1 for each i.
Since the formation of a direct product preserves equational
identities, we can be sure that the direct product of BA's is itself a BA.
In case each C_1 is a Boolean algebra of subsets of some set V_1, then
V_1 = V_1 for some u, v_i, w_i, for all i ∈ I.

It is easy to infer that D will be closed under each c_j, from the fact that
if X_1 ∈ D, then c_j(X_1) = X_1 for some u, v_i, w_i, for all i ∈ I.

In this case the definitions for q_j, q_j, c_j show that q_j is an isomorphism
of D onto the BA c_j of D for each j ∈ I. Also, in this case, we have
q_j c_j = 1 and c_j q_j = 1 whenever q_j is an element of D with q_j ≠ 0.
Writing c_j for c_j, for convenience, we express these results by the following.

(1.4) Proposition. Let D_i, D_j be arbitrary BA's, and let D be their
Boolean product with canonical homomorphisms q_i of D_i into D for i ∈ I.
Then there exist operations q_i, q_j on D satisfying the laws (C_1) - (C_6),
such that q_i is an isomorphism of D_i onto the BA c_j of D for j ∈ I.

Furthermore, for any element g ∈ D, if g ≠ 0 then c_j q_j g = q_j q_j g = 1.
We call the system (D_i, q_i, q_j) the cylindrification of the product
of the BA's D_i and D_j.

§ 2. Throughout this section we shall use the word structure to refer
to any system X of the same similarity type as CAs's and CRAs's, i.e., to
systems

\[ X = \langle A, +, ·, 0, 1, c_0, d_{ω, ω} | ω > 1 \],

where A is a set, + and · are binary operations on A, 0, 1, c_0, c_1 are unary
operations on A, and 0, 1, c_0, c_1, d_0, d_1 ∈ A.

(2.1) Definition. We let A be the class of all structures X satisfying
the equational identities (C_1) - (C_6), (C_7), (C_8), and (1.1) (iv).
Our aim is to show that $\Omega = C_{\mathcal{A}}$. Since we already know that $C_{\mathcal{A}} \subseteq \Omega$ by (1.3), we concentrate on demonstrating the opposite inclusion. We shall need several elementary facts about structures in $\Omega$, which we collect in the following.

(2.2) Lemma. Let $\mathfrak{A}$ be any structure in $\Omega$. Then the following hold (for any $x, y \in A$ and any $\alpha = 0, 1$):

(i) $c_{\alpha}(0) = c_{\alpha}0$,
(ii) $c_{\alpha}(x+y) = c_{\alpha}x + c_{\alpha}y$,
(iii) $c_{\alpha}(-x) = -c_{\alpha}x$,
(iv) $c_{\alpha}(x) < c_{\alpha}(y)$ whenever $x < y$,
(v) $c_{\alpha}(x-d_{\alpha}x) = c_{\alpha}d_{\alpha}x - c_{\alpha}(x-d_{\alpha}x)$,
(vi) $c_{\alpha}(x-d_{\alpha}y) = c_{\alpha}d_{\alpha}y - c_{\alpha}(x-d_{\alpha}y)$,
(vii) the structure $\mathfrak{A}^{\alpha}$ formed as in (1.2) is a $\mathcal{A}$,
(viii) if $x \in c_{\alpha}A$ then $c_{\alpha}(x+y) = x+c_{\alpha}y$.

Proof. Parts (i)-(iii) are copied from (1.1); their proofs in the theory of $\mathcal{A}$'s depend only on (C1)-(C2), which hold for $\mathfrak{A} \in \Omega$ by (2.1). Part (iv) is an immediate consequence of (i), since from $x < y$ follows $x+y = y$.

To obtain (v) we first use (ii) to get

$$c_{\alpha}(x-d_{\alpha}x) + c_{\alpha}(-x-d_{\alpha}x) = 0;$$

then by (C0)

$$c_{\alpha}(x-d_{\alpha}x) - c_{\alpha}(-x-d_{\alpha}x) = 0;$$

(v) then follows from the theory of $\mathcal{A}$'s.

To obtain (vi) we first use (C6) to get $x-d_{\alpha}x < c_{\alpha}(x-d_{\alpha}x)$, from which we see that $x-d_{\alpha}x < c_{\alpha}(x-d_{\alpha}y)$. The opposite inequality may be obtained by computing

$$c_{\alpha}[c_{\alpha}(x-d_{\alpha}y)-c_{\alpha}(-x-d_{\alpha}y)];$$

$$= c_{\alpha}(x-d_{\alpha}y) - c_{\alpha}d_{\alpha}y - c_{\alpha}(x-d_{\alpha}y)$$

by (C0)

$$= c_{\alpha}(x-d_{\alpha}y) - c_{\alpha}d_{\alpha}y - c_{\alpha}(x-d_{\alpha}y)$$

by (C0)

$$= 0,$$

whence

$$c_{\alpha}(x-d_{\alpha}y) - c_{\alpha}d_{\alpha}y = 0$$

by (C0),

and so $c_{\alpha}(x-d_{\alpha}y)-c_{\alpha}d_{\alpha}y < x-d_{\alpha}y$, completing the proof of (vi).

Turning to part (vii), we see that $c_{\alpha}A$ is closed under $+$ by (ii) and under $-$ by (iii), which, with (C1), is enough to assure (vii). Finally, as to (viii), if $x \in c_{\alpha}A$ then $x = c_{\alpha}x$ by (i), hence $c_{\alpha}(x+y) = x+c_{\alpha}y$ by (C0).

(2.2) proved, let us return to the task of showing that $\Omega \subseteq C_{\mathcal{A}}$.

To do this we choose any $\mathfrak{A} \in \Omega$ and must produce a $\mathfrak{A} \in C_{\mathcal{A}}$, and an element $\alpha$ of $\mathfrak{A}$, such that $\mathfrak{A} = \mathfrak{A}^{\alpha}$. To construct this $\mathfrak{A}$ we shall form various direct products and cylindric products of $\mathcal{B}$'s, but in addition we need a new type of product for structures in $\Omega$, which we now describe.

(2.3) Definition. Let $\mathfrak{B}$, $\mathfrak{B}' \in \Omega$, where

$$\mathfrak{B} = \langle \mathcal{B}, +, -, 0, 1, c_{\alpha}, d_{\alpha} \rangle_{\alpha \in \omega \rightarrow 0};$$

$$\mathfrak{B}' = \langle \mathcal{B}', +', -, '0', '1, d', \rangle_{\alpha \in \omega \rightarrow 0};$$

Suppose that for $\alpha = 0, 1$ we have a one-one map $h_{\alpha} : c_{\alpha}\mathcal{B} \rightarrow c_{\alpha}\mathcal{B}'$, and let $h = h_{0}(1)$.

Let

$$\langle A, +', -, '0', '1, d' \rangle_{\alpha \in \omega \rightarrow 0};$$

be the direct product

$$\langle \mathcal{B}, +', -, '0', '1, d' \rangle \times \langle \mathcal{B}', +', '-', '0', '1, d' \rangle;$$

and define operations $c_{\alpha}'$ on $A$ as follows: For $\alpha = 0, 1$ and any $\langle x, y \rangle \in A$,

$$c_{\alpha}^{\prime}(x, y) = \langle c_{\alpha}(x), c_{\alpha}(y) \rangle + h_{\alpha}^{-1}(h_{\alpha}(c_{\alpha}(x)), h_{\alpha}(c_{\alpha}(y)),$$

Then the structure

$$\mathfrak{A} = \langle A, +', '-', '0', '1, d' \rangle_{\alpha \in \omega \rightarrow 0};$$

is called the superposition product of $\mathfrak{B}$ and $\mathfrak{B}'$ (with respect to $h_{\alpha}$ and $h_{\alpha}$).

In case $h_{\alpha}$ and $h_{\beta}$ are fixed in advance we denote this $\mathfrak{B} \otimes \mathfrak{B}'$.

(2.4) Lemma. Let $\mathfrak{B}$, $\mathfrak{B}'$, $h_{\alpha}$, $h_{\beta}$, $b_{\alpha}$, $b_{\beta}$, be as in Definition (2.3), and let $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{B}'$ as defined there. Assume that for $\alpha = 0, 1$, the map $h_{\alpha}$ is an isomorphism of the $\mathcal{B}$ on $c_{\alpha}\mathcal{B}$ onto the $\mathcal{B}$ obtained by relativizing $c_{\alpha}B'$ to $b_{\alpha}$. Assume, furthermore, that for $\alpha = 0, 1$, we have $h_{\alpha}(c_{\alpha}(a)) \in d_{\alpha}$.

Let $\mathfrak{A} \in \Omega$. If, for some $\alpha = 0, 1$ and $y \in A$ such that $y \neq 0'$ in $\mathfrak{A}$, then also $c_{\alpha}^{\prime}(x, y) = 1$ whenever $x \neq \alpha$ and

$$\langle x, y \rangle \in A.$$
In order to demonstrate that this condition holds for \( \mathcal{A} \) we first show that for \( \alpha = 0, 1 \) and \( \langle u, v \rangle \in \mathcal{A} \) we have:

\[
(c_\alpha(d_{\alpha 1} - x), h_{\alpha 1}^{-1}(b_{\alpha 1} \cdot c_\alpha(d_{\alpha 1}' - y)), h_{\alpha 1}^{-1}(b_{\alpha 1} \cdot c_\alpha(d_{\alpha 1}' - y))) = 0.
\]

To obtain (*) we begin with the hypothesis \( h_{\alpha}(c_\alpha(d_{\alpha 1} - x), c_\alpha(d_{\alpha 1}' - y)) = 0 \) of (2.4). Multiplying through by \( e' \), applying \( c_\alpha \), and using (C_1) gives

\[
(c_\alpha(h_{\alpha}(c_\alpha(d_{\alpha 1} - x), c_\alpha(d_{\alpha 1}' - y))) = 0.
\]

Since range \( (h_{\alpha} \subseteq c_\alpha \mathcal{B}' \) we can use (2.2) (viii) to get

\[
h_{\alpha}(c_\alpha(d_{\alpha 1} - x), c_\alpha(d_{\alpha 1}' - y)) = 0.
\]

Now multiplying by \( d_{\alpha 1} \) and recalling that \( h_{\alpha} \) an isomorphism, we get

\[
c_\alpha(d_{\alpha 1}; h_{\alpha}^{-1}(b_{\alpha 1} \cdot c_\alpha(d_{\alpha 1}' - y))) = 0.
\]

But \( d_{\alpha 1} \leq c_\alpha(d_{\alpha 1}) \) by (C_3) so

\[
(d_{\alpha 1}; h_{\alpha}^{-1}(b_{\alpha 1} \cdot c_\alpha(d_{\alpha 1}' - y))) = 0.
\]

Finally apply \( c_\alpha \) and use (2.2) (viii) again to get the first equation of (*). The second equation is derived from the first by applying the isomorphism \( h_{\alpha} \) and recalling that \( h_{\alpha} = h_{\alpha}(1) \).

Now let us use (*) to derive (C_1). For any \( \alpha = 0, 1 \) and \( \langle x, y \rangle \in \mathcal{A} \) we have:

\[
(c_\alpha(d_{\alpha 1} - x), c_\alpha(d_{\alpha 1}' - y)) = 0.
\]

The proof above that (C_1) holds in \( \mathcal{A} \) completes our proof that \( \mathcal{A} \in \mathcal{Q} \), and we turn to the second part of Lemma (2.4). Assume, therefore, that
for \( x \neq \lambda \) and any \( y \in B' \) we have \( c'_n c'_n y = 1' \), and consider any \( \langle x, y \rangle \in A \) with \( \langle x, y \rangle \neq 0^d \). Then:

\[
\begin{align*}
\langle x, y \rangle &= \langle x, y \rangle^4 + \langle h_{-1}^{-1}(b'_k c'_n y), h_c c_n x \rangle \\
&= \langle x, y \rangle + \langle h_{-1}^{-1}(b'_k c'_n y), h_c c_n x \rangle \\
&\quad + \langle h^{-1}(b'_k c'_n y), h^{-1}(b'_k c'_n x) \rangle.
\end{align*}
\]

Now from \( \langle x, y \rangle \neq 0^d \) we either have \( x \neq 0 \) or \( y \neq 0' \). In case \( y \neq 0' \), then by hypothesis \( c'_n c'_n y = 1' \), and hence \( h_{-1}^{-1}(b'_k c'_n y) = h_{-1}^{-1} b'_k = h_{-1}^{-1} h_c(1) = 1 \), and thus (**) gives \( c'_n c'_n \langle x, y \rangle = 1' \) by definition of \( A' \) and by (C0) for \( A \).

On the other hand, suppose that \( x \neq 0 \). Then \( c_n x \neq 0 \) by (C1) for \( A \), and hence \( h_c c_n x = 0' \) because \( h_c \) is an isomorphism. But \( h_c c_n x = c_n h_c c_n x \) by (1.1)(i) for \( A' \), since the range of \( h_c \) is \( c_n^{*} B' \). Thus \( c'_n c'_n c_n x = 1' \) by the hypothesis; and from this we have also \( h_{-1}^{-1}(b'_k c'_n c_n x) = h_{-1}^{-1}(b'_k) = 1 \). Using these in (**) gives \( c'_n c'_n \langle x, y \rangle = 1' \) again.

This completes the proof of Lemma (2.6).

Having defined superposition products in (2.3) and derived their basic properties in (2.4), let us see how we can use them. Recall that we wish to show that \( B \subset C \), by considering any \( B \in \Omega \) and showing that \( B = \left\{ \alpha \right\} \) for some \( C \in \gamma \) and element \( \alpha \) of \( A \). The way in which we shall find \( \alpha \) and \( A \) is suggested in the following lemma.

\[ \text{(2.5) Lemma.} \quad \text{Suppose that} \quad B, \quad B' \in \Omega. \quad \text{Assume that for each} \quad x = 0, 1 \quad \text{there is an isomorphism} \quad h, \quad c_n^B \quad \text{onto} \quad (c_n^* B)'_{h, \alpha}, \quad \text{where} \quad b, \quad h_{\alpha} = h_c(1). \quad \text{Let} \quad A' \quad \text{be the superposition product} \quad B \otimes B' \quad \text{(with respect to} \quad b, \quad h_{\alpha}), \quad \text{and suppose that} \quad \alpha' = c_n^B \quad \text{onto} \quad (c_n^A A)'_{\alpha}, \quad \text{where} \quad a, \quad h_{\alpha} = h_c(1).
\]

\[ \text{Proof.} \quad \text{Let} \quad B, \quad B', \quad A' \quad \text{be as in (2.3), and let} \quad f \quad \text{be the map of} \quad B \quad \text{into} \quad A \quad \text{such that} \quad f(x) = \langle x, 0 \rangle \quad \text{for all} \quad x \in B. \quad \text{Clearly} \quad f \quad \text{is a one-one map of} \quad B \quad \text{onto} \quad A \quad \text{by (2.4).} \]

It is obvious that \( f(0) = 0^d, \quad f(1) = a, \quad f \) carries \( + \) into \( + \) and \( * \) respectively, and for any \( x \in B \):

\[
\begin{align*}
\langle x, 0 \rangle &= \langle x, 0 \rangle = \langle 1, 0 \rangle = \langle x, x \rangle, \\
\langle x, 1 \rangle &= a = a^* \cdot \langle x, 0 \rangle = a^* \cdot \langle x, 0 \rangle \\
\langle x, x \rangle &= a = a^* \cdot \langle x, 0 \rangle.
\end{align*}
\]

Thus it remains only to show that \( f \) carries \( d_n \) to \( (d_n^d)^n \) and \( e_n \) into \( (e_n^d)^n \).

Let us compute

\[
\begin{align*}
\langle d_n, 0 \rangle &= \langle d_n, 0 \rangle = \langle 1, 0 \rangle = \langle d_n, d_n \rangle \\
&= a^* \cdot \langle d_n, 0 \rangle.
\end{align*}
\]
Finally, let us define a map $m$ of $c_{\beta}^\alpha \mathfrak{D}'$ to $c_{\beta}^\alpha \mathfrak{D}'$, as follows. Consider any element $z \in c_{\beta}^\alpha \mathfrak{D}'$, say $z = g_\beta(x, y)$ with $x \in \mathfrak{C}_0 = c_{\beta}^\alpha \mathfrak{B}$ and $y \in \mathfrak{C}_1 = (c_{\beta}^\alpha \mathfrak{B})_{\overline{0}}, \ldots_{\overline{0}}$. Then we set

$$m(z) = m_{g_\beta}(z, y) = g_\beta(\langle x \cdot \delta_{\alpha} \rangle + y, x) = c_{\alpha}^\beta \delta_{\alpha}.$$  

To establish (2.6) (iv) we wish to show that $m$ is an isomorphism of the BA $c_{\beta}^\alpha \mathfrak{D}'$ onto $c_{\beta}^\alpha \mathfrak{D}'$. Since (1), the definition of $m$, leads directly to the conclusion

$$m(g_\beta(x, y) + D_{\delta_{\alpha}}(x', y')) = m_{g_\beta}(x, y) + D_{\delta_{\alpha}}(x', y')$$

for all $g_\beta(x, y), g_\beta(x', y') \in c_{\beta}^\alpha \mathfrak{D}'$, using (2.2) (ii) for $\mathfrak{B}$, let us show that $m$ preserves negation.

Recalling that

$$-D_{\delta_{\alpha}}(x, y) = g_\beta(-x, y', -x \cdot \delta_{\alpha})$$

for $x \in c_{\beta}^\alpha \mathfrak{C}_1$, we easily get

$$m_{\beta}(x, y) + D_{\delta_{\alpha}}(x, y') = g_\beta(1, -x \cdot \delta_{\alpha}) = 1^\beta$$

by (3), definition of $+\beta$, (3), and (1), since $y = -c_{\alpha}^\beta \delta_{\alpha}$. Now

$$m_{\beta}(x, y) = m_{\beta}(g_\beta(x, y), y') = +D_{\delta_{\alpha}}(g_\beta(x, y), y')$$

by (1) and (3), since $g_\beta$ is a homomorphism, and

$$g_\beta(\langle x \cdot \delta_{\alpha} \rangle + y, \langle x \cdot \delta_{\alpha} \rangle + y', -c_{\alpha}^\beta \delta_{\alpha}) = 0.$$  

Hence

$$m_{\beta}(x, y) = m_{\beta}(g_\beta(x, y), y') = g_\beta(0, 0) = 0^\beta,$$

which, together with (4), shows that

$$m(-D_{\delta_{\alpha}}(x, y)) = -D_{\delta_{\alpha}}(m(x, y)).$$

From (2) and (5) we see that $m$ is a homomorphism, so let us now show that it is one-one. Suppose, then, that $g_\beta(x, y)$ and $g_\beta(x', y')$ are elements of $c_{\beta}^\alpha \mathfrak{D}'$ such that

$$m_{\beta}(x, y) = m_{\beta}(x', y').$$

By (1) and the fact that $g_\beta$ is an isomorphism we can infer from (6) that

$$c_{\alpha}^\beta \delta_{\alpha} = c_{\alpha}^\beta \delta_{\alpha}$$

and

$$z = c_{\alpha}^\beta \delta_{\alpha} = c_{\alpha}^\beta \delta_{\alpha}.$$

Since $c_{\alpha}^\beta \delta_{\alpha} < c_{\alpha}^\beta \delta_{\alpha}$ by (2.2) (iv), while $y < -c_{\alpha}^\beta \delta_{\alpha}$, and similarly $c_{\alpha}^\beta \delta_{\alpha} < c_{\alpha}^\beta \delta_{\alpha}$, we obtain from (7a) that

$$c_{\alpha}^\beta \delta_{\alpha} = c_{\alpha}^\beta \delta_{\alpha}$$

and

$$y = y'.$$

From (8a), multiplying through by $c_{\alpha}^\beta \delta_{\alpha}$ and using (2.2) (vi), we get $x \cdot c_{\alpha}^\beta \delta_{\alpha} = x' \cdot c_{\alpha}^\beta \delta_{\alpha}$, which, upon using $c_{\alpha}^\beta \delta_{\alpha}$ to both sides and using (2.2) (vii), gives $x \cdot c_{\alpha}^\beta \delta_{\alpha} = x' \cdot c_{\alpha}^\beta \delta_{\alpha}$. When this is combined with (7b) we get, finally, $x = x'$. Since this equation, as well as (8b), have been derived from (6), we have completed the proof that $m$ is one-one, and so by (2) and (5) we know that

$$m$$

is an isomorphism of $c_{\beta}^\alpha \mathfrak{D}'$ onto $c_{\beta}^\alpha \mathfrak{D}'$.

Let us now show that the range of $m$ is the whole of $c_{\beta}^\alpha \mathfrak{D}'$. Consider, therefore, an arbitrary element $g_\beta(u, v) \in c_{\beta}^\alpha \mathfrak{D}'$, where $u \in \mathfrak{C}_0 = c_{\beta}^\alpha \mathfrak{B}$ and $v \in \mathfrak{C}_1$ (so that $v \in c_{\beta}^\alpha \mathfrak{B}$ and $v < -c_{\alpha}^\beta \delta_{\alpha}$). Put

$$x = c_{\alpha}^\beta \delta_{\alpha} + v, \quad y = -c_{\alpha}^\beta \delta_{\alpha}.$$

Using (1) we compute

$$m_{\beta}(x, y) = m_{\beta}(c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha})$$

so that

$$m_{\beta}(x, y) = m_{\beta}(c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha} + v, -c_{\alpha}^\beta \delta_{\alpha})$$

by (2.2) (ii) and (2.2) (vii). But $c_{\alpha}^\beta \delta_{\alpha} < c_{\alpha}^\beta \delta_{\alpha}$ by (2.2) (vii), $v \cdot c_{\alpha}^\beta \delta_{\alpha} < v \cdot c_{\alpha}^\beta \delta_{\alpha}$ by (1), so $v \cdot c_{\alpha}^\beta \delta_{\alpha} = 0$ (since $v < -c_{\alpha}^\beta \delta_{\alpha}$), and $c_{\alpha}^\beta \delta_{\alpha} - c_{\alpha}^\beta \delta_{\alpha} = 0$ by (2.2) (iv). Thus (11) yields $m_{\beta}(x, y) = g_\beta(u, v)$.

Since $g_\beta(u, v)$ was an arbitrary element of $c_{\beta}^\alpha \mathfrak{D}'$ we can now conclude from (9) that

$$m$$

is an isomorphism of $c_{\beta}^\alpha \mathfrak{D}'$ onto $c_{\beta}^\alpha \mathfrak{D}'$.

Thus, the following computation completes the proof of (2.6) (iv):

$$m_{\beta}(c_{\alpha}^\beta \delta_{\alpha}) = m_{\beta}(c_{\alpha}^\beta \delta_{\alpha}, 0) = 0^\beta$$  

(by definition of $m$).

This finishes the proof of Lemma (2.6).
We are now ready, starting from any $B \in \Omega$, to form a structure $B'$ of the kind described in Lemma (2.5).

(2.7) Lemma. Let $\mathcal{B}$ be any structure in $\Omega$,
$$\mathcal{B} = \langle B, +, \cdot, \cdot, -1, 0, 1, c, a, d, \rangle_{\alpha, \lambda = 0, 1, \ldots}.$$ Then we can find another structure $\mathcal{B}' \in \Omega$,
$$\mathcal{B}' = \langle B', +', \cdot', \cdot', -1', 0', 1', c', a', d', \rangle_{\alpha, \lambda = 0, 1, \ldots}$$
with the following properties:

(i) $c'_0 c'_0 = 1'$ and $c'_0 c' = 1'$ whenever $\omega \in B'$ and $\omega \neq 0'$;
(ii) there is an isomorphism $\mathcal{h}_\alpha$ of the $\mathbb{B}A_\alpha \mathcal{B}$ onto $(c'_\alpha \mathcal{B}')_{\mathcal{h}_\alpha}$ where $\mathcal{h}_x = \mathcal{h}_x(1)$ for each $x = 0, 1$;
(iii) $c'_\alpha d'_\alpha = -\mathcal{h}_x(c_\alpha d_\alpha)$ for each $x = 0, 1$;
(iv) the superposition product $\mathcal{B} \otimes \mathcal{B}'$ (with respect to $\mathcal{h}_0$, $\mathcal{h}_1$) is a CA.$\mathcal{h}$.

Proof. Starting with the given structure $\mathcal{B}$, we first find a structure $\mathcal{F}'$, mappings $f_0$, $f_1$, $m$, and elements $c_0$, $c_1$, satisfying the conditions of Lemma (2.6).

For $x = 0, 1$, let $i_x = -f_x(c_\alpha d_\alpha) \in D$, let $\mathcal{M}$ be the $\mathbb{B}A_\alpha \mathcal{F}'$ over $\mathcal{M}$, and write
$$\mathcal{M} = \langle \mathcal{I}, +', \cdot', \cdot', -1, 0, 1, i_x \rangle$$
in particular, we have $I = \{c_\alpha d_\alpha \in D \mid c_\alpha d_\alpha \notin \mathcal{I}\}$, $I^2 = i_x$, and $-1 = i_x^{0,0} = -1$ for each $x \in I$.

Now for $x = 0, 1$, define $d_x = i_x^{1,0}$ and $c_x y = y$ for any $y \in I$. It is then a trivial matter to verify that

(1) the structure $\mathcal{M}' = \langle \mathcal{I}, +', \cdot', \cdot', -1, 0, 1, i_x, d_x, c_x' \rangle_{\alpha, \lambda = 0, 1, \ldots}$ is an element of $\Omega$.

Next we need certain mappings $j_0$ and $j_1$ of $\mathcal{M}$ into $D$. In fact, we can take $j_0$ to be the identity map on $I$ (recalling that $I \subseteq D$). As for $j_1$, for any $x \in I$ we define $j_1(x) = m(x)$. Since $j_1(1^{x}) = m(1^{x}) = m(-f_x(c_\alpha d_\alpha)) = -f_x(c_\alpha d_\alpha) = i_x$, we easily see that

(2) for $x = 0, 1$, $j_1$ is an isomorphism of the $\mathbb{B}A_\alpha \mathcal{F}'$ onto $(c_\alpha \mathcal{D}')_{\mathcal{h}_x}$.

Furthermore, by (2.6) (i) we have, of course,

(3) $j_1(c'_\alpha d'_\alpha) = d'_\alpha$ $\alpha = 0, 1$.

By combining (1), (2), and (3) we are able to apply Lemma (2.4). Letting $\mathcal{B}'$ be the superposition product $\mathcal{M}' \otimes \mathcal{M}'$ (with respect to $j_0$ and $j_1$) we thus see that $\mathcal{B}' \in \Omega$. Also (2.7) (i) holds by (2.4), in view of (2.6) (ii).

To obtain (2.7) (ii), we define $\mathcal{h}_x$ from $c'_\alpha B$ to $c'_\alpha B'$ for each $x = 0, 1$, by setting

(4) $\mathcal{h}_x(z) = c'_\alpha (f_0 z, f_1 z)$ for all $z \in c'_\alpha B$.

Using the definition of superposition products (2.3) we get

$$\mathcal{h}_x(z) = \langle c'_\alpha (f_0 z, f_1 z), \mathcal{h}_x(0), c'_\alpha (f_0 z, f_1 z) \rangle$$
and so, by (2.6) (iii),

(5) $\mathcal{h}_x(z) = \langle j_0^{-1}(f_0 z, f_1 z), f_0 z \rangle$ for all $z \in c'_\alpha B$.

From this, using (2.6) (iii) and (2) above, we obtain (2.7) (ii).

Next, using (2.3) and the definitions of $\mathcal{M}'$ and $\mathcal{M}''$, we compute

$$c'_\alpha (d'_\alpha, d'_\alpha) = c'_\alpha (d'_0, d'_0) = c'_\alpha (1', 0')$$
$$= \langle c'_\alpha (f_0^0, f_0^0), \mathcal{h}_x(0), c'_\alpha (f_0^0, f_0^0) \rangle$$
$$= \langle 1', j_0 1', j_0 1' \rangle$$
$$= -\langle 0', 0', 0' \rangle$$
$$= -\mathcal{h}_x(c_\alpha d_\alpha)$$

(by 4) and definition of $i_x$),

which proves (2.7) (iii).

Now let $\mathcal{U}$ be the superposition product $\mathcal{M}' \otimes \mathcal{M}'$ (with respect to $\mathcal{h}_0$, $\mathcal{h}_1$). Since $\mathcal{h}_x(c_\alpha d_\alpha) = 0'$ by (2.7) (iii), and using (2.7) (ii), we can apply (2.4) to obtain $\mathcal{U} \in \Omega$. Again by (2.4), this time with (2.7) (i), we obtain $c'_\alpha c'_\alpha = c'_\alpha c'_\alpha = 1'$ whenever $\alpha \neq 0$, so that by $(C_4)$ we have $c'_\alpha c'_\alpha = c'_\alpha c'_\alpha$ for all $\alpha \in A$; thus $(C_4)$ holds for $\mathcal{U}$. Finally, by (2.3) we compute

$$c'_\alpha (d'_\alpha, d'_\alpha) = c'_\alpha (d'_0, d'_0)$$
$$= \langle c_\alpha d_\alpha, c_\alpha d_\alpha \rangle$$
$$= \langle 1', 1' \rangle$$

(by (2.7) (iii))

$$= 1'$$

so that $(C_4)$ holds for $\mathcal{U}$.

Putting together the facts that $\mathcal{U} \in \Omega$ and that $(C_4)$ holds for $\mathcal{U}$, we obtain $\mathcal{M} \in C_A$. By definition of $\Omega$ and $C_A$,

All the lemmas are now at hand for our equational characterization of the class $C_A$ of 2-dimensional cylindric relativised algebras.

(2.8) Theorem (4). For any structure $\mathcal{M}$ we have $\mathcal{M} \in C_A$ iff $\mathcal{M} \in \Omega$.  

(5) This theorem was first shown by Henkin in 1969 by a method involving the embedding of a given structure of $\Omega$ into an associated structure of $\Omega$. The present proof involving the notion of superposition products is new, and is also due to Henkin.
Proof. By (1.3) we have $Cr_3 \subseteq \Omega$. To obtain the opposite inclusion we take any $\mathcal{B} \in \Omega$ and, by (2.7) (ii), (2.7) (iv), and (2.5), we conclude that $\mathcal{B}$ is isomorphic to some $Cr_4 \subseteq \mathcal{A}_5$. But then, by the “exchange principle” of the general theory of structures, $\mathcal{B}$ is itself a $Cr_3$.

§ 3. In the preceding section we presented a set of equational identities (2.1) which characterize the class $Cr_3$ of 2-dimensional cylindric relation algebras. In the present section we shall show that no such characterization is possible for $Cr_4$ (or for any $Cr_n$ with $n > 2$).

(3.1) Theorem. There is a structure $\mathcal{B} \in Cr_4$, and a subalgebra $\mathcal{D}$ of $\mathcal{B}$, such that $\mathcal{D} \not\subseteq Cr_4$.

Proof. In § 1 we introduced the notion of an $n$-dimensional cylindric set algebra (CSA$_n$), from which the notion of a cylindric algebra is abstracted. To construct the $\mathcal{B}$ of (3.1) we begin with an $\mathcal{A} \in$ CSA$_4$, chosen as follows. We take $1^A$ to be the set $\{0, 1\}$ of all ordered triples $(x_1, x_2, x_3)$ of natural numbers, we let $A$ be the set of all subsets of $1^A$, and we take

$$\mathcal{B} = \langle A, \cup, \cap, \otimes, A, 1^A, a, d_{01}, a_{00} \rangle_{n=1,2,3}.$$  

Next we choose an element $a \in A$ by specifying that for any $(x_1, x_2, x_3) \in a$:

1. $(x_1, x_2, x_3) \in a$ if and only if:
   a. Either $x_1 = x_2 = x_3 = 0$, or
   b. $x_1 = x_2 = 0$, or
   c. $x_2 = x_3 = 0$.

And then we take $\mathcal{B} = \mathcal{A}_a$, the relativization of $\mathcal{A}$ to $a$. Thus, writing

$$\mathcal{B} = \langle A, \cup, \cap, \otimes, 1^A, a, d_{01}, a_{00} \rangle_{n=1,2,3},$$

we see that $B$ is the set of all subsets of $\mathcal{A}$; that $a_{01}$ is $a$ for all $Y \in B$ (so $(x_1, x_2, x_3) \in d_{01}Y$ if and only if $(x_1, x_2, x_3) \in a$ and $(y_1, y_2, y_3) \in Y$ for some $y \in a$); and that $d_{01} = a \land d_{11} = d_{00} = \emptyset$, so that by (1) we get

$$d_{01} = d_{11} = d_{00} = \emptyset.$$

Now let $D$ be the set of all those elements $X$ of $B$ such that either $X$ or $\otimes X$ is finite. It is well known (and obvious) that $D$ is closed under $\cup$, $\cap$, $\otimes$, and that it contains $\emptyset$ and $a$ among its elements; and from (2) we see also that each $d_{01} 

(\text{iv})$ The results of this section (but for $Cr_3$ instead of $Cr_4$) were obtained by Rosek in 1968. His proof was simplified in 1969 by Henkin, and was then soon to yield the sharper results for $Cr_3$. It is this proof that is given here for (3.1).
And by similar reasoning we obtain:

(7) Whenever \( \langle y_0, y_1, 1 \rangle \in a \) we have

\[
\langle 1, 2, 1 \rangle \subseteq^{E} c_1^{a^B} B(\langle y_0, y_1, 1 \rangle).
\]

Let us assume, now, that there is some \( \langle y_0, y_1, 1 \rangle \in a \) such that

(8) \( \langle y_0, y_1, 1 \rangle \subseteq^{E} c_1^{a^B} B(\langle 0, 1, 0 \rangle). \)

Applying \( c_1^a \) and then \( c_1^b \) to both sides of this inequality (as we may by (2.2) (iv)), and using (C_4) and (1.1) (i) for \( \mathbb{E} \), we get

\[
c_1^{a^B} B(\langle y_0, y_1, 1 \rangle) \subseteq^{E} c_1^{a^B} B(\langle 0, 1, 0 \rangle)
\]

and hence, by (7),

\[
\langle 1, 2, 1 \rangle \subseteq^{E} d_2^{a^B} (\langle 0, 1, 0 \rangle).
\]

But \( \langle 1, 2, 1 \rangle \subseteq^{E} d_2^{a^B} \), and similarly \( \langle 0, 1, 0 \rangle \subseteq^{E} d_2^{a^B} \). Thus

\[
\langle 1, 2, 1 \rangle \subseteq^{E} c_1^a c_1^b (\langle 0, 1, 0 \rangle),
\]

and hence, by (7),

\[
\langle 1, 2, 1 \rangle \subseteq^{E} c_1^a c_1^b (\langle 0, 1, 0 \rangle).
\]

But \( \langle 1, 2, 1 \rangle \subseteq a \), so by (4) we get

\[
\langle 1, 2, 1 \rangle \subseteq^{E} d_2^{a^B} (\langle 0, 1, 0 \rangle).
\]

But this is impossible, since by definition of \( c_1^a \) each triple in \( c_1^a (\langle 0, 1, 0 \rangle) \) has the form \( \langle 0, x, 0 \rangle \). This contradiction arises from assuming that (8) holds for some \( \langle y_0, y_1, 1 \rangle \in a \).

It follows, by (1), that whenever \( \langle y_0, y_1, 1 \rangle \in a \) and

\[
\langle y_0, y_1, 1 \rangle \subseteq^{E} c_1^a (\langle 0, 1, 0 \rangle),
\]

we must have \( y_1 = 0 \). This fact, together with (6), completes the demonstration that (5) holds — assuming (4).

Of course \( a^B c_1^a (\langle 0, 1, 0 \rangle) \) is an element \( Y \) of \( \mathbb{D} \), according to (4), and so by definition of \( D \) we must have either \( Y \) or \( a^B Y \) finite. On the other hand from (5) we infer that \( Y \) consists of all those \( \langle y_0, y_1, 1 \rangle \in a \) such that \( y_1 = 0 \), whence by (1) both \( Y \) and \( a^B Y \) are infinite.

Thus a contradiction has been obtained from the assumption (4), and this shows that \( D \cap C_1 \). This fact, together with (3), completes the proof of Theorem (3.1).

Since any equational class is closed under formation of substructures, (3.1) leads directly to the following.

(3.2) COROLLARY. There is no set of equational identities that characterizes the class of structures \( C_R \).

It is a simple matter to modify the proof of (3.1) so as to establish a counterpart of this theorem for any dimension \( a > 3 \); and hence its Corollary (3.2) may be similarly extended.

Although, for \( a > 3 \), we see that there are subalgebras of \( C_R^a \)'s which are not themselves \( C_R^a \)'s, it is an open question whether every homomorphic image of a \( C_R \) is itself a \( C_R \).

§ 4. In this section we mention several results without presenting proofs.

Let \( S \mathcal{C}_R \) be the class of all subalgebras of \( C_R^a \). In the preceding section we have seen that for \( a > 3 \) we can find a structure \( \mathbb{F} \in \mathcal{S}_{C_R} \) such that \( \mathbb{F} \cap C_R \), and hence we have concluded that \( C_R \) is not an equational class for \( a > 3 \) (although it is for \( a < 3 \)). What about the class \( \mathcal{S}_{C_R} \) — is it equational?

(4.1) THEOREM. For each \( a > 3 \) an infinite set \( I_{\mathbb{F}} \) of equational identities is known which characterizes the class of structures \( \mathcal{S}_{C_R} \). This class cannot be characterized by any finite set of such identities.

In addition to the equations defining \( \mathbb{F} \), \( I_{\mathbb{F}} \) contains (1.1) (vi) and the equation

\[
\eta_0 (d_{x^\lambda}, d_{x^\mu}) \leq d_{x^\lambda} \quad \text{for} \quad \lambda \neq \mu,
\]

a weakened form of (C_1). The remaining equations of \( I_{\mathbb{F}} \) are grouped into various infinite bundles, of which one example is:

\[
\eta_0 \eta_0 \cdots \eta_0 (x^\lambda, d_{x^\mu}) \leq \eta_0 (x^\lambda)
\]

whenever \( \lambda, \mu \) are distinct. By a general theorem of Jónsson and Tarski on Boolean algebras with operators \((\mathbb{F})\), it follows that any structure satisfying the equational identities of \( I_{\mathbb{F}} \) can be extended to a complete and atomistic structure satisfying those same identities. The proof of (4.1) is obtained by showing that any complete atomistic structure satisfying \( I_{\mathbb{F}} \) is a \( C_R \), and then applying the Jónsson-Tarski result. This method of proof gives the following.

\[(\mathbb{F})\] This theorem was proved by Rees in 1968; a full description of the set \( I_{\mathbb{F}} \), and a proof of the theorem, will be included in her doctoral dissertation. An indirect proof of the equational character of \( \mathcal{S}_{C_R} \) was given by Pigozzi in 1969, who showed that the class \( \mathcal{S}_{C_R} \) is closed under formation of homomorphisms; since closure under direct products was known, and closure under formation of subalgebras is obvious, this allows the application of Garrett Birkhoff's characterization of equational classes (See [2], pp. 11 and 266). Pigozzi's proof does not furnish the set \( I_{\mathbb{F}} \) explicitly, nor deal with the question of finite sets of equations.

\[(\mathbb{F})\] See [3], Theorem 2.18.
(4.2) COROLLARY. Every complete atomistic structure of $\mathcal{S}_{\alpha}$ is a $\mathcal{C}_{\alpha}$.

It is natural to conjecture that (4.2) can be strengthened, namely, that every complete structure in $\mathcal{S}_{\alpha}$, whether atomistic or not, is a $\mathcal{C}_{\alpha}$. However, this remains an open question at present. In this connection recall that the structure $2$ of § 3, which is in $\mathcal{S}_{\alpha}$ but not in $\mathcal{C}_{\alpha}$, is incomplete, and that this incompleteness plays a central role in showing that $2 \not\in \mathcal{C}_{\alpha}$.

We now turn our attention to the operation relativization on the cylindric set algebras (CSA's) introduced in § 1. Although every BA is isomorphic to a Boolean set algebra, there can be no similar result for CA's in general since for finite a a CSA is always simple, and hence a direct product of two such algebras, while obviously a CA, is not isomorphic to any CSA. A CA which is isomorphic to a subdirect product of CSA's is called representable. Equivalently, an $\mathfrak{M} \subseteq \mathbb{C}_{\alpha}$ is representable if and only if, for every non-zero element $x$ of $\mathfrak{M}$ there is a homomorphism $l$ of $\mathfrak{M}$ to some CSA such that $l(x) \neq 0$.

For every $\alpha \geq 2$ there are non-representable CA's, and the relativization of CSA's is one of the principal ways of constructing these (14). Another method of constructing non-representable CA's that has been employed is a method of "adjoining elements" which is closely related to the formation of superposition products (introduced in 2.3 above). However, all non-representable CA's that can be constructed by adjoining elements to a CSA turn out to be isomorphic to relativized CSA's.

The last observation obviously raises the question whether every CA may be isomorphic to a relativized CSA (14). But the answer is negative. There are equational identities which hold in every CA which is a relativized CSA, but which fail in certain CA's obtained from CSA's by a process of "twisting cylindrications." These identities are related to certain operations called substitutions which can be defined in every CA.

If $x \not= y$, and $\lambda < \alpha$, we define the operation $S^x_\lambda$ on any CA $\mathbb{C}$ by the rule $S^x_\lambda(x) = c(x_\lambda, x)$ for all elements $x$ of $\mathbb{C}$. This operation is called a substitution because if $\mathfrak{M}$ is a CSA whose elements are subsets of $\mathbb{U}$, then for every $x \in \mathbb{U}$ we have $\omega \in S^x_\lambda(x)$ if $\omega' \in x_\lambda$, where $\omega'$ is obtained from $\omega$ by substituting the $\lambda$th coordinate for the $\lambda$th coordinate. The identities to which we have referred in the preceding paragraph can be expressed in terms of these substitution operations as follows:

For all distinct $\alpha_1, \ldots, \alpha_m$, $\lambda < \alpha$ and all $x \in \mathbf{A}$,

$$S^{x_{\lambda}}_{\alpha_1} S^{x_{\lambda}}_{\alpha_2} \cdots S^{x_{\lambda}}_{\alpha_m} S^{x_{\lambda}}_{\alpha} = S^{x_{\lambda}}_{\alpha_1} S^{x_{\lambda}}_{\alpha_2} \cdots S^{x_{\lambda}}_{\alpha_m} S^{x_{\lambda}}_{\alpha}.$$  

(14) See [1] for a description of such a construction in a metalogical, rather than a set-theoretical, context.

(15) In particular, every representable CSA is known to be isomorphic to a relativized CSA. This is an unpublished result of Henkin's.