Pinning countable ordinals

by

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Abstract. We determine all countable ordinals α for which there is a map into ω^3 such that the image of any subset of α of order type α has order type ω^3.

Let A and B be well-ordered sets. A function $\pi\colon A \to B$ is called a *pinning map* in case, for every subset $X \subset A$ which is order-isomorphic to A, its image $\pi(X)$ is order-isomorphic to B. If α and β are ordinals, we say α *can be pinned* to β, in symbols $\alpha \to \beta$, if there is a pinning map from α into β. Clearly, if A and B have order-type α and β, respectively, then $\alpha \to \beta$ if and only if there is a pinning map from A into B.

Specker introduced this notion in [8], where he studies partition relations of the form $\alpha \to (\alpha, \varkappa)^2$ where α is an ordinal and $\varkappa$ a cardinal. (See [2] for a definition of this partition relation.) To rule out trivial cases, we may assume that $\alpha > 1$ and $\varkappa \geqslant 3$. Positive partition relations of this sort have been proved for only three countable ordinals. Ramsey's theorem [6] says that $\omega \to (\omega, \omega)^2$. Specker [8] showed that $\omega^2 \to (\omega^2, n)^2$ for every $n < \omega$. Chang [1] showed that $\omega^\omega \to (\omega^\omega, 3)^2$, and E. C. Milner (unpublished) generalized Chang's result by showing that $\omega^\omega \to (\omega^\omega, n)^2$ for every $n < \omega$. (See [3] for a proof of this result.)

Specker [8] observed that, if $\alpha \to \beta$ and $\alpha \to (\alpha, \varkappa)^2$, then $\beta \to (\beta, \varkappa)^2$. He proved that $\omega^3 \not\to (\omega^3, 3)^2$ and that $\omega^m \to \omega^3$ for $3 \leqslant m < \omega$, thus proving that $\omega^m \not\to (\omega^m, 3)^2$ for $3 \leqslant m < \omega$.

In this paper, we answer a question raised by Specker in [8], by characterizing the countable ordinals which can be pinned to ω^3. It follows from our results that, if α is a countable ordinal such that $\alpha \to (\alpha, 3)^2$, then either $\alpha \in \{0, 1, \omega^2\}$ or else $\alpha = \omega^{\omega^\beta}$ for some $\beta < \omega_1$. One can conjecture that $\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, n)^2$ for all $\beta < \omega_1$ and $n < \omega$. As we have already remarked, this has been proved only for $\beta = 0$ and $\beta = 1$.

Rotman [7] has also done some work on pinning countable ordinals; the notation $\alpha \to \beta$ is due to him.

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We assume that the reader is familiar with basic properties of ordinals and ordinal arithmetic. The *order type* of a well-ordered set A is the unique ordinal isomorphic to A, and is denoted $\operatorname{tp} A$. An ordinal is *decomposable* if it is the sum of two smaller ordinals. An *indecomposable* ordinal is a nonzero ordinal which is not decomposable. The indecomposable ordinals are just the ordinal powers of ω. Milner and Rado [5] proved that, if $\operatorname{tp} A = \alpha$ is an indecomposable ordinal, then, for any partition $A = B \cup C$, either $\operatorname{tp} B = \alpha$ or $\operatorname{tp} C = \alpha$. Every nonzero ordinal can be uniquely expressed in the form $\alpha_0 + \alpha_1 + \ldots + \alpha_n$ where $n < \omega$; $\alpha_0, \ldots, \alpha_n$ are indecomposable; and $\alpha_0 \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_n$. If A and B are subsets of an ordered set, the notation $A < B$ means that $a < b$ for all $a \in A$ and $b \in B$.

The following theorem reduces the study of the relation $\alpha \to \beta$ to the case where both ordinals are indecomposable. The proof is left to the reader.

THEOREM 1. *Let α and β be nonzero ordinals. Write $\alpha = \alpha_0 + \ldots + \alpha_m$, $\beta = \beta_0 + \ldots + \beta_n$, where $m, n < \omega$; $\alpha_0, \ldots, \alpha_m$, $\beta_0, \ldots, \beta_n$ are indecomposable; $\alpha_0 \geqslant \ldots \geqslant \alpha_m$ and $\beta_0 \geqslant \ldots \geqslant \beta_n$. Then $\alpha \to \beta$ if and only if there is a one-to-one function f: $\{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ such that $\alpha_{f(i)} \to \beta_i$ for each $i \in \{0, 1, \ldots, n\}$.*

Our main theorem characterizes the countable indecomposable ordinals that can be pinned to ω^3.

THEOREM 2. *For every ordinal $\alpha < \omega_1$, we have $\omega^\alpha \to \omega^3$ if and only if α is decomposable and $\alpha \geqslant 3$.*

Proof. In Theorem 3 below we prove that, if α is decomposable and $3 \leqslant \alpha < \omega_1$, then $\omega^\alpha \to \omega^3$. In Theorem 8 we prove that, if α is indecomposable and $\alpha < \omega_1$, then $\omega^\alpha \not\to \omega^3$. Obviously $\omega^0 \not\to \omega^3$. Since Specker [8] has proved that $\omega^2 \not\to \omega^3$, the theorem follows.

Using Theorems 1 and 2, we can characterize the countable ordinals that can be pinned to ω^3. Namely, suppose $\alpha = \omega^{\varepsilon_0} + \omega^{\varepsilon_1} + \ldots + \omega^{\varepsilon_n} < \omega_1$, where $n < \omega$ and $\varepsilon_0 \geqslant \varepsilon_1 \geqslant \ldots \geqslant \varepsilon_n$. Then $\alpha \to \omega^3$ if and only if some ε_i is decomposable and $\geqslant 3$.

THEOREM 3. *If $3 \leqslant \alpha < \omega_1$ and α is decomposable, then $\omega^\alpha \to \omega^3$.*

Proof. The proof for the case of a successor ordinal $\alpha = \delta + 1$ consists of Lemmas 4 and 5 below, since $\omega^\alpha = \omega^{\delta+1} = \omega^\delta \cdot \omega$. The proof for the case of α a limit ordinal is Lemma 6 below.

LEMMA 4. (Specker [8]). *If $\omega^2 \leqslant \alpha < \omega_1$ then $\alpha \to \omega^2$.*

LEMMA 5. *If α is an indecomposable ordinal and $\alpha \to \beta$, then $\alpha\omega \to \beta\omega$.*

Proof. Let $\alpha\omega = \bigcup_{n<\omega} A_n$, where $A_0 < A_1 < \ldots$ and $\operatorname{tp} A_n = \alpha$ for each $n < \omega$. Let $\beta\omega = \bigcup_{n<\omega} B_n$, where $B_0 < B_1 < \ldots$ and $\operatorname{tp} B_n = \beta$ for each $n < \omega$. For each $n < \omega$, there is a pinning map π_n: $A_n \to B_n$. Let $\pi = \bigcup_{n<\omega} \pi_n$. We claim that π: $\alpha\omega \to \beta\omega$ is a pinning map. Suppose $X \subset \alpha\omega$ and $\operatorname{tp} X = \alpha\omega$. Let $X_n = X \cap A_n$ and let $N = \{n < \omega\colon \operatorname{tp} X_n = \alpha\}$. Then N is infinite, since $\operatorname{tp} X = \alpha\omega$ and α is indecomposable. Since $\pi(X_n) = \pi_n(X_n)$ and π_n is a pinning map, we have $\pi(X_n) = \beta$ for all $n \in N$. So $\operatorname{tp} \pi(X) = \beta\omega$.

LEMMA 6. *If $\alpha < \omega_1$ and α is a decomposable limit ordinal, then $\omega^\alpha \to \omega^3$.*

Proof. Let $\alpha = \beta + \gamma$, where $\alpha > \beta \geqslant \gamma \geqslant \omega$. Then $\omega^\alpha = \omega^{\beta+\gamma} = \omega^\beta\omega^\gamma$. Let $\omega^\alpha = \bigcup_{\mu<\omega^\gamma} A_\mu$, where $\operatorname{tp} A_\mu = \omega^\beta$ and $A_\mu < A_\nu$ for $\mu < \nu < \omega^\gamma$. Let $\omega^3 = \bigcup_{\mu<\omega^2} B_\mu$, where $\operatorname{tp} B_\mu = \omega$ and $B_\mu < B_\nu$ for $\mu < \nu < \omega^2$. By Lemma 4, there is a pinning map ϱ: $\omega^\gamma \to \omega^2$. For each $\mu < \omega^\gamma$, let π_μ: $A_\mu \to B_{\varrho(\mu)}$ be a one-to-one function. Let $\pi = \bigcup_{\mu<\omega^\gamma} \pi_\mu$. We claim that π: $\omega^\alpha \to \omega^3$ is a pinning map.

Suppose $X \subset \omega^\alpha$ and $\operatorname{tp} X = \omega^\alpha$. For each $\mu < \omega^\gamma$, let $X_\mu = X \cap A_\mu$. Let $N = \{\mu < \omega^\gamma\colon X_\mu \text{ is infinite}\}$. Note that $\operatorname{tp} \bigcup_{\mu\notin N} X_\mu \leqslant \omega^\gamma < \omega^\alpha$. Since $\operatorname{tp} X = \omega^\alpha$ and ω^α is indecomposable, it follows that $\operatorname{tp} \bigcup_{\mu\in N} X_\mu = \omega^\alpha$, so $\operatorname{tp} N = \omega^\gamma$. Therefore, since ϱ is a pinning map, $\operatorname{tp} \varrho(N) = \omega^2$. For each $\mu \in N$, since $\pi(X_\mu) = \pi_\mu(X_\mu)$ and π_μ is one-to-one, $\pi(X_\mu)$ is an infinite subset of $B_{\varrho(\mu)}$. So $\operatorname{tp} \pi(X) \geqslant \omega \cdot \omega^2 = \omega^3$.

The following lemma will be used in the proof of Theorem 8.

LEMMA 7. *Given any ordinal $\alpha < \omega_1$, any limit ordinal $\beta > \omega$, and any function f: $\alpha^2 \to \beta$, there is a set $X \subset \alpha^2$ with $\operatorname{tp} X = \alpha$ and $\operatorname{tp} f(X) < \beta$.*

Proof. Let $\alpha^2 = \bigcup_{\mu<\alpha} A_\mu$, where $\operatorname{tp} A_\mu = \alpha$ for $\mu < \alpha$, and $A_\mu < A_\nu$ for $\mu < \nu < \alpha$. Let $\alpha = \{\nu_n\colon n < \omega\}$ be an enumeration of α. If $\operatorname{tp} f(A_\nu) < \beta$ for some ν, then $X = A_\nu$ works; so we can assume that $\operatorname{tp} f(A_\nu) = \beta$ for all $\nu < \alpha$. Then $f(A_\nu)$ is cofinal in β, which is a limit ordinal. Therefore, we can choose $x_n \in A_{\nu_n}$ for $n < \omega$, so that $f(x_0) < f(x_1) < \ldots$ Let $X = \{x_n\colon n < \omega\}$; then $\operatorname{tp} X = \alpha$ and $\operatorname{tp} f(X) = \omega < \beta$.

THEOREM 8. *If $\alpha < \omega_1$ and α is indecomposable, then $\omega^\alpha \not\to \omega^3$.*

Proof. The case $\alpha = 1$ is easy, so we assume $\alpha \geqslant \omega$. Then there is a sequence of ordinals $\beta(0) < \beta(1) < \ldots$ such that $\sup_{n<\omega} \beta(n) = \alpha = \sup_{n<\omega} \beta(n) \cdot 4$. Hence $\omega^\alpha = \sum_{n<\omega} \omega^{\beta(n)} = \sum_{n<\omega} \omega^{\beta(n)\cdot 4}$. Let $\omega^\alpha = \bigcup_{n<\omega} A(n)$, where $A(0) < A(1) < \ldots$, and $\operatorname{tp} A(n) = \omega^{\beta(n)\cdot 4}$ for each $n < \omega$. Let $\omega^3 = \bigcup_{n<\omega} B(n)$, where $B(0) < B(1) < \ldots$ and $\operatorname{tp} B(n) = \omega^2$ for each $n < \omega$.

Let a function $f\colon \omega^{\alpha} \to \omega^{3}$ be given. By Lemma 7, for each $n<\omega$, we can choose $A_0(n) \subset A(n)$ so that $\operatorname{tp} A_0(n) = \omega^{\beta(n)\cdot 2}$ and $\operatorname{tp} f\big(A_0(n)\big) < \omega^3$. Since $\omega^{\beta(n)\cdot 2}$ is indecomposable, we can choose $A_1(n) \subset A_0(n)$ so that $\operatorname{tp} A_1(n) = \omega^{\beta(n)\cdot 2}$ and $\operatorname{tp} f\big(A_1(n)\big) \leqslant \omega^2$. Repeating this argument, we obtain $A_2(n) \subset A_1(n)$ with $\operatorname{tp} A_2(n) = \omega^{\beta(n)}$ and $\operatorname{tp} f\big(A_2(n)\big) \leqslant \omega$. Finally, using the indecomposability of $\omega^{\beta(n)}$, we can choose $W(n) \subset A_2(n)$ so that $\operatorname{tp} W(n) = \omega^{\beta(n)}$ and, either $f\big(W(n)\big) \subset B(i)$ for some $i \leqslant n$, or else $f\big(W(n)\big) \subset \bigcup_{i>n} B(i)$.

Note that, for any infinite $N \subset \omega$, we have $\operatorname{tp} \bigcup_{n\in N} W(n) = \omega^{\alpha}$.

For $i<\omega$, let $N_i = \{n\colon f\big(W(n)\big) \subset B(i)\}$. Let $N_\omega = \{n\colon f\big(W(n)\big) \subset \bigcup_{i>n} B(i)\}$. Now we consider three cases.

Case 1. N_i is infinite for some $i<\omega$. Let $X = \bigcup_{n\in N_i} W(n)$; then $\operatorname{tp} X = \omega^{\alpha}$, and $\operatorname{tp} f(X) \leqslant \omega^2$ since $f(X) \subseteq B(i)$.

Case 2. $N_i \neq \emptyset$ for infinitely many $i<\omega$. Let $I = \{i<\omega\colon N_i \neq \emptyset\}$. For each $i \in I$, choose $n_i \in N_i$. Let $X = \bigcup_{i\in I} W(n_i)$. Then $\operatorname{tp} X = \omega^{\alpha}$; and $\operatorname{tp} f(X) \leqslant \omega^2$, since $\operatorname{tp} f(X) \cap B(i) \leqslant \omega$ for each $i<\omega$.

Case 3. N_ω is infinite. Let $X = \bigcup_{n\in N_\omega} W(n)$; then $\operatorname{tp} X = \omega^{\alpha}$. For each $i<\omega$, we have $f(X) \cap B(i) \subset \bigcup_{n<i} f\big(W(n)\big)$; hence $\operatorname{tp} f(X) \cap B(i) < \omega^2$ for each $i<\omega$; hence $\operatorname{tp} f(X) \leqslant \omega^2$.

Finally, we give an application to the partition calculus.

THEOREM 9. *If* $\alpha < \omega_1$ *and* $\alpha \to (\alpha, 3)^2$, *then, either* $\alpha \in \{0, 1, \omega^2\}$, *or else* $\alpha = \omega^{\omega^{\beta}}$ *for some* $\beta < \omega_1$.

Proof. Clearly α cannot be decomposable. Hence, either $\alpha = 0$, or $\alpha = \omega^0 = 1$, or $\alpha = \omega^1 = \omega^{\omega^0}$, or $\alpha = \omega^2$, or $\alpha = \omega^{\varepsilon}$ where $3 \leqslant \varepsilon < \omega_1$. Suppose $\alpha = \omega^{\varepsilon}$, $3 \leqslant \varepsilon < \omega_1$. By the results of Specker [8], α cannot be pinned to ω^3. It follows by Theorem 3 that ε is indecomposable, i.e., $\varepsilon = \omega^{\beta}$ for some $\beta < \omega_1$; so $\alpha = \omega^{\omega^{\beta}}$.

References

[1] C. C. Chang, *A partition theorem for the complete graph on* ω^{ω}, J. Combinatorial Theory (A) 12 (1972), pp. 396–452.

[2] P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. 62 (1956), pp. 427–489.

[3] J. Larson, *A short proof of a partition theorem for the ordinal* ω^{ω}, Ann. Math. Logic 6 (1973), pp. 129–145.

[4] — *On some arrow relations*, Ph. D. Thesis, Dartmouth College, Hanover, New Hampshire, 1972.

[5] E. C. Milner and R. Rado, *The pigeon-hole principle for ordinal numbers*, Proc. London Math. Soc. 15 (1965), pp. 750–768.

[6] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1930), pp. 264–286.

[7] B. Rotman, *A mapping theorem for countable well-ordered sets*, J. London Math. Soc. 2 (1970), pp. 509–512.

[8] E. Specker, *Teilmengen von Mengen mit Relationen*, Comment. Math. Helv. 31 (1957), pp. 302–314.

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