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Relations on lines as primitive notions for Euclidean geometry

by

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Abstract. Euclidean geometry is usually treated as a theory of relations on points. It is proved that this geometry may be formulated as a theory of relations on lines. The simplest systems of primitive notions for dimension-free, two-dimensional and n-dimensional (with $n > 3$) geometries are given. For $n = 3$ the problem of defining such a system remains open.

The possible systems of primitive notions for geometry were studied by Royden [5], Beth and Tarski [1], Scott [6]. Some negative results are contained in Robinson [4] and Tarski [8]. In all these systems variables are ranging over points, i.e. geometry is treated as a theory of structures with universa consisting of points only. In Menger [3] two universa are used: that of points and an additional one of lines. In Tarski [7] elements of the universum are open discs. In this paper we are concerned with possibilities of taking as the universum the set of lines only and relations on them as primitive notions ([1]).

We shall prove that the binary relation of perpendicularity together with the ternary relation of copunctuality may be used as a system of primitive notions for dimension-free elementary Euclidean geometry. Perpendicularity alone suffices for all dimensions higher than three. In dimension two, a ternary relation is essential: there is no system of primitives for plane Euclidean geometry consisting of binary relations only. In the three-dimensional case, we know that perpendicularity together with the binary relation of intersection lines may be used as system of primitives. However, we do not know whether the notion of intersection of lines is superfluous or not.

We do not know whether exists at all a binary line-relation, which may be used as the only primitive notion for space Euclidean geometry.

([1]) Most of the results have been obtained when both authors attended the Meeting on Foundations of Geometry in Oberwolfach 15–21. 7. 1973. We wish to express our thanks to the organisers of that meeting.

§ 1. Preliminaries. Let $\mathfrak{F} = \langle F, 0, 1, +, \cdot, \leqslant \rangle$ be an ordered commutative field. We assume that the field $\mathfrak{F}$ is Euclidean, i.e. that for any $\alpha \in F$, if $0 \leqslant \alpha$ then there exists a $\beta \in F$ such that $\alpha = \beta^2$ and $0 \leqslant \beta$. The element β will be denoted as usual by $\sqrt{\alpha}$. Later on, when referring to a field, we shall always mean an ordered commutative Euclidean field $\mathfrak{F}$.

The elements of F^n will be called *points* and denoted by small Latin letters a, b, c, d, x, y, z. As is well known, F^n with appropriate operations forms an n-dimensional linear space. We shall denote the sum of a and b by $a \oplus b$, the difference of a and b by $a \ominus b$, the inner product of a and b by $a \odot b$ and the product of α and b by αb. As usual, we shall write a^2 instead of $a \odot a$. By the norm of a we mean $|a| = \sqrt{a^2} \in F$.

The *n-dimensional Cartesian* space over $\mathfrak{F}$ is the structure

(1) $$\mathfrak{C}^n_{\mathfrak{F}} = \langle F^n_{\mathfrak{F}}, B^n_{\mathfrak{F}}, D^n_{\mathfrak{F}} \rangle$$

where

$$B^n_{\mathfrak{F}}(abc) \leftrightarrow |a \ominus b| + |b \ominus c| = |a \ominus c|$$

and

$$D^n_{\mathfrak{F}}(abcd) \leftrightarrow |a \ominus b| = |c \ominus d|\,.$$

By a line in $\mathfrak{C}^n_{\mathfrak{F}}$ we mean any set of points

(2) $$L(ab) = \{a \oplus \xi(a \ominus b) \colon \ \xi \in F\}$$

where $a \neq b$. The point $a \ominus b$ is called the *direction vector of the line* $L(ab)$. Lines will be denoted by capital Latin letters K, L, M, N. The set of all lines in $\mathfrak{C}^n_{\mathfrak{F}}$ will be denoted by $L^n_{\mathfrak{F}}$.

Let $\mathfrak{P} = \langle P_0, \ldots, P_p \rangle$ be a sequence of relations on $L^n_{\mathfrak{F}}$. We put $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P}) = \langle L^n_{\mathfrak{F}}, P_0, \ldots, P_p \rangle$.

By a geometrical notion we mean either a point-geometrical notion, i.e. a function assigning to a field $\mathfrak{F}$ and a number $n \geqslant 2$ a relation on F^n, or a line-geometrical notion, i.e. a function assigning to a field $\mathfrak{F}$ and a number $n \geqslant 2$ a relation on $L^n_{\mathfrak{F}}$. Thus, for example B, is the point-geometrical notion of betweenness, and $B^n_{\mathfrak{F}}$ is the betweenness relation in $\mathfrak{C}^n_{\mathfrak{F}}$. We shall usually omit subscripts $\mathfrak{F}$ and superscripts n. This should not cause any misunderstanding. We shall write $\equiv$ instead of D.

We shall use several line-geometrical notions defined as follows [2]:

[2] In the following we put a dot over a symbol of a geometrical notion if it requires that lines have a point in common. In case a notion requires lines to be disjoint, we put the sign ˇ over the symbol. For an arbitrary binary relation P we use the following abbreviations:

We often write xPy instead of $P(xy)$.

We write $x_0x_1 \ldots x_m P y_0 y_1 \ldots y_n$ whenever $x_i P y_j$ for all $i \leqslant m$ and $j \leqslant n$.

We write $P(x_0x_1 \ldots x_n)$ whenever $x_i P x_j$ for all $i, j \leqslant n$ with $i \neq j$. We use the same symbols to denote mathematical objects and their symbols.

Lines $L(ab)$ and $L(a'b')$ are weakly perpendicular:

(3) $$L(ab) \perp L(a'b') \leftrightarrow (a \ominus b) \odot (a' \ominus b') = 0\,.$$

Lines K, L are (strongly) perpendicular:

(4) $$K \dot{\perp} L \leftrightarrow K \perp L \wedge \exists a \ \ a \in K \cap L\,.$$

Lines $L(ab)$, $L(cd)$ are parallel:

(5) $$L(ab) \| L(cd) \leftrightarrow \exists \alpha\beta\,[\alpha^2 + \beta^2 \neq 0 \wedge \alpha(a \ominus b) = \beta(c \ominus d)]\,,$$

in other words, two lines are parallel iff their direction vectors are linearly dependent.

Lines K, L are strongly parallel:

(6) $$K \check{\|} L \leftrightarrow K \| L \wedge K \neq L\,.$$

Lines K, L intersect each other:

(7) $$K \dot{\times} L \leftrightarrow \exists a [\{a\} = K \cap L]\,.$$

Lines K, L, M are co-punctual:

(8) $$p(KLM) \leftrightarrow \exists a \ [a \in K \cap L \cap M]\,.$$

Lines K, L, M are co-planar (there is a two-dimensional hyperplane in $\mathfrak{C}^n_{\mathfrak{F}}$ containing all of them):

(9) $$P(KLM) \leftrightarrow \exists abc\,[K, L, M \subseteq \{a \oplus \alpha b \oplus \beta c \colon \ \alpha, \beta \in F\}]\,.$$

Let us restrict ourselves to $\mathfrak{C}^n_{\mathfrak{R}}$ — the n-dimensional Cartesian space over the field of real numbers. We assume it is known what the measure of an angle between two lines (not necessarily intersecting) means. We shall write $K \dot{\times}_\alpha L$ if the lines K and L do intersect and the measure of the angle between them is α, analogously $K \check{\times}_\alpha L$ if they do not intersect and the measure of the angle between them is α. It is easy to see that

(10) $$K \dot{\perp} L \leftrightarrow K \dot{\times}_{\pi/2} L\,,$$

(11) $$K \perp L \leftrightarrow K \dot{\times}_{\pi/2} L \vee K \check{\times}_{\pi/2} L\,.$$

We shall use point-geometrical notions:

Points a, b, c are co-linear:

(12) $$L(abc) \leftrightarrow \exists K\,[a, b, c \in K]\,.$$

Points a, b, c form a right angle:

(13) $$\llcorner(abc) \leftrightarrow \exists KL\ [K\perp L \wedge a, b \in K \wedge b, c \in L]\,.$$

By the n-dimensional Euclidean geometry $\mathbf{E}^n$ we mean the non-elementary theory of $\mathfrak{C}^n_{\mathfrak{R}}$. We shall write $\mathbf{E}^{n\leqslant}$ for the common part of all $\mathbf{E}^m$ with $n \leqslant m$. By $\mathbf{EE}^n$ we mean the elementary theory of all $\mathfrak{C}^n_{\mathfrak{F}}$ where F is any ordered, commutative Euclidean field. We shall write $\mathbf{EE}^{n\leqslant}$ for the common part of all $\mathbf{EE}^m$ with $n \leqslant m$.

A set of points is a line iff there are two different points a and b such that it is of the form $L(ab)$. Thus instead of lines we may speak of pairs of different points, and of relations on such pairs instead of line-relations. A line L may be determined by different pairs of points; thus in such an approach we have to treat some pairs as equal, namely

$$\langle ab\rangle =' \langle a'b'\rangle \leftrightarrow L(aba') \wedge L(abb')\,.$$

In other words there is a one-to-one correspondence between lines and elements of the quotient set $S'/_{='}$ where

$$S' = \{\langle ab\rangle :\ a \neq b\}\,.$$

This approach is especially convenient to describe the notion of a definable line-relation (line-notion). We say that the sequence $\mathfrak{P}$ of line-notions $P_0, \ldots, P_p$ is *definable in* $\mathbf{EE}^n$ if and only if there are the formulas $\Phi_0, \ldots, \Phi_p$ formulated in terms of B and D such that for any field $\mathfrak{F}$ we have

(14) $$\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P}) \cong \langle (F^n)^2, P'_0, \ldots, P'_p\rangle \upharpoonright S'/_{='}$$

where $P'_0, \ldots, P'_p$ are relations defined in $\mathfrak{C}^n_{\mathfrak{F}}$ by the formulas $\Phi_0, \ldots, \Phi_p$. We say that the sequence $\mathfrak{P}$ is *definable in* $\mathbf{E}^n$ if (14) holds at least for $\mathfrak{F} = \mathfrak{R}$. We say that it is *definable in* $\mathbf{EE}^{m\leqslant}$ (respectively $\mathbf{E}^{m\leqslant}$) if (14) holds for every $n \geqslant m$.

On the other hand, we say that the notions of betweenness and equidistance are *definable in a structure* (*a class of structures*) $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P})$ if there is a natural number k and formulas $\Phi_B, \Phi_D, \Psi, \Xi$ from the language of $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P})$ such that

$$\mathfrak{C}^n_{\mathfrak{F}} \cong \langle (L^n_{\mathfrak{F}})^k, B', D'\rangle \upharpoonright S'/_{='}$$

where B' and D' are relations defined in $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P})$ by formulas Φ_B and Φ_D, $S' \subseteq (L^n_{\mathfrak{F}})^k$ is defined in $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P})$ by Ψ, and $='$ is the equivalence relation defined by Ξ.

We say that a sequence of line-notions $\mathfrak{P}$ is a *system of primitives for* $\mathbf{EE}^n$ ($\mathbf{E}^n$, $\mathbf{EE}^{m\leqslant}$, $\mathbf{E}^{m\leqslant}$) if $\mathfrak{P}$ is definable in $\mathbf{EE}^n$ ($\mathbf{E}^n$, $\mathbf{EE}^{m\leqslant}$, $\mathbf{E}^{m\leqslant}$) and moreover the notions of betweenness and equidistance are definable in an appropriate class of structures of the form $\mathfrak{L}^n_{\mathfrak{F}}(\mathfrak{P})$.

§ 2. Possible systems of primitives. We shall use the following two lemmas:

LEMMA 2.1. *The line-geometrical notions*

$$\perp,\ \dot{\perp},\ \dot{\times},\ p$$

are definable in $\mathbf{EE}^{2\leqslant}$.

Proof. It will be enough to indicate formulas $\Phi_{\perp}, \Phi_{\dot{\perp}}, \Phi_{\dot{\times}}$ and Φ_p such that isomorphism (14) holds for all Euclidean fields and all $n \geqslant 2$. The required formulas are provided by the following equivalences:

$$L(abc) \leftrightarrow B(abc) \vee B(bca) \vee B(cab)\,,$$

$$\llcorner(abc) \leftrightarrow \exists x\, B(xbc) \wedge xb \equiv bc \wedge xa \equiv ac\,,$$

$$\begin{aligned} L(ab)\perp L(cd) \leftrightarrow \exists xy\ [&L(abx) \wedge L(cdy) \wedge \llcorner(axy) \wedge \llcorner(axc) \wedge \llcorner(axd) \wedge \\ &\wedge \llcorner(bxy) \wedge \llcorner(bxc) \wedge \llcorner(bxd) \wedge \llcorner(cxy) \wedge \llcorner(cya) \wedge \llcorner(cyb) \wedge \\ &\wedge \llcorner(dyx) \wedge \llcorner(dya) \wedge \llcorner(dyb) \wedge a \neq b \wedge c \neq d] \end{aligned}$$

(see Fig. 1),

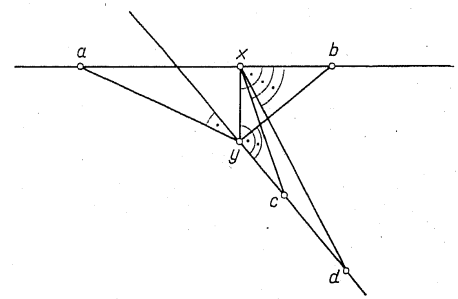

Fig. 1

$$\begin{aligned} L(ab)\,\dot{\perp}\,L(cd) \leftrightarrow \exists x\ [&L(abx) \wedge L(cdx) \wedge \llcorner(axc) \wedge \llcorner(axd) \wedge \llcorner(bxc) \wedge \\ &\wedge \llcorner(bxd) \wedge a \neq b \wedge c \neq d]\,, \end{aligned}$$

$$L(ab)\,\dot{\times}\,L(cd) \leftrightarrow \exists!x\ [L(abx) \wedge L(cdx)]\,,$$

$$p\big(L(ab)L(a'b')L(a''b'')\big) \leftrightarrow \exists x\ [L(abx) \wedge L(a'b'x) \wedge L(a''b''x)]\,.$$

This completes the proof.

LEMMA 2.2. *The point notions of collinearity L and right angle $\llcorner$ may be used as primitive notions for* $\mathbf{EE}^2$.

Proof. It will suffice to give the following equivalences (the first one is due to Jenks [2]):

$$B(abc) \leftrightarrow \exists x\,[\llcorner(axc) \wedge \llcorner(abx) \wedge \llcorner(cbx) \wedge L(abc)]$$

(see Fig. 2),

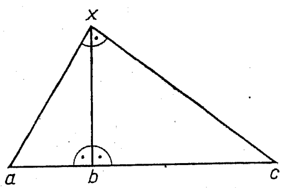

Fig. 2

$$ab \equiv'' ac \leftrightarrow \exists xy\;\; b \neq c \wedge \llcorner(abx) \wedge \llcorner(acx) \wedge \llcorner(ayb) \wedge L(ayx) \wedge L(byc)$$

(see Fig. 3),

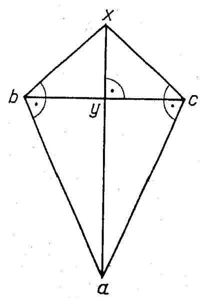

Fig. 3

$$ab \equiv' ac \leftrightarrow \exists x\,[ab \equiv'' ax \wedge ac \equiv'' ax]\,.$$

$$M(abc) \leftrightarrow (a = b \vee a \neq c) \wedge ba \equiv' bc \wedge L(abc)$$

(midpoint relation),

$$ab \equiv cd \leftrightarrow \exists xy\,[M(axy) \wedge M(bxc) \wedge cd \equiv' cy]$$

(see Fig. 4). This completes the proof.

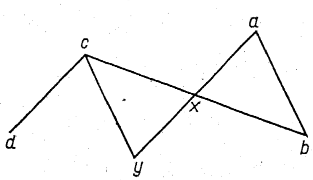

Fig. 4

THEOREM 2.3. *It is possible to use perpendicularity* $\dot{\perp}$ *and co-punctuality* p *as a system of primitives for* $\mathsf{EE}^{2\leqslant}$.

Proof. Because of Lemma 2.1 it will suffice to prove that the point notions of collinearity and right angle are definable in the theory of all structures $\mathfrak{L}^n_{\mathfrak{F}}(\dot{\perp}, p)$ with any field $\mathfrak{F}$ and $n \geqslant 2$. We put $k = 2$. Thus we shall consider pairs of lines, in fact not all pairs, but only pairs of different intersecting lines:

$$S' = \{\langle K, L\rangle : K \neq L \wedge p(KKL)\}\,.$$

We shall identify all pairs of lines having a common intersection point:

$$\langle KL\rangle =' \langle MN\rangle \leftrightarrow p(KLM) \wedge p(KLN)\,.$$

In other words, by a "point" we mean, loosely speaking, pencils of co-punctual lines. We shall denote these points by small Latin letters, just like points in $\mathfrak{C}^n_{\mathfrak{F}}$. Thus it may happen that for some L we have $\langle KL\rangle \in a$. This is to be understood as "the point a is lying on the line K" and will be denoted by $a \in' K$. Let

$$L(abc) \leftrightarrow \exists K\,[a, b, c \in' K]$$

(points a, b, c are collinear)

$$\llcorner(abc) \leftrightarrow \exists KL\,[K \dot{\perp} L \wedge ab \in' K \wedge bc \in' L]$$

(points a, b, c form a right angle). This by Lemma 2.2 proves the theorem.

THEOREM 2.4. *The pair of line-notions* $\dot{\perp}, \dot{\times}$ *may be used as a system of primitives for* $\mathsf{EE}^{3\leqslant}$.

Proof. Because of Lemma 2.1 and Theorem 2.3, it will suffice to define the ternary relation of co-punctuality p. First we shall define the notion of co-planarity in terms of perpendicularity and intersection:

$$P(KLM) \leftrightarrow \exists NN'N''\,[\dot{\perp}(NN'N'') \wedge KLM \dot{\times} N'N'' \wedge \wedge \neg K \dot{\times} N \wedge \neg L \dot{\times} N \wedge \neg M \dot{\times} N \wedge KLM \neq N],$$

and then the notion of co-punctuality:

$$p(KLM) \leftrightarrow K = L = M \vee (K = L \wedge L \dot{\times} M) \vee (K = M \wedge M \dot{\times} L) \vee \vee (L = M \wedge M \dot{\times} K) \vee \{\neq(KLM) \wedge [P(KLM) \rightarrow \exists N\,(KLM \dot{\perp} N)]\}\,.$$

This completes the proof of the theorem.

THEOREM 2.5. *The notion of perpendicularity* $\dot{\perp}$ *may be used as the only primitive notion for* $\mathsf{EE}^{4\leqslant}$.

Proof. Because of Theorem 2.4 it suffices to give a definition of $\dot{\times}$ in terms of $\dot{\perp}$:

$$K \dot{\times} L \leftrightarrow \exists MN\;\,[KL \dot{\perp} MN \wedge M \dot{\perp} N \wedge K \neq L]\,.$$

3. Negative results. Now we intend to prove some results on the impossibility of taking some sequences of notions as systems of primitives for certain geometries. In fact, we shall prove these theorems for E^n only, since this implies analogous results for $\mathsf{E}^{n\leqslant}$, EE^n and $\mathsf{EE}^{n\leqslant}$. To prove these theorems, we first have to give a classification of all binary relations on lines in $\mathfrak{C}^n_{\mathfrak{R}}$.

Let P be any binary relation (in particular the identity relation); we put

$$P^{(0)} = \emptyset\,, \qquad P^{(1)} = P\,.$$

Thus, e.g. $K =^{(1)} K$ always holds and $K =^{(0)} L$ never holds.

In $\mathfrak{C}^2_{\mathfrak{R}}$, all definable binary line-relations are of the form P_{ijA} where $i, j = 0, 1$, $A \subseteq (0, \pi/2\rangle \subseteq R$ and

$$P_{ijA}(KL) \leftrightarrow [K =^{(i)} L] \vee [K \dot{\parallel}^{(j)} L] \vee \exists\alpha[\alpha \in A \wedge K \dot{\times}_\alpha L]\,. \tag{15}$$

In fact, let P be any binary definable relation. Since it is definable, it has to be closed under similarity transformations, i.e., if $P(KL)$ holds and σ is a similarity transformation of $\mathfrak{C}^2_{\mathfrak{R}}$, then $P(\sigma K \sigma L)$ has to hold. Let $P(KK)$ for some line K. For any line L, there is a similarity transformation mapping K on L, $P(LL)$ holds for any line L. Similarly, we may prove that if the relation P holds for two distinct parallel lines, it holds for any two distinct parallel lines, and that if P holds for two lines intersecting at the angle α, then P contains $\dot{\times}_\alpha$.

Employing similar methods, we may prove that for all dimensions higher than 2, all definable binary line-relations are of the form P_{ijAB} where $i, j = 0, 1$, $A, B \subseteq (0, \pi/2\rangle \subseteq R$ and

$$P_{ijAB}(KL) \leftrightarrow [K =^{(i)} L] \vee [K \dot{\parallel}^{(j)} L] \vee \exists\alpha[\alpha \in A \wedge K \dot{\times}_\alpha L] \vee \vee \exists_\beta[\beta \in B \wedge K \dot{\times}_\beta L]\,. \tag{16}$$

THEOREM 3.1. *There is no system of primitives for E^2 consisting of binary line-relations only.*

Proof. Let K_0 be a fixed line and σ any one-to-one mapping such that $K \parallel \sigma K$ for each K and $\sigma K_0 \neq K_0$ (e.g. a translation). We put

$$\varphi K = \begin{cases} \sigma K & \text{if} \quad K \parallel K_0\,, \\ K & \text{otherwise}\,. \end{cases}$$

It is easy to see that the mapping preserves all definable binary line-relations (see (15)) but does not preserve co-punctuality. Thus, by Padoa's method, the notion of co-punctuality is not definable in terms of definable binary line-relations.

By an analogous argument, for $n \geqslant 3$, copunctuality is not definable in terms of the relations P_{ijAA} (i.e. P_{ijAB} for $A = B$). As a corollary we get

THEOREM 3.2. *For any $n \geqslant 2$, it is not possible to use $\langle \dot{\perp}, \parallel \rangle$ as a system of primitives for E^n.*

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