Non-extensional equality

by

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Abstract. Intuitionistically, two mathematical objects are said to be *intensionally equal*, if they are given to us as the same object (i.e. are identical). Extensional equality (as in classical set theory) is a much coarser equivalence relation, and there are many equivalence relations (notions of non-extensional equality) intermediate between intensional equality. This paper discusses non-extensional notions of equality and their uses in proof theory of intuitionistic formal systems and foundational discussions.

§ 1. Introduction

1.1. The purpose of this paper is to present a rather detailed discussion of non-extensional concepts of equality, which arise quite naturally in the discussion of the foundations of constructive mathematics. The discussion is mostly self-contained, appart from the proofs of the technical results quoted, for which the reader should consult the references.

For definiteness, the discussion may be supposed to be in terms of intuitionistic concepts, although most of the discussion is applicable in a much wider context.

There is no deliberate attempt to say anything new on the subject; we only try to explore the concept and its possible uses from different angles, and to correct erroneous views concerning the notion of intensional equality.

I am indebted to G. Kreisel for detailed criticism of earlier versions of this paper.

1.2. Intensional equality may be considered as the extreme, basic form of non-extensional equality: two objects are intensionally equal, if they are *given to us* as the same object. (When interpreted literally, the preceding description of "intensionally equal" is almost tautologous, it seems to define "intensionally equal" by "same". Note however, that the stress is on "given to us". The aim is to *convey* to the reader the intended basic concept.)

In standard intuitionistic terminology "a is intensionally equal to b" is equivalent to saying "a and b are the same mental construction". (The concept of sameness as intended here involves of course already a certain abstraction and idealization: the natural number 3 constructed today "is the same as" the natural number 3 constructed tomorrow.)

A typical example is the following: the binary functions $+$, $+'$ on the natural numbers, defined by the recursion equations (S for successor, x, y numerical variables)

$$\begin{cases} x+0 = x\,, \\ x+\mathrm{S}y = \mathrm{S}(x+y)\,, \end{cases} \qquad \begin{cases} 0+'y = y\,, \\ \mathrm{S}x+'y = \mathrm{S}(x+'y)\,, \end{cases}$$

represent extensionally the same function, but are given to us by distinct sets of equations, corresponding to distinct computation procedures, hence not intensionally equal.

Since "intensional equality" is a concept of the same degree of generality as "proposition", "property" etc. we can say very little about it, but two aspects seem to be obvious: 1^0 intensional equality is decidable, and 2^0 the decision should be immediate, i.e. should not require a non-trivial mathematical argument.

1.3. Before we proceed, the following remarks are in order. It is not to be expected that intensional equality as such is mathematically manageable, or that it possesses mathematical interest. The very fact that considering objects from a strictly intensional point of view means carrying along *all* information about these objects, that is we are not permitted to abstract from any of their properties, clearly indicates that, in general, strict intensional equality is not mathematically useful. Part of the success of set theory is in the fact that we often need consider *extensions* of properties only.

On the other hand, thorough-going extensional equality is usually too "coarse" from a constructive point of view, as illustrated by the example at the end of this section. So it is to be expected that in constructive mathematics there attaches a certain interest to concepts of equality intermediate between strict intensional equality and extensional equality.

The "applications" of non-extensional concepts of equality can be divided into two categories:

a) the use of the concept of intensional equality in foundational discussions, e.g. in establishing the validity of axioms for informally given concepts, and

b) mathematical uses of non-extensional concepts, e.g. in proof theory.

Let us illustrate how the need for a non-extensional concept of equality appears naturally in recursion theory. Let W_x denote the recursively enumerable set with gödelnumber x. It is a well-known theorem that there exist primitive recursive functions φ, ψ such that

$$(1)\qquad W_x \cup W_y = W_{\varphi(x,y)} \cup W_{\psi(x,y)}\ \& \\ W_x \subseteq W_{\varphi(x,y)}\ \&\ W_y \subseteq W_{\psi(x,y)}\ \&\ W_{\varphi(x,y)} \cap W_{\psi(x,y)} = \emptyset\,.$$

If we define intensionally $W_x =_i W_y \equiv_{\mathrm{def}} x = y$, then φ, ψ may be regarded as effective mappings. But there are no φ, ψ satisfying (1) with respect to extensional equality, i.e. we cannot satisfy

$$W_x = W_{x'}\ \&\ W_y = W_{y'} \rightarrow W_{\varphi(x,y)} = W_{\varphi(x',y')}\ \&\ W_{\psi(x,y)} = W_{\psi(x',y')}\,.$$

If we would permit ourselves to talk about extensional operations on r.e. sets only, we could not have effective versions of the theorem.

§ 2. Hereditarily recursive operations HRO

2.1. To make the discussion more concrete, we shall now discuss intuitionistic arithmetic in all finite types and the model of the hereditarily recursive operations HRO.

We first describe a "neutral" theory **N-HA$^\omega$** ("neutral" because it permits extensional as well as intensional interpretations of equality at higher types).

The basic type structure $\mathbf{T}$ is given by $0 \in \mathbf{T}$; $\sigma, \tau \in \mathbf{T} \Rightarrow (\sigma)\tau \in \mathbf{T}$ (0 is the type of the natural numbers, $(\sigma)\tau$ is a collection of mappings of objects of type σ to objects of type τ).

We have variables in each type: $x^\sigma, y^\sigma, z^\sigma, u^\sigma, v^\sigma, w^\sigma$. Constants are 0 (of type 0), S (successor), $\Pi_{\sigma,\tau} \in (\sigma)(\tau)\sigma$ (projection), $\Sigma_{\varrho,\sigma,\tau} \in \big((\varrho)(\sigma)\tau\big)\big((\varrho)\sigma\big)(\varrho)\tau$ (substitution), $\mathrm{R}_\sigma \in (\sigma)\big((\sigma)(0)\sigma\big)(0)\sigma$ (recursor), and $\mathrm{App}_{\sigma,\tau}$ (application of an object of type $(\sigma)\tau$ to an object of type σ); there are constants $=_\sigma$ for equality at type σ. Application is indicated by juxtaposition, $t_1 t_2 \ldots t_n$ abbreviates $\big(\ldots\big((t_1 t_2)t_3\big)\ldots\big)t_n$. Terms, prime formulae and formulae are defined as usual; the logical operators are $\forall$, $\exists$, &, $\vee$, $\rightarrow$ ($\neg A$ is defined as $A \rightarrow 1 = 0$).

N-HA$^\omega$ is axiomatized by intuitionistic predicate logic, the usual axioms for successor, induction, defining axioms for the constants Π, Σ, R, and for equality, besides symmetry, reflexivity and transitivity, also substitutivity:

$$x^\sigma = y^\sigma \rightarrow z^{(\sigma)\tau} x^\sigma = z^{(\sigma)\tau} y^\sigma\,,$$

$$x^{(\sigma)\tau} = y^{(\sigma)\tau} \rightarrow x^{(\sigma)\tau} z^\sigma = y^{(\sigma)\tau} z^\sigma\,.$$

We shall usually omit type sub- and superscripts in the sequel; types will be assumed to be "coherent".

The defining axioms for the constants are specified as

$$\Pi xy = x\,, \quad \Sigma xyz = xz(yz)\,, \quad \mathrm{R}xy0 = x\,, \quad \mathrm{R}xy(\mathrm{S}z) = y(\mathrm{R}xyz)z\,.$$

2.2. This theory is made into an intensional theory $\mathbf{I\text{-}HA}^\omega$ by adding a constant for equality $\mathrm{E}_\sigma \in (\sigma)(\sigma)0$ satisfying

$$\mathrm{E}xy = 0 \leftrightarrow x = y\,, \qquad \mathrm{E}xy \leqslant 1\,.$$

The extensional theory $\mathbf{E\text{-}HA}^\omega$ is obtained by defining extensional equality $=_e$ inductively:

$$x^{(\sigma)\tau} =_e y^{(\sigma)\tau} \leftrightarrow \forall z\,(xz =_e yz)$$

and adding to $\mathbf{N\text{-}HA}^\omega$ $x = y \leftrightarrow x =_e y$.

Alternatively, we may restrict ourselves to equality at type 0 as a primitive and treat $=_\delta$ as defined.

2.3. A model HRO (Hereditarily Recursive Operations) for $\mathbf{I\text{-}HA}^\omega$ is defined as follows. We define, for each $\sigma \in \mathbf{T}$, an arithmetical predicate V_σ:

$$V_0(x) \equiv_{\mathrm{def}} x = x\,, \qquad V_{(\sigma)\tau}(x) \equiv_{\mathrm{def}} \forall y \in V_\sigma \exists z \in V_\tau\big(\{x\}(y) \simeq z\big)$$

($\{\cdot\}\cdot$, $\simeq$ as in [2]). Our objects of type σ are then the pairs (x, σ), where $x \in V_\sigma$. Application is essentially partial recursive function application:

$$\big(x, (\sigma)\tau\big)(y, \sigma) = \big(\{x\}(y), \tau\big)\,.$$

By elementary recursion theory, we can find numerals [S], $[\Pi_{\sigma,\tau}]$, $[\Sigma_{\varrho,\sigma,\tau}]$, $[\mathrm{R}_\sigma]$, $[\mathrm{E}_\sigma]$ such that $\big([\mathrm{S}], (0)0\big)$, $\big([\Pi_{\sigma,\tau}], (\sigma)(\tau)\sigma\big)$, $\Big([\Sigma_{\varrho,\sigma,\tau}], \big((\varrho)(\sigma)\tau\big)\big((\varrho)\sigma\big)(\varrho)\tau\Big)$, $\Big([\mathrm{R}_\sigma], (\sigma)\big((\sigma)(0)\sigma\big)(0)\sigma\Big)$, $\big([\mathrm{E}_\sigma], (\sigma)(\sigma)0\big)$ represent S, $\Pi_{\sigma,\tau}$, $\Sigma_{\varrho,\sigma,\tau}$, R_σ, E_σ in the model. Equality is interpreted as (essentially) equality between gödelnumbers: $(x, \sigma) = (y, \sigma) \equiv_{\mathrm{def}} x = y$.

2.4. The model illustrates quite clearly that

(A) It makes perfectly good sense to think about a concept of *decidable* equality with intensional character, at all finite types.

(B) It is by no means necessary that every object should have a *name* (description) in the language under consideration, for the concept of intensional equality to make good sense: each V_σ $(\sigma \neq 0)$ contains many elements not corresponding to closed terms in the model. This also refutes the hypothesis that intensional equality is really a syntactical notion, or that there is a confusion between "use" and "mention" if intensional equality is introduced in the language.

It should be noted that the equality in HRO is not intensional equality in the *strict* sense: an object of type σ is *given* to us by a *proof* that $x \in V_\sigma$ for some x. We have, in defining $=_\sigma$ in the model, abstracted from the proof and only taken x into account. Thus $=_\sigma$ as defined in our model only approximates intensional equality. (For a further discussion of this point see § 7.)

It is also not necessary to restrict the attention to mechanically computable functions such as the partial recursive functions in HRO; one might just as well introduce a model HHA of hereditarily hyperarithmetic operations.

2.5. Since, by coding types as numbers, HRO is in fact definable in intuitionistic arithmetic **HA**, and since formulae of $\mathbf{HA} \subseteq \mathbf{I\text{-}HA}^\omega$ are interpreted in HRO by themselves (modulo logical equivalence), it is obvious that $\mathbf{I\text{-}HA}^\omega$ is a conservative extension of **HA**.

2.6. Let us define a term of $\mathbf{N\text{-}HA}^\omega$ to be normal, if it does not contain subterms of the form $\Pi t_1 t_2$, $\Sigma t_1 t_2 t_3$, $\mathrm{R} t_1 t_2 0$, $\mathrm{R} t_1 t_2 (\mathrm{S} t_3)$. It can be shown that in $\mathbf{N\text{-}HA}^\omega$ each term is equal to a normal term (e.g. [15] § 2.2). Taking additional care in the choice of numbers for [S], $[\Pi_{\sigma,\tau}]$, $[\Sigma_{\varrho,\sigma,\tau}]$, $[\mathrm{R}_\sigma]$, it can be achieved that closed terms in distinct normal forms are represented by distinct elements in HRO. In fact, the axioms IE, listed in 4.2 can be satisfied.

2.7. A quite interesting property of HRO is given by the following theorem ([13] 8.5 and § 7 below): It is consistent relative **HA** to assume that HRO satisfies the axiom of choice $\mathrm{AC}_{\sigma,\tau}$ for all σ, τ, where

$$\mathrm{AC}_{\sigma,\tau} \qquad \forall x^\sigma \exists y^\tau A(x, y) \to \exists z^{(\sigma)\tau} \forall x^\sigma A(x, zx).$$

This contrasts with the properties of the term models, discussed in § 4 below, where even $\mathrm{AC}_{0,0}$ does not hold.

§ 3. Some applications to proof theory

3.1. In this section we present some examples of proof-theoretic results which are established quite naturally with the help of HRO, its variants and analogues, in cases where the corresponding extensional models (such as the hereditarily effective operations HEO and its analogues, cf. [13] § 7 or [15] 3.6.15) would not do. We have selected examples of results not referring in their statement to a concept of non-extensional equality.

3.2. $\mathbf{HA} + \mathrm{IP} + \mathrm{CT}_0$ is conservative over **HA** with respect to negative formulae (i.e. formulae in the $\forall$, &, $\to$ fragment) (See [13] 6.3 (iv)). Here IP denotes the following schema:

$$\mathrm{IP} \qquad (\neg A \to \exists y\, B) \to \exists y (\neg A \to B)$$

(y not free in A), and CT_0 is a form of Church's thesis:

$$\mathrm{CT}_0 \qquad \forall x\, \exists y\, A(x, y) \to \exists z \forall x\, \exists u \big(T(z, x, u)\ \&\ A(x, Uu)\big)$$

(T Kleene's T-predicate, U the result-extracting function).

The proof uses modified realizability. Modified realizability assigns to each formula A of $\mathbf{I\text{-}HA}^\omega$ a formula $\exists x_1^{\sigma_1} \ldots \exists x_n^{\sigma_n} A'(x_1, \ldots, x_n)$ (A' negative); negative formulae are left unchanged by the assignment. If we then interprete the objects of finite type in the assigned formulae as elements of HRO, all theorems of $\mathbf{HA} + \mathrm{IP} + \mathrm{CT}_0$ become modified realizable (provably in **HA**), i.e. the formulae assigned to theorems of this theory become provable in **HA** on interpretation in HRO. HEO, the hereditarily effective operations would not do, since CT_0 is not modified-realizable w.r.t. HEO.

3.3. $\mathbf{HA}+\mathrm{M}+\mathrm{IP}_0+\mathrm{CT}_0$ is conservative w.r.t. prenex formulae of **HA**. This is proved in a similar way, using the Dialectica-interpretation instead of modified realizability ([13] 6.3 (v)). Here M is Markov's schema, and IP_0 a weaker version of IP:

M $\qquad \forall x(A\vee\neg A)\ \&\ \neg\neg\exists x\,A\to\exists x\,A\,,$

IP_0 $\qquad \forall x(A\vee\neg A)\ \&\ (\forall x\,A\to\exists y\,B)\to\exists y(\forall x\,A\to B)$

(y not free in A).

3.4. Extending HRO to a model HRO^2 with "second-order types" (cf. [15] 2.9.7; [14] § 4) one obtains, similarly to the result in 3.2: $\mathbf{HAS}+\mathrm{IP}+\mathrm{CT}_0+\mathrm{UP}$ is conservative over **HAS** with respect to negative first order formulae. Here **HAS** is intuitionistic second-order arithmetic with species variables and full impredicative comprehension, IP, CT_0 are as before, and UP is the schema

UP $\qquad \forall X\exists x\,A(X, x)\to\exists x\forall X\,A(X, x)$

(x a numerical variable, X a species variable).

3.5. For a suitable closed term F^3 of type 3 of $\mathbf{N}\text{-}\mathbf{HA}^\omega$ (where type $n+1$ is defined as $(n)0$):

(1) $\qquad$ not $\mathbf{N}\text{-}\mathbf{HA}^\omega\vdash x^2=_e y^2\to F^3x^2=F^3y^2$,

where $=_e$ is defined extensional equality. This expresses that the closed terms of $\mathbf{N}\text{-}\mathbf{HA}^\omega$, although extensional on extensional arguments, are not absolutely extensional. To establish (1) by a counterexample, take for F^3: $\lambda x^2\cdot x^2[\lambda z^0\cdot x^2(\lambda w^0\cdot z^0)]$, and take for x^2 and y^2, in the model HRO as described in 2.6, $(n, 2)$ and $(m, 2)$ respectively; n, m gödelnumbers of extensionally equal, total, 1-1 recursive functions, $n\neq m$. λ is interpreted as the combinatorially defined λ-operator. The example is due to H. P. Barendregt and R. Statman.

3.6. Let **EL** denote a system of elementary intuitionistic analysis, with variables for numbers $(x, y, z, \ldots)$ and unary number-theoretic functions $(\alpha, \beta, \gamma, \ldots)$, equality between numbers only, definition of functions by λ-abstraction, a constant for defining unary functions primitive recursive in given functions, with the appropriate axioms for this constant, a pairing function with inverses and pairing axioms, and besides the axioms and schemata of intuitionistic arithmetic, a weak axiom of choice:

$$\forall x\exists y\,A(x, y)\to\exists\alpha\forall x\,A(x, \alpha x)$$

(A quantifier-free) (cf. [15] 1.9.10). **EL** is a conservative extension of **HA**, as can be seen by interpreting the function variables as ranging over total recursive functions.

Let us define continuous function application $\cdot|\cdot$ by

$$\alpha|\beta\simeq\gamma\equiv_{\mathrm{def}}\forall x\exists z\big(\alpha(\langle x\rangle*\bar\beta z)\neq 0\big)\ \&$$
$$\&\ \forall x\Big(\alpha\big(\langle x\rangle*\bar\beta(\min_z[\alpha(\langle x\rangle*\bar\beta z)\neq 0])\big)=\gamma x+1\Big)\,,$$

where $\langle x_1, \ldots, x_n\rangle$ codes the finite sequence $x_1, \ldots, x_n$, $*$ indicates concatenation, $\bar\beta z\equiv_{\mathrm{def}}\langle\beta 0, \ldots, \beta(z-1)\rangle$.

Then, relative to an ω-model $\mathfrak{U}$ of **EL**, we can define models $ICF(\mathfrak{U})$, $ECF(\mathfrak{U})$ which are exactly analogous to HRO, HEO, but with partial recursive function application replaced by continuous function application.

Specifically we define

$$V_0^1(x)\equiv_{\mathrm{def}}x=x\,,\qquad V_1^1(\alpha)\equiv_{\mathrm{def}}\alpha=\alpha\,,$$

and with $\sigma\neq 0$

$$V^1_{(\sigma)0}(\alpha)\equiv_{\mathrm{def}}\forall\gamma\in V^1_\sigma\exists x\big(\alpha(\gamma)\simeq x\big)$$

where

$$\alpha(\gamma)\simeq x\equiv_{\mathrm{def}}\alpha\big(\bar\gamma\min_z[\alpha(\bar\gamma z)\neq 0]\big)=x+1\,.$$

$$V^1_{(0)\sigma}(\alpha)\equiv_{\mathrm{def}}\forall x\exists\gamma\in V^1_\sigma(\alpha|\lambda z\cdot x\simeq\gamma)\,,$$

and for $\sigma, \tau\neq 0$

$$V^1_{(\sigma)\tau}(\alpha)\equiv_{\mathrm{def}}\forall\beta\in V_\sigma\exists\gamma\in V_\tau(\alpha|\beta\simeq\gamma)\,.$$

Objects of type 0 are then pairs $(x, 0)$, objects of type $\sigma\neq 0$ are pairs (α, σ) with $\alpha\in V^1_\sigma$. Application is interpreted by $(\sigma\neq 0, \tau\neq 0)$:

$$(\alpha, 1)(x, 0)=(\alpha x, 0)\,,\qquad \big(\alpha, (0)\sigma\big)(x, 0)=(\alpha|\lambda z\cdot x, \sigma)\,,$$
$$\big(\alpha, (\sigma)0\big)(\beta, \sigma)=\big(\alpha(\beta), 0\big)\,,\qquad \big(\alpha, (\sigma)\tau\big)(\beta, \sigma)=(\alpha|\beta, \tau)\,.$$

Equality is interpreted as $(\alpha, \sigma)=(\beta, \sigma)\leftrightarrow\forall x(\alpha x=\beta x)$. From the formalized theory of recursive functionals ([3]; [15], 2.6.2) one then obtains interpretations for [0], [S], $[\Pi_{\sigma,\tau}]$, $[\Sigma_{\varrho,\sigma,\tau}]$, $[\mathrm{R}_\sigma]$; thus we have obtained a model for $\mathbf{N}\text{-}\mathbf{HA}^\omega$, definable in **EL**. So $\mathbf{N}\text{-}\mathbf{HA}^\omega$ is also a conservative extension of **EL**.

Similarly, one defines $ECF(\mathfrak{U})$ (extensional continuous functionals), defining simultaneously with the objects of each type a relation of extensional equivalence at each type ([15], 2.6.5).

By means of these models we can obtain analogues to some of the results in 3.2–3.5, a continuity axiom schema replacing CT_0.

3.7. $\mathbf{EL}+\mathrm{IP}^1+\mathrm{CONT}$ is conservative over **EL** w.r.t. negative formulae. Here the schema CONT is given by

CONT $\quad \forall\alpha\exists\beta\, A(\alpha,\beta)\to\exists\gamma\forall\alpha\, A(\alpha,\gamma|\alpha)$.

IP^1 is now the second order form of IP:

$\mathrm{IP}^1 \quad (\neg A\to\exists\alpha\, B)\to\exists\alpha(\neg A\to B)$

(α not free in A). IP is a special case of IP^1. This result is analogous to 3.2 ([15], 3.6.18 (ii)).

3.8. $\mathbf{EL}+\mathrm{M}^1+\mathrm{IP}^1_0+\mathrm{CONT}$ is conservative over **EL** w.r.t. prenex formulae. Here M^1, IP^1_0 are second-order generalizations of M, IP_0, function variables taking the place of numerical variables (3.6.18 (iii) in [15]).

3.9. The preceding results may be extended to cases where **EL** is replaced by stronger systems such as $\mathbf{EL}+\mathrm{BI_D}$ (bar-induction of type 0 for decidable predicates).

§ 4. Term models; comparison with HRO

4.1. $\mathbf{N\text{-}HA}^\omega$ possesses a property which is lost by extension to $\mathbf{I\text{-}HA}^\omega$: each closed term t^0 of type 0 can be shown to be equal to a numeral $\bar{n}$ in $\mathbf{N\text{-}HA}^\omega$, i.e. $\mathbf{N\text{-}HA}^\omega\vdash t^0=\bar{n}$. This result does not hold for $\mathbf{I\text{-}HA}^\omega$: we can find two closed terms t_1^1, t_2^1 of type 1, representing extensionally equal functions, and two versions HRO′, HRO″ (i.e. defined as in 2.3, but with different choices for the numbers representing S, $\Pi_{\sigma,\tau}$, $\Sigma_{\varrho,\sigma,\tau}$, R_σ) such that in HRO′ t_1^1, t_2^1 are interpreted by the same object, and in HRO″ by distinct objects (cf. [15], § 2.5); therefore in the first model $\mathrm{E}_\sigma t_1^1 t_2^1$ is interpreted as $(0,0)$, and in the second model by $(1,0)$; therefore $\mathrm{E}_\sigma t_1^1 t_2^1$ cannot be evaluated in $\mathbf{I\text{-}HA}^\omega$.

4.2. A term of $\mathbf{I\text{-}HA}^\omega$ is said to be in *normal form*, if it does not contain a subterm of one of the following forms:

$$\Pi t_1 t_2\,,\quad \Sigma t_1 t_2 t_3\,,\quad \mathrm{R}t_1t_2 0\,,\quad \mathrm{R}t_1t_2(\mathrm{S}t_3)\,,$$

$$\mathrm{E}t_1t_2 \text{ with } t_1, t_2 \text{ closed.}$$

If we add the following rule IE_0 to $\mathbf{I\text{-}HA}^\omega$

$$\mathrm{IE}_0 \qquad \mathrm{E}t_1t_2=\begin{cases}1 & \text{if } t_1, t_2 \text{ are closed, normal and distinct,}\\ 0 & \text{if } t_1, t_2 \text{ are closed, normal and identical,}\end{cases}$$

then it can be shown ([15], § 2.3) that each closed term of $\mathbf{I\text{-}HA}^\omega+\mathrm{IE}_0$ is provably equal to a term in normal form. Since the only closed terms of type 0 in normal form are numerals, in $\mathbf{I\text{-}HA}^\omega+\mathrm{IE}_0$ we again have

the property that each closed term of type 0 can be shown to be equal to a numeral.

The somewhat syntactical-looking schema IE_0 can be replaced by the following axioms IE_1 (which imply IE_0), the negative diagram as it were of intensional equality ([15], § 2.3):

IE_1 Let t_1, t_2 be two distinct terms with the same type taken from the set of terms containing Π_{σ_1,σ_2}, $\Sigma_{\sigma_3,\sigma_4,\sigma_5}$, S, R_{σ_6}, E_{σ_7}, $\Pi_{\sigma_8,\sigma_9}x_1$, $\Sigma_{\sigma_{10},\sigma_{11},\sigma_{12}}x_2$, $\mathrm{R}_{\sigma_{13}}x_3$, $\mathrm{E}_{\sigma_{14}}x_4$, $\Sigma_{\sigma_{15},\sigma_{16},\sigma_{17}}x_5x_6$, $\mathrm{R}_{\sigma_{18}}x_7x_8$, for all $\sigma_1, \ldots, \sigma_{18}\in\mathsf{T}$. Then $t_1\neq t_2$ is an axiom. $x\neq x'\vee y\neq y'\to \mathrm{R}xy\neq\mathrm{R}x'y'$; $x\neq x'\vee y\neq y'\to\Sigma xy\neq\Sigma x'y'$. If $\sigma\neq\sigma'$, then $\Sigma_{\varrho,\sigma,\tau}xy\neq\Sigma_{\varrho,\sigma',\tau}x'y'$.

As has been remarked in 2.6, a variant of HRO can be defined in which IE_1 is satisfied. From this version of HRO it is also immediate that the normal form of a closed term of $\mathbf{I\text{-}HA}^\omega+\mathrm{IE}_0$ is uniquely determined, since closed terms in normal form are interpreted by distinct elements in the model.

4.3. We are now in a position to describe the term models for $\mathbf{I\text{-}HA}^\omega+\mathrm{IE}_0$ (cf. [9]; [5] Appendix I; [15], § 2.5). In the first term model, the objects of type σ are the (gödelnumbers of) closed terms in normal form of type σ. Application is defined as follows: $t_1^{(\sigma)\tau}t_2^\sigma$ is interpreted as the uniquely determined term t_3^τ in normal form such that $t_1^{(\sigma)\tau}t_2^\sigma=t_3^\tau$. Equality is interpreted as (literal) identity; the constants constitute their own interpretation.

Some of the principal differences with HRO are: the domain of objects of type σ is recursive (in fact primitive recursive, for a standard gödelnumbering); each object of the model has a name (description) in the theory $\mathbf{I\text{-}HA}^\omega+\mathrm{IE}_0$; equality is also primitive recursive, but application is not; in fact the application operation in the model is recursive but not provably recursive in arithmetic.

4.4. The second term model is obtained by taking the closed terms of type σ as objects of type σ, application as juxtaposition. Equality is interpreted as equality of normal form; otherwise as the first model. Now application is primitive recursive, but equality, although recursive, is not provably recursive in arithmetic.

4.5. Neither term model satisfies AC, in contrast to the consistency of AC for HRO relative **HA**. The invalidity of AC for the term models is seen as follows. Let $\bar{n}$ be a numeral which is a gödelnumber for a recursive, but not arithmetically provably recursive function. Then $\forall x\exists y\,\mathrm{T}\bar{n}xy$ holds; the validity of AC in the model would require a closed term t^1 of type 1 such that $\forall x\,\mathrm{T}(\bar{n},x,t^1x)$, which is impossible, all closed terms of type 1 being interpreted by provably recursive functions of **HA**.

§ 5. Foundational "applications" of intensional equality

5.1. Axioms of choice. If x is a numerical variable, D some intuitionistically meaningful domain, and $N \to D$ is the species of all mappings from N to D, then we can justify a countable axiom of choice

$$\forall x \exists y \in D\big(A(x,y)\big) \to \exists f \in N \to D \forall x\big(A(x,fx)\big) \tag{1}$$

as follows. On the intuitionistic meaning of the quantifier combination $\forall x \exists y \in D$, giving a proof of $\forall x \exists y \in D\, A(x,y)$ means that we have a construction of $y \in D$ from x such that $A(x,y)$, for any x. This construction is nothing else than the required mapping from N to D.

That intensional equality enters implicitly into this consideration, is seen by a comparison with another more general axiom of choice

$$\forall x \in D\, \exists y \in D'\big(A(x,y)\big) \to \exists \psi \in D \to D' \forall x \in D\big(A(x,\psi x)\big); \tag{2}$$

here D, D' are arbitrary intuitionistically meaningful domains. (2) cannot be justified in the same manner as (1): a proof of $\forall x \in D\, \exists y \in D'\big(A(x,y)\big)$ implies that for each $x \in D$ we can find a $y \in D'$ such that $A(x,y)$; but this does not necessarily imply the existence of a function ψ, since the procedure for constructing a $y \in D'$ from any given $x \in D$ might also depend on the *proof* that $x \in D$.

This problem does not enter into the justification of (1), for if we view natural numbers as very special mental constructions, obtained by iteration of the process of adding abstract units, then not only does the usual equality between natural numbers correspond to intensional equality, but also it is decidable (in an absolute sense) whether a construction given to us is a natural number or not. Natural numbers carry, in a manner of speaking, their own proof that they are natural numbers.

Note also that for the justification of (1) it is essential that $N \to D$ consists of all mapping from N to D. (Of course, special restrictions on the premiss in (1) carry over to the conclusion: if $A(x,y)$ does not contain choice parameters, f may be assumed to be lawlike, i.e. also not to depend on "choice".)

5.2. Axioms for lawlike sequences. One of the essential axioms for the theory of lawless sequences **LS**, as described e.g. in [4] or in [10], § 9, is formulated as

$$\forall x(\alpha x = \beta x) \vee \neg \forall x(\alpha x = \beta x), \tag{1}$$

if we choose a formulation of **LS** with equality between terms of type 0 only. Although in this form **LS** does not explicitly refer to intensional equality, intensional equality enters quite essentially in the justification of (1). For let us write $\equiv$ for intensional equality between lawless sequences, then obviously $\alpha \equiv \beta \vee \neg \alpha \equiv \beta$. If $\alpha \equiv \beta$, then $\forall x(\alpha x = \beta x)$. Now assume $\alpha \not\equiv \beta$ and $\forall x(\alpha x = \beta x)$. Since for any lawless sequence at any moment only an initial segment is known, it would follow that for distinct lawless sequences α, β $\forall x(\alpha x = \beta x)$ would have to be asserted from our knowledge of *initial* segments of α, β only, which is obviously impossible. Hence $\alpha \not\equiv \beta \to \neg \forall x(\alpha x = \beta x)$. This yields (1).

5.3. Discussion of intensional identity of proofs. Consider first elementary intuitionistic arithmetic formulated with zero, successor, addition and multiplication. Each closed term can be interpreted as a *description* of a natural number; the numerals are *canonical* descriptions of the natural numbers which reflect directly the construction of the natural numbers themselves. A not entirely trivial argument is necessary to show that each closed term can be evaluated (i.e. proven to be equal to a numeral). It is this non-trivial argument (which amounts to showing that addition and multiplication as operations on natural numbers are always defined) which permits us to regard closed terms as descriptions of natural numbers. (This also explains why the syntactical equivalence relation between descriptions corresponding to intensional equality between objects need not be trivially decidable, although intensional equality is: in comparing the descriptions, we have to rely on the non-trivial proof that certain operations on the objects are always defined.)

It is also possible to interpret the closed terms as descriptions of computations (namely the standard evaluation procedure for closed terms). Then obviously intensional equality between the computations corresponds to literal equality between their descriptions (i.e. closed terms), and arithmetical equality $t = t'$ corresponds to: the computations t and t' have the same result. Arithmetical $=$ is, on this interpretation ([1]), coarser than intensional equality.

After these preliminary discussions, we are in a position to discuss briefly a conjecture on intensional identity between proofs which has been put forward in the literature ([8], 3.5.6).

We distinguish between proofs (the objects) and deductions; deductions are syntactical objects: descriptions of proofs. For example, let us consider a natural deduction system for intuitionistic implicational logic, with a single reduction rule for deductions: $\to$ contraction. (For all terminology see [8].) For this example the conjecture may be stated as follows:

> two proofs corresponding to deductions π, π' are intensionally the same $\Leftrightarrow$ π, π' reduce to the same normal form.

In one direction ($\Rightarrow$) the conjecture is doubtful: if it is possible that different formulae express the same proposition, then it is also possible that different normal deductions represent the same proof. But even if we assume $\Rightarrow$, the conjecture also present difficulties in the other direction.

([1]) We do not wish to suggest that these interpretations are the only possible ones!

If we think of the objects of the theory as "direct proofs" whose canonical descriptions are normal deductions (cf. numbers and numerals in the case of arithmetic), then the inference rules $\to$I, $\to$E correspond to certain operations on direct proofs; the normalization theorem then establishes that these operations are always defined, and permits us to regard any deduction as a description of a direct proof. The conjecture in the direction $\Leftarrow$ then becomes trivially true.

But if the deductions are seen as descriptions of a more general concept of proof (see footnote (1)) (in the same manner as closed terms are descriptions of computations in the case of arithmetic) it may be argued that intensional equality should correspond to (literal) equality of deductions; and then the conjecture is false.

So, without further information about the intended concept of proof, the conjecture in the direction $\Leftarrow$ is meaningless because, as just explained, ambiguous (2).

Finally it should be added that for a natural deduction system for arithmetic, where, besides proper reductions, also induction reductions are considered, the conjecture is manifestly false, as pointed out by G. Kreisel. First proving $\forall xy(x+y=y+x)$, then specializing to $27+52 = 52+27$ is a proof basically different from the proof of $27+52=52+27$ obtained by evaluating both sides of the equation; basically different inasmuch as the second argument does not use the insight that addition in general is commutative. Nevertheless, the deduction corresponding to the first proof reduces to a numerical computation.

§ 6. Non-extensional, decidable equality for incomplete objects

6.1. Our principal example of a decidable, non-extensional notion of equality was provided by HRO. It might be instructive to present another model, involving "incomplete" objects (objects which are not completely fixed in advance, such as choice sequences and lawless sequences). We do not claim any mathematical interest for the non-extensional equality in the model below, only pedagogical interest.

6.2. Let α denote a single lawless sequence, and let us put $(\alpha)_n \equiv_{\text{def}} \lambda y \cdot j(n, y)$ (j a pairing function onto the natural numbers). Then $\mathfrak{U} \equiv_{\text{def}} \{n * (\alpha)_n |\ n \in N\}$ is a model for **LS**, the theory of lawless sequences (without species variables). Here $n * (\alpha)_n$ indicates the sequence obtained by concatenating n and $(\alpha)_n$. Now $n * (\alpha)_n \equiv m * (\alpha)_m$ (intensional equality in our model) is interpreted as $n = m$ ([12]).

Therefore (although of course the enumeration of $\mathfrak{U}$ cannot be expressed in **LS**) we have an easily visualizable idea of "intensional equality" in this case. The example might help to remove the doubts of those who feel uneasy about the idea of many distinct lawless sequences.

6.3. We may go one step further, and consider the species

$$\mathfrak{U}^* \equiv_{\text{def}} \big\{\{x\} | \nu_u(\varepsilon_1, \ldots, \varepsilon_u) \colon \{x\} \in \mathrm{K},\ u \in \mathrm{N},\ \varepsilon_1, \ldots, \varepsilon_u \in \mathfrak{U},\ \varepsilon_1, \ldots, \varepsilon_u \text{ all distinct}\big\}.$$

Here K is the species of (lawlike) Brouwer operations (i.e. neighbourhood functions of continuous functionals, cf. [6], § 3), $\{x\}$ is the (partial) recursive function with gödelnumber x, $\cdot|\cdot$ is the continuous function application already defined in 3.6, ν_u is a standard coding of u-tuples of sequences into a single sequence.

Two elements of $\mathfrak{U}^*$

$$\{x\}|\nu_u(\varepsilon_1, \ldots, \varepsilon_u), \quad \{x'\}|\nu_v(\varepsilon_1', \ldots, \varepsilon_v')$$

are intensionally equal in our model, iff

$$x = x', \quad u = v, \quad \varepsilon_i = \varepsilon_i' \quad \text{for} \quad 1 \leqslant i \leqslant u .$$

$\mathfrak{U}^*$ provides an ω-model for a substantial fragment of intuitionistic analysis with choice sequences (see [11]), modulo the assumption of a generalization of Church's thesis (to be precise: ECT_0 for **IDB**; for ECT_0 see 7.3, for **IDB** see [6], § 3, where it is shown that ECT_0 is consistent relative **IDB**).

§ 7. Concluding remarks about HRO

7.1. The natural intended model for **N-HA**$^\omega$ (cf. [1], "Berechenbare Funktion") consists of the constructive functionals of finite type. The natural numbers are the objects of type 0. Suppose we have already understood the concept of a constructive functional of type σ and of type τ. Such functionals are assumed to be *given* as such, and for any construction it should be clear whether it is presented as such a functional or not. The constructive functionals of type $(\sigma)\tau$ then consist of all mappings (completely described by a rule) from objects of type σ to objects of type τ with respect to (strict) intensional equality.

For this notion, the axiom of choice AC should hold (by arguments similar to those used in the special case of 5.1).

7.2. HRO is not a precise rendering of this informally described model: as we remarked before, the mappings of $V_{(\sigma)\tau}$ are mappings which depend, for any argument $x \in V_\sigma$, on x only, not on the proof that $x \in V_\sigma$.

Hence it is also not obvious that HRO should satisfy AC. On the intended intuitionistic interpretation of implication, an implication

$$x \in V_\sigma \to A$$

(2) A similar discussion is found in [7]. However, the discussion there takes its point of departure from a syntactical notion of definitional equality, whereas our concept of intensional equality is a relation between objects, and not between their descriptions.

is established giving a rule transforming proofs of $x \in V_\sigma$ into proofs of A. However, if we can find a re-interpretation of the logical operations such that implications $x \in V_\sigma \to A$ can be assumed to be established by rules depending on the *truth* of $x \in V_\sigma$ only, not on the *proof* of $x \in V_\sigma$, we might expect to show consistency of AC for HRO relative to this interpretation.

7.3. This is achieved by Kleene's numerical realizability (cf. [13]; [15], § 3.2). Let us call a formula of **HA** *almost negative* if it does not contain $\vee$, and $\exists$ only in subformulae $\exists x(t = s)$. Let ECT_0 (Extended Church's Thesis) be the following schema:

$$\mathrm{ECT}_0 \qquad \forall x[Ax \to \exists y\, Bxy] \to \exists u \forall x[Ax \to \exists v(\mathrm{T}uxv \,\&\, B(x, \mathrm{U}v))]$$

(A almost negative).

Then ([15], § 3.2; [13]), if $x\mathrm{r}A$ expresses "x realizes A", we can show:

$$\mathbf{HA} + \mathrm{ECT}_0 \vdash A \Leftrightarrow \mathbf{HA} \vdash \exists x(x\mathrm{r}A),$$

$$\mathbf{HA} + \mathrm{ECT}_0 \vdash A \leftrightarrow \exists x(x\mathrm{r}A).$$

Since $x \in V_\sigma$ is also expressible by an almost negative formula, the realizability interpretation is an interpretation of the required kind, and one easily shows that AC for HRO is derivable in $\mathbf{HA} + \mathrm{ECT}_0$. $\mathrm{AC}_{1,0}$ is obviously false for the corresponding extensional model HEO, since

$$\forall x^1 \exists y^0 \forall u^0 \exists v^0[\mathrm{T}yuv \,\&\, x^1u = \mathrm{U}v]$$

but we cannot find an extensional z^2 such that

$$\forall x^1 \forall u^0 \exists v^0[\mathrm{T}(z^2x, u, v) \,\&\, x^1u = \mathrm{U}v].$$

§ 8. Digressions

8.1. "Fixpoints" of the Dialectica interpretation. Let us call a theory **H** a *fixpoint of the Dialectica interpretation* if, for any sentence A such that $\mathbf{H} \vdash A$, with Dialectica translation

$$\exists x_1 \dots x_n \forall y_1 \dots y_m\, A_D(x_1, \dots, x_n, y_1, \dots, y_m)$$

(A_D quantifier-free), we can find a sequence $t_1, \dots, t_n$ of closed terms of **H** such that

$$\mathbf{H} \vdash A_D(t_1, \dots, t_n, y_1, \dots, y_m).$$

$\mathbf{N\text{-}HA}^\omega$ is not a fixpoint, as we can see by taking for A the formula $\neg\neg\exists z^0(z^0 = 0 \leftrightarrow x^\sigma = y^\sigma)$, suggested by an example of W. A. Howard. $\mathbf{N\text{-}HA}^\omega \vdash A$ since, of course, $\vdash \neg\neg(x^\sigma = y^\sigma \vee x^\sigma \neq y^\sigma)$ and

$$\mathbf{N\text{-}HA}^\omega \vdash (x^\sigma = y^\sigma \vee x^\sigma \neq y^\sigma) \to \exists z^0(z^0 = 0 \leftrightarrow x^\sigma = y^\sigma).$$

But

$$\mathbf{N\text{-}HA}^\omega \vdash [tx^\sigma y^\sigma = 0 \leftrightarrow \neg\neg(x^\sigma = y^\sigma)]$$

is false, for each t of type $(\sigma)(\sigma)0$, since otherwise we would have

$$\mathbf{N\text{-}HA}^\omega \vdash \neg\neg(x^\sigma = y^\sigma) \vee \neg(x^\sigma = y^\sigma).$$

In fact the natural proof of the soundness theorem for the Dialectica interpretation has to appeal to the decidability of prime formulae to show the interpretability of $A \to A \,\&\, A$.

$\mathbf{I\text{-}HA}^\omega$ is a fixpoint for the Dialectica interpretation, and so are $\mathbf{I\text{-}HA}^\omega + \mathrm{IE}_0$, $\mathbf{I\text{-}HA}^\omega + \mathrm{IE}_1$ (IE_1 is described in 4.2).

On the other hand, $\mathbf{E\text{-}HA}^\omega$ formulated as a theory with equality of type 0 only, is not a fixpoint: W. A. Howard has shown that already $\forall x^1 y^1 z^2(\forall u^0(xu = yu) \to z^2x^1 = z^2y^1)$, the simplest non-trivial instance of the extensionality axiom, does not have a Dialectica interpretation by means of functionals of $\mathbf{E\text{-}HA}^\omega$. To obtain a fixpoint, one has to weaken the extensionality axiom to a rule, in the simplest form given by

$$\vdash t =_e s \Rightarrow \vdash F[t] = F[s]$$

($=_e$ is defined extensional equality, as before). The resulting theory is called $\mathbf{WE\text{-}HA}^\omega$.

If we interpret the objects of finite type by ECF (extensional continuous functionals), then the extensionality axiom becomes interpretable; but the axiom stating that functionals of type 2 are continuous is itself not interpretable relative to ECF ([15], 2.6.7, 3.5.12).

However, there is a natural sub-theory of $\mathbf{N\text{-}HA}^\omega$ which is a fixpoint; let us call this theory $\mathbf{HA}^\omega$. $\mathbf{HA}^\omega$ is formulated with equality of type 0 only, and instead of the defining axioms for the functional constants we have schemata

$$F[\Pi xy] = F[x], \quad F[\Sigma xyz] = F[xz(yz)], \quad F[\mathrm{R}xy0] = F[x],$$

$$F[\mathrm{R}xy(\mathrm{S}z)] = F[y(\mathrm{R}xyz)z].$$

8.2. A $\beta\eta$-conversion variant of HRO. The theory $\mathbf{I\text{-}HA}^\omega$ as described before was based on combinators, and the reduction relation and normal form corresponded to weak reduction and weak normal form in combinatory logic.

If we consider instead a theory $\lambda\mathbf{I\text{-}HA}^\omega$ with λ instead of Π, Σ as a primitive, and with a rule: if t reduces to t', then $t = t'$, where reduction is interpreted as reduction w.r.t. $\beta\eta\delta$-conversion for typed terms, then HRO as it stands is not a model for this theory; but it is possible to construct an HRO-like model for $\lambda\mathbf{I\text{-}HA}^\omega$ too (i.e. $=$ is, for each type, interpreted by equality between natural numbers, and the model contains many objects besides those denoted by closed terms of $\lambda\,\mathbf{I\text{-}HA}^\omega$). For details, see [15], 2.4.18.

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MATHEMATISCH INSTITUUT DER UNIVERSITEIT VAN AMSTERDAM

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Multirelation et âge 1-extensifs

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Résumé. Une relation R' extension d'une autre R, est dite une 1-extension lorsque, pour toute formule logique universelle ou combinaison booléenne d'universelles, la valeur de vérité est la même pour R et R', les individus libres étant substitués par des éléments de la petite base $|R|$. Nous obtenons ici une condition algébrique d'existence d'une 1-extension commune à deux relations (à l'isomorphie près). Comme application, deux chaînes, ou ordres totaux, de bases infinies, admettent toujours une 1-extension commune.

1. Rappels sur la 1-extension. Rappelons qu'une multirelation, ou suite finie de relations de base commune, soit R, admet R' comme 1-*extension*, lorsque R' est une extension de R et que, pour toute partie finie F de la base $|R'|$, il existe un isomorphisme f de la restriction R'/F sur une restriction de R, avec $f(x) = x$ pour chaque x de l'intersection $F \cap |R|$.

Rappelons qu'un *isomorphisme local* d'une multirelation R vers une autre R', est un isomorphisme d'une restriction de R sur une restriction de R'. Etant donné un entier naturel p, un isomorphisme local f est dit un $(1, p)$-*isomorphisme* de R vers R' lorsque, pour tout $q \leqslant p$ et tous éléments $a_1, \ldots, a_q$ de la base $|R|$, il existe $a'_1, \ldots, a'_q$ de $|R'|$ la transformation f augmentée de la transformation de chaque a_i en a'_i $(i = 1, \ldots, q)$ étant un isomorphisme local de R vers R', et inversement en échangeant R, R' et remplaçant f par f^{-1}. On voit qu'une extension R' de R est une 1-extension si et seulement si, pour toute partie finie F de $|R|$, et tout entier p, l'identité sur F est un $(1, p)$-isomorphisme de R vers R'.

Une traduction logique de la définition précédente, dit que R' est une 1-extension de R lorsque, pour toute formule logique P du premier degré: formule prénexe universelle, ou existentielle, ou combinaison booléenne des deux; avec des prédicats substituables par R ou par son extension R'; et avec n individus libres substitués par des éléments quelconques $a_1, \ldots, a_n$ de la petite base $|R|$; les valeurs de vérités pour R et pour R' sont les mêmes: $P(R)(a_1, \ldots, a_n) = P(R')(a_1, \ldots, a_n)$.

Si R' est une 1-extension de R, il en est évidemment de même de toute restriction de R' à un ensemble intercalé entre $|R|$ et $|R'|$.

Pour toute R de base infinie, il existe une restriction dénombrable