That completes our sketch. We should remark that, as we have written this out, it has taken up $3\frac{1}{2}$ sides of paper. The full argument, as we have it written out (on the same kind of paper, etc.) takes up some 24 sides. The reader has been warned. (Though, indeed, most of the details required *are* buried in [4], at some point or other.)

Remark. Since we have stated that all our proofs generalize to arbitrary $\varkappa$, we should perhaps mention how the above generalizes, since the observant reader may have noticed that ω figured heavily in our definition of $\mathcal{S}$ (more precisely, ω^+ so figured). In order to construct a $\varkappa$-morass for $\varkappa > \omega$, one would work entirely above $\varkappa$ and carry $\varkappa$ as a constant of all structures concerned, whence it would be entirely analogous to speak of $\varkappa^+$ (since $\varkappa$ would, for the purpose in hand, be "absolute").

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Degrees of dependence in the theory of semisets

by

Petr Hájek (Prague)

> Probably we shall have in the future essentially differen intuitive notions of sets just as we have different notion of space, and will base our discussions of sets on axiom that correspond to the kind of sets which we want to study
>
> *A. Mostowski* (*)

Abstract. The relation of dependence between semisets is studied through the special outlook of the theory of degrees of unsolvability. Degrees of dependence are compared with similar notions of Recursion theory and Set theory; the Kleene-Post-Spector theorem on bounds is transferred to the theory of semisets; some consequences concerning transitive models of set theory are deduced.

The aim of this paper is to study the relation of dependence between semisets through the special outlook of the theory of degrees of unsolvability. We shall not attempt to create any systematic theory of degrees of dependence; we only show that it is possible and perhaps useful to consider them. In Section 1 we give a summary of used notions of the theory of semisets and try (again) to formulate the relations of the theory of semisets to the set theory. In Section 2 we compare degrees of dependence with similar notions of Recursion theory and Set theory. In Section 3 we transfer a proof of a (Kleene-Post-Spector) theorem on upper and lower bounds of degrees of unsolvability into the theory of semisets and obtain a result on bounds of degrees of dependence. So we have an application of Recursion theory to the theory of semisets. In Section 4 we formulate some consequences concerning model-classes ("transitive models containing all the ordinals") in Set theory in the style of usual applications of the theory of semisets to Set theory. It is of some interest that we obtain a model-class without the axiom of choice by a construction that does not use any notion of symmetry.

(*) A. Mostowski, *Recent results in set theory*, in: Problems in the Philosophy of Mathematics, 1967, p. 24.

§ 1. Preliminaries: The theory of semisets and notions of dependence.

(a) The theory of semisets. The theory of semisets (denoted by **TSS**) is developed in [7]; one can find axioms of **TSS** and some statements of relations between **TSS** and Set theory (both the Zermelo–Fraenkel system **ZF** and the Gödel–Bernays system **TS**) in [2] and [3].

We shall summarize the basic facts in **TSS**. **TSS** can be viewed as a theory with three sorts of variables — for *sets*, *real classes* and *classes* (both real and imaginary). Sets are denoted by lower-case letters $x, y, \ldots$ and classes by capital letters $X, Y, \ldots$ The axioms assure that sets with real classes behave as sets and classes of the Gödel–Bernays set theory (with the axiom of regularity; in the present paper we use **TSS** and **TS** to denote the theories denoted by **TSS**''' and **TS**''' in [7]). *Semisets* are subclasses of sets; the axioms assure that a class is imaginary (non-real) iff it intersects some set in a proper semiset (nonset). Semisets are denoted by $\sigma, \varrho, \tau, \ldots$ (A powerful axiom on "systems of semisets" — (C2) of [7] — will not be repeated here.) We have two notions of equivalence of sets: $x \approx y$ (x is *equivalent* to y) iff there is a set which is a 1-1 mapping of x onto y, and $\sigma \hat{\approx} \varrho$ (σ is *absolutely equivalent* to ϱ) iff there is a semiset which is a 1-1 mapping of σ onto ϱ.

By [7] 1456, **TSS** is a conservative extension of **TS**. One can add various axioms on existence of proper semisets to obtain various (proper) extensions **TSS*** of **TSS** that are still conservative extensions of **TS**, i.e. a formula φ is provable in **TS** iff it is provable in **TSS** as a statement on sets and real classes. We shall give some examples in the next subsection.

(b) Dependence. We restrict ourselves to dependence of semisets on semisets. By [7] 1460 and 4103, ϱ is (*disjointedly*) *dependent* on σ if there is a (disjointed) relation r such that $\varrho = r''\sigma$ (i.e. if

$$(\forall x)(x \in \varrho \equiv (\exists y)(\langle xy\rangle \in r \ \& \ y \in \sigma);$$

r is *disjointed* if the converse of r is a function, i.e. if

$$(\forall xyz)(\langle xy\rangle \in r \ \& \ \langle xz\rangle \in r . \rightarrow y = z)) .$$

Since both notions are reflexive and transitive, we shall write $\varrho \leqslant_D \sigma$ and $\varrho \leqslant_{DD} \sigma$ to denote dependence and disjointed dependence respectively. $\varrho \equiv_D \sigma$ means $\varrho \leqslant_D \sigma \ \& \ \sigma \leqslant_D \varrho$ (ϱ and σ are *similar*) and analogously for $\equiv_{DD}$ (*disjointedly similar*). A semiset σ is a *support* $(\mathrm{Supp}(\sigma)$; cf. [7] 1462, 4130) if semisets dependent on σ are closed under difference; a support σ is *disjointed* if $(\forall \varrho)(\varrho \leqslant_D \sigma \equiv \varrho \leqslant_{DD} \sigma)$. σ is a *Boolean support* if there is a complete Boolean algebra b (a set) such that σ is an ultrafilter on b closed under intersections of subsets. By [7] 4219, a Boolean support is a disjointed support.

A semiset σ is a *total support* $(\mathrm{TSupp}(\sigma))$ if all semisets depend on σ.

In accordance with [7], 4131, (S3) denotes the axiom "there is a σ which is a total support".

Let $\boldsymbol{\sigma}$ be a fixed support; call classes intersecting each set in a semiset dependent on $\boldsymbol{\sigma}$ "*$\boldsymbol{\sigma}$-comprehensive classes*". Then sets, real classes and $\boldsymbol{\sigma}$-comprehensive classes satisfy all the axioms of **TSS** and $\boldsymbol{\sigma}$ becomes a total support. We denote the interpretation (direct syntactic model) just described by $\mathfrak{Supp}(\boldsymbol{\sigma})$; it is a model of (**TSS**, S3) in (**TSS**, $\mathrm{Supp}(\boldsymbol{\sigma})$). Semisets of the model are just the semisets dependent on $\boldsymbol{\sigma}$.

By [7] 4107 and 4114, we have the following

1.1. LEMMA (**TSS**). *Let $\sigma \subseteq a$ and put $\sigma^* = \mathbf{P}(a) - \mathbf{P}(a - \sigma)$ ($\mathbf{P}$ denotes the power-class operation). Then, for each ϱ,*

(1) $\varrho \leqslant_D \sigma \equiv \varrho \leqslant_{DD} \sigma^*$;

(2) $\sigma \equiv_D \sigma^*$;

(3) *σ is a support iff σ^* is a disjointed support*;

(4) *σ is a support iff $\mathbf{P}(a-\sigma) \leqslant_D \sigma$.*

Consequently, each support is similar to a disjointed support. Let (St) be the axiom "each non-empty semiset of ordinals has a first element". By [1], (S3) implies (St) in **TSS** and hence, by [7] 4241, each support is similar to a Boolean support.

(AS) denotes the axiom "each complete Boolean algebra bears a Boolean support (complete ultrafilter)". By [7] 5321, (AS) is equivalent to the axiom (GC) of general collaps saying "each infinite set is absolutely equivalent to the set ω of all natural numbers". The consistency of (AS) with **TSS** was proved independently by B. Balcar and the author (unpublished). (BS) is the axiom saying "each semiset depends on a support".

We shall now give three typical examples of strengthened theories of semisets. In this paper,

$\mathbf{TSS}_1$ is the theory (**TSS**, S3),

$\mathbf{TSS}_2$ is the theory (**TSS**, AS, BS),

$\mathbf{TSS}_3$ is the theory (**TSS**, AS, $\neg$St).

All these theories extend **TS** conservatively and seem to be theories with a reasonable theory of degrees of dependence. Evidently, we have the following:

$$\mathbf{TSS}_1 \vdash (\mathrm{BS}), \neg(\mathrm{AS}), \ \mathbf{TSS}_2 \vdash (\mathrm{St}), \ \mathbf{TSS}_3 \vdash \neg(\mathrm{BS}) .$$

In the next subsection we formulate the relation of these theories to **TS** from another point of view.

(c) TSS and TS from another point of view. Recall from [7] 1442 that a *modell-class* is a real complete (transitive) class M which is closed under Gödelian operations and is almost universal, i.e. each subset of M is included in an element of M. $\mathrm{Mcl}(M)$ means that M is a model-class. Let $\mathbf{M}$ be a *fixed model-class* and call a subclass X of $\mathbf{M}$ $\mathbf{M}$=comprehensive if $(\forall x \in \mathbf{M})(X \cap x \in \mathbf{M})$. Then the elements of $\mathbf{M}$, $\mathbf{M}$=comprehensive subclasses of $V\mathbf{M}$ and arbitrary subclasses of $\mathbf{M}$ satisfy all the axioms of **TSS.** We denote the model just described by $\mathfrak{St}(\mathbf{M})$. It is a model of **TSS** in $(\mathbf{TSS}, \mathrm{Mcl}(\mathbf{M}))$ and a model of $(\mathbf{TSS}, \mathrm{St})$ in $(\mathbf{TS}, \mathrm{Mcl}(\mathbf{M}))$. The last fact can also be expressed by saying that **TS** (with a constant fixed for a model-class) becomes an extension of $(\mathbf{TSS}, \mathrm{St})$ if variables for sets, real classes and arbitrary classes of **TSS** are identified with variables for elements of $\mathbf{M}$, $\mathbf{M}$=comprehensive subclasses of $\mathbf{M}$ and arbitrary subclasses of $\mathbf{M}$. This justifies the following

1.2. METADEFINITION. Let $\mathbf{TSS}^+$ and $\mathbf{TS}^+$ be extensions of **TSS** and $(\mathbf{TS}, \mathrm{Mcl}(\mathbf{M}))$ respectively not enlarging the respective language. $\mathbf{TSS}^+$ is said to be *extensible* to $\mathbf{TS}^+$ if $\mathfrak{St}(\mathbf{M})$ is a model of $\mathbf{TSS}^+$ in $\mathbf{TS}^+$. $\mathbf{TSS}^+$ is *conservatively extensible* to $\mathbf{TS}^+$ if $\mathfrak{St}(\mathbf{M})$ is a faithful model of $\mathbf{TSS}^+$ in $\mathbf{TS}^+$, i.e. if $\mathbf{TSS}^+ \vdash \varphi$ is equivalent to $\mathbf{TS}^+ \vdash \varphi^{\mathfrak{St}(\mathbf{M})}$ for each **TSS**-statement φ. $\mathbf{TSS}^+$ is *fully extensible* to $\mathbf{TS}^+$ if, after the identifications of variables described above, each $\mathbf{TS}^+$-formula whose only free variables are $\mathbf{TSS}^+$-variables is $\mathbf{TS}^+$-equivalent to a $\mathbf{TSS}^+$-formula with the same variables. $\mathbf{TSS}^+$ is not consistently extensible to a set theory if each $\mathbf{TS}^+$ such that $\mathbf{TSS}^+$ is extensible to $\mathbf{TS}^+$ is contradictory.

1.3. METATHEOREM. (1) $\mathbf{TSS}_1$ *is conservatively and fully extensible to* $\mathbf{TS}, \mathrm{Mcl}_{S3}(\mathbf{M})$, *where* $\mathrm{Mcl}_{S3}(\mathbf{M})$ *is* $\mathrm{Mcl}(\mathbf{M})$ & $(\exists z \subseteq \mathbf{M})(\forall x \subseteq \mathbf{M})(\exists r \in \mathbf{M})$ $(x = r''z)$ (read: $\mathbf{M}$ is a *model-class with a total support*).

(2) *Neither* $\mathbf{TSS}_2$ *nor* $\mathbf{TSS}_3$ *are consistently extensible to a set theory.*

The demonstration of (1) is implicit in demonstrations of [7] 5129 and 6410. In fact, by the demonstration of 6410 for each **TS**-formula φ one finds a **TSS**-formula ψ with an additional variable σ such that

$$\mathbf{TS} \vdash \varphi(x^{\square}, \varrho^{\square}, X^{\square}) \equiv (\forall \sigma^{\square})\big(\mathrm{TSupp}^{\square}(\sigma^{\square}) \rightarrow \psi^{\square}(x^{\square}, \varrho^{\square}, X^{\square}, \sigma^{\square})\big)$$
$$\equiv (\exists \sigma^{\square})\big(\mathrm{TSupp}^{\square}(\sigma^{\square}) \;\&\; \psi^{\square}(x^{\square}, \varrho^{\square}, X^{\square}, \sigma^{\square})\big)\,.$$

(The superscript $^{\square}$ means that the corresponding notion or formula is to be interpreted in $\mathbf{TS}^+$.) Moreover, if φ is a normal formula (only set variables are quantified) then ψ is also a normal formula. (See demonstration of 6410.)

The assertion (2) follows from the more or less evident fact that $(\mathbf{TSS}, \mathrm{AS})$ is not consistently extensible to a set theory since in a $\mathbf{TS}^+$ such that $\mathbf{TSS}$, AS is extensible to $\mathbf{TS}^+$ one can prove that all ordinals are countable.

The moral of the metatheorem is that sometimes speaking on semisets is only an axiomatic way of speaking on subsets of a model-class (e.g. if we work in $\mathbf{TSS}_1$) but *sometimes not* (e.g. if we work in $\mathbf{TSS}_2$ or $\mathbf{TSS}_3$).

§ 2. **Degrees of dependence.** We have the quasiorderings $\leqslant_D$ and $\leqslant_{DD}$ between semisets and think of degrees of dependence and degrees of disjoint dependence as of "factors of the equivalence $\equiv_D$ and $\equiv_{DD}$ respectively". There is a certain difficulty in that we do not know how to define degrees of dependence within **TSS**. The difficulty is the same as when one tries to define cardinality in the set theory without axioms of regularity and choice; but in both cases the difficulty is not too serious and our theorems on degrees of dependence can be always decoded into theorems on the quasiorderings $\leqslant_D$ and $\leqslant_{DD}$.

Note in passing that degrees are definable in $\mathbf{TSS}_1$: if σ is a fixed total support then one can put $\mathrm{dg}_D(\varrho) = \{r;\ r''\sigma \equiv_D \varrho\}$, i.e. degrees are pairwise disjoint (proper) classes. The following is more interesting:

2.1. LEMMA (**TSS**). *Let σ be an arbitrary semiset and let $\sigma \subseteq a$. If $\varrho \leqslant_D \sigma$ then there is a $\tau \subseteq \mathbf{P}(a)$ such that $\varrho \equiv_D \tau$.*

Proof. By 1.1, there is a function f such that $\varrho = f^{-1}{}''(\sigma^*)$ and $\sigma^* \subseteq \mathbf{P}(a)$. Hence $\varrho \equiv_D f''\varrho \subseteq \mathbf{P}(a)$.

2.2. COROLLARY (**TSS**). *For each $\sigma \subseteq a$, each D-degree less than $\mathrm{dg}_D(\sigma)$ is the degree of a subsemiset of $\mathbf{P}(a)$; each DD-degree less than $\mathrm{dg}_{DD}(\sigma)$ is the degree of a subsemiset of a.*

We are interested not only in the analogy of definitions of D-degrees (DD-degrees) and the degrees of unsolvability; our aim is the transfer of some proofs of theorems on degrees of unsolvability to proofs of theorems on D-degrees. We shall base our transfers on the beautiful Shoenfield's exposition [5]. We shall often need lists of semisets; this notion is easily explicated by the notion of an exact functor (see [7] 1408) and all constructions of lists of semisets by induction are based on the metatheorem [7] 4228.

In the rest of the present section, we shall compare D-degrees with DD-degrees and both D-degrees and DD-degrees with degrees of unsolvability (R-degrees) and degrees of (many-one) reducibility (red-degrees) from Recursion theory and degrees of constructibility (C-degrees) from Set theory (considered by Sacks, see e.g. [4] Section 23). The reader not interested in this comparison can turn to Section 3.

Since $\varrho \leqslant_{DD} \sigma$ implies $\varrho \leqslant_D \sigma$, each D-degree is a union of some DD-degrees. We shall show that DD-degrees in general do not coincide with D-degrees. Note that if $\sigma \subseteq a$ and $\sigma^* = \mathbf{P}(a) - \mathbf{P}(a-\sigma)$ then $\mathrm{dg}_D(\sigma) = \mathrm{dg}_D(\sigma^*)$ and $\mathrm{dg}_{DD}(\sigma) \leqslant \mathrm{dg}_{DD}(\sigma^*)$ (by 1.1); hence to find a σ such that $\mathrm{dg}_D(\sigma) \neq \mathrm{dg}_{DD}(\sigma)$ it is sufficient to find a σ such that $\mathrm{dg}_{DD}(\sigma) < \mathrm{dg}_{DD}(\sigma^*)$.

2.3. LEMMA (**TSS**). *If $\mathbf{P}(\mathbf{P}(a)) \mathrel{\hat{\approx}} \aleph_0$ then there is a $\sigma \subseteq a$ such that for $\sigma^* = \mathbf{P}(a) - \mathbf{P}(a-\sigma)$ we have $\sigma^* \not\leqslant_{DD} \sigma$.*

Proof. Let $\mathbf{exp}(\mathbf{P}(a), a)$ be the set of all set-mappings of $\mathbf{P}(a)$ into a. Note that $\mathbf{P}(\mathbf{P}(a)) \mathrel{\hat{\approx}} \aleph_0$ implies $\mathbf{exp}(\mathbf{P}(a), a) \mathrel{\hat{\approx}} \aleph_0$ and we can suppose

$a = \aleph_0$. Hence let $\{f_n\}_{n\in\omega}$ be a list of all elements of $\mathbf{exp}(\mathbf{P}(a), a)$. (Each f_n is a set, but the whole list is a proper semiset.) Furthermore, let $\{x_n\}_{n\in\omega}$ be a list of all subsets of a. We construct a semiset $\sigma \subseteq a$ and assure the following conditions:

(1_n) $\sigma \neq x_n$,

(2_n) $(\exists x \subseteq a)\big(x \cap \sigma = 0 \equiv f_n(x) \in \sigma\big)$.

The conditions imply that σ is a proper semiset and $\sigma^* \not\leqslant_{DD} \sigma$. We construct two disjoint sequences $\{u_s\}_{s\in\omega}$, $\{v_s\}_{s\in\omega}$ of elements of a each having infinite number of distinct elements and such that each σ containing all u_s and no v_s satisfies all the conditions. Let $\{r_s\}_{s\in\omega}$ be a list of all the conditions and let $u_0, \ldots, u_{s-1}, v_0, \ldots, v_{s-1}$ be defined. We describe the step s which assures the condition r_s.

If r_s is (1_n) then take the least $w \in a$ distinct from all the u_i, v_i $(i < s)$ (a list of elements of a being fixed; by (St), $\{u_0, \ldots, u_{s-1}, v_0, \ldots, v_{s-1}\}$ is a finite set and hence one can find the w). If $w \in x_n$ put $v_s = w$ and let u_s be the first element of a distinct from $\{u_0, \ldots, u_{s-1}, v_0, \ldots, v_s\}$. If $w \notin x_n$ then put $u_s = w$ and let v_s be the first element of a distinct from $\{u_0, \ldots, u_s, v_0, \ldots, v_{s-1}\}$. Then (1_n) is assured.

If r_s is (2_n) then find the first $x \subseteq a$ such that $x \nsubseteq \{v_0, \ldots, v_s\} \cup \{f_n(x)\}$ and $f_n(x) \notin \{u_0, \ldots, u_{s-1}\}$ if there is such an x. Then take $u_s \in x - \{v_0, \ldots, v_{s-1}\}$ and $v_s = f_n(x)$. This assures (2_n). If there is no such x then put $u_s = u_{s-1}$ and $v_s = v_{s-1}$. We must prove that (2_n) is assured also in this case. Whenever $x \subseteq a$ and x has more than $s+1$ elements, $f_n(x) \in \{u_0, \ldots, u_{s-1}\} \subseteq \sigma$. Evidently, one can find an $(s+2)$-element subset x of $\{v_s;\ s \in \omega\}$; then $x \cap \sigma = 0$ and $f_n(x) \in \{u_0, \ldots, u_{s-1}\}$, hence $f_n(x) \in \sigma$, which proves (2_n).

2.4. Corollary (**TSS**). *If there is a support τ and a one-one mapping $\varrho \leqslant_D \tau$ of $\mathbf{P}(\mathbf{P}(a))$ onto a then there is a $\sigma \subseteq a$ such that for $\sigma^* = \mathbf{P}(a) - \mathbf{P}(a - \sigma)$ we have $\sigma^* \not\leqslant_{DD} \sigma$.*

Proof. Let a support τ be fixed. Then in the sense of $\mathfrak{Supp}(\tau)$ we have (S1) and $\mathbf{P}(\mathbf{P}(a)) \approx a$; hence, by 2.3, we have $\sigma^* \not\leqslant_D \sigma$ in the sense of $\mathfrak{Supp}(\tau)$, which implies $\sigma^* \not\leqslant_{DD} \sigma$.

Remark. We see that by 2.4 we can prove e.g. in (**TSS**, AS) that D-degrees and DD-degrees do not coincide.

Since each semiset similar to a support is a support we can speak on *support degrees*. The degree of all non-empty sets is the trivial support degree; existence of non-trivial support degrees is guaranteed by various axioms, e.g. by (AS). We ask whether there is a degree that is not a support degree.

2.5. Lemma (**TSS**$_1$). *If $\mathbf{P}(\mathbf{P}(a)) \approx \aleph_0$ then there is a $\sigma \subseteq a$ that is not a support.*

Proof. By 1.1, we must assure $\mathbf{P}(a - \sigma) \not\leqslant_D \sigma$ or, equivalently, $\mathbf{P}(a - \sigma) \not\leqslant_{DD} \sigma^*$. So we have the following conditions (x_n as above, $\{f_n\}_{n\in\omega}$ is a list of $\mathbf{exp}(\mathbf{P}(a), \mathbf{P}(a))$):

(1_n) $\sigma \neq x_n$,

(2_n) $(\exists x \subseteq a)(x \cap \sigma = 0 \equiv f_n(x) \cap \sigma = 0)$.

Let $\{r_s\}_{s\in\omega}$ be the list of conditions. We construct two disjoint infinite semisets $\sigma^+, \sigma^- \subseteq a$ such that each σ such that $\sigma^+ \subseteq \sigma$, $\sigma \cap \sigma^- = 0$ satisfies our conditions. At each step we put finitely many elements into σ^+ and σ^-. σ^+_s denotes the finite set of elements put into σ^+ before step s, similarly for σ^-_s. (1_n) is assured as in the proof of 2.3. To assure $(2_n) = r_s$ we look for the first $x \subseteq a$ such that $x \nsubseteq \sigma^-_s$ and $f_n(x) \nsubseteq \sigma^-_s$; if we have such x, we let $\sigma^+ \cap x \neq 0$ and $\sigma^+ \cap f(x) \neq 0$. If there is no such x we do nothing at step s. We show that (2_n) is assured in this case. If $x \subseteq a$ and x has more elements than σ^-_s then $f_n(x) \subseteq \sigma^-_s$. In particular, take an $x \subseteq \sigma^-$ with more elements than σ^-_s. This x satisfies (2_n).

2.6. Corollary (**TSS**). *If there is a support τ and a 1-1-mapping $\varrho \leqslant_D \tau$ of a onto $\mathbf{P}(\mathbf{P}(a))$ then there is a $\sigma \subseteq a$ which is not a support.*

2.7. Remark. (1) Cf. 2.12 below.

(2) If $\sigma \subseteq a$ is not a support then there is a $\sigma^* \subseteq \mathbf{P}(a)$ which is not a disjoint support and hence $\mathbf{P}(a) - \sigma^* \not\leqslant_D \sigma^*$. This contrasts with R-degrees since if A^C denotes the complement of A (A a set of natural numbers) then $A^C \leqslant_R A$. One could say that D-degrees should be compared with red-degrees rather than with R-degrees; but cf. 2.14 below.

2.8. Discussion. We want to compare D-degrees with C-degrees of Set theory. In this case we can even ask if C-degrees (of semisets) are reasonably definable in the theory of semisets. Since our **TSS**$_1$ is fully conservatively extensible to **TS**, $\mathrm{Mcl}_{S3}(\mathbf{M})$, we have a **TSS**-formula $\tau_1 \leqslant_C \tau_2$ such that

$$\mathbf{TS}, \mathrm{Mcl}_{S3}(\mathbf{M}) \vdash (\tau_1 \leqslant_C \tau_2)^\square \equiv \tau_1^\square \in \mathbf{M}[\tau_2^\square],$$

where $\mathbf{M}[\tau_2^\square] = \mathbf{Cstr}([\mathbf{M}, \tau_2^\square])$ is the smallest model class containing $\mathbf{M}$ (as a subclass) and $\tau_2^\square$ (as element) (cf. [7] Sect. III — 6). One readily proves in **TSS**$_1$ that $\leqslant_C$ is a quasi-ordering and other reasonable things. By 1.3, $\tau_1 \leqslant_C \tau_2$ can be expressed in two equivalent forms, namely,

$$(\forall \sigma)\big(\mathrm{TSupp}(\sigma) \to \tau_1 \leqslant_C^\sigma \tau_2\big) \quad \text{and} \quad (\exists \sigma)\big(\mathrm{TSupp}(\sigma)\ \&\ \tau_1 \leqslant_C^\sigma \tau_2\big),$$

where $\tau_1 \leqslant_C^\sigma \tau_2$ is a normal formula with three free variables τ_1, σ, τ_2. This enables us to define $\leqslant_C$ reasonably also in (**TSS**, BS) (and hence e.g. in **TSS**$_2$). We are lead to the following definition 2.9; Lemma 2.10 shows that our definition is reasonable.

2.9. Definition (**TSS**, BS). $\tau_1 \leqslant_C \tau_2 \equiv (\forall \sigma)\big(\mathrm{Supp}(\sigma)\ \&\ \tau_1, \tau_2 \leqslant_D \sigma . \to \tau_1 \leqslant_C^\sigma \tau_2\big)$ ($\tau_1 \leqslant_C^\sigma \tau_2$ is the formula described in 2.8).

2.10. LEMMA (TSS, BS). *If* σ_1, σ_2 *are supports and if* $\tau_1, \tau_2 \leqslant_D \sigma_1, \sigma_2$ *then* $\tau_1 \leqslant_C^{\sigma_1} \tau_2 \equiv \tau_1 \leqslant_C^{\sigma_2} \tau_2$. (*And, consequently,* $\tau_1 \leqslant_C \tau_2$ *is equivalent to* $(\exists\sigma)(\mathrm{Supp}(\sigma) \,\&\, \tau_1, \tau_2 \leqslant_D \sigma \,\&\, \tau_1 \leqslant_C^{\sigma} \tau_2$.)

Proof. By (BS), take a support σ such that $\sigma_1, \sigma_2 \leqslant_D \sigma$; we prove our assertion for σ_1 and σ instead of σ_1 and σ_2. Let $\boldsymbol{\sigma}_1 \leqslant_D \boldsymbol{\sigma}$ be fixed and consider the following theories:

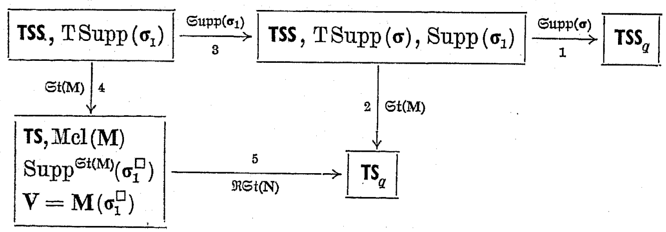
$$\mathbf{TSS}_q = (\mathbf{TSS}, \mathrm{Supp}(\boldsymbol{\sigma}_1) \,\&\, \mathrm{Supp}(\boldsymbol{\sigma}) \,\&\, \boldsymbol{\sigma}_1 \leqslant_D \boldsymbol{\sigma})$$

(the theory of semisets in question),

$$\mathbf{TS}_q = (\mathbf{TS}, \mathrm{Mcl}(\mathbf{M}), \mathrm{Supp}^{\mathfrak{St}(\mathbf{M})}(\boldsymbol{\sigma}^{\square}), \mathbf{V} = \mathbf{M}[\boldsymbol{\sigma}^{\square}], \mathrm{Supp}^{\mathfrak{St}(\mathbf{M})}(\boldsymbol{\sigma}_1^{\square}), \mathbf{N} = \mathbf{M}[\boldsymbol{\sigma}_1^{\square}])$$

(the theory of sets in question).

In the following diagram arrows 2 and 4 are faithful models and, by [7] 6313, the diagram commutes:



By the definition of $\leqslant_C$, we have

$$\mathbf{TS}_q \vdash (\tau_1 \leqslant_C^{\sigma} \tau_2)^{\mathfrak{St}(\mathbf{M})} \equiv \tau_1 \in \mathbf{M}[\tau_2]$$

and

$$\mathbf{TS}_q \vdash (\tau_1 \leqslant_C^{\sigma_1} \tau_2)^{\mathfrak{Supp}(\sigma_1) * \mathfrak{St}(\mathbf{M})} \equiv (\tau_1 \leqslant_C^{\sigma} \tau_2)^{\mathfrak{St}(\mathbf{M}) * \Re\mathfrak{St}(\mathbf{N})}$$
$$\equiv (\tau_1 \in \mathbf{M}[\tau_2])^{\Re\mathfrak{St}(\mathbf{N})} \equiv \tau_1 \in \mathbf{M}[\tau_2]$$

(the last equivalence holds because relative constructibility is a notion absolute in $\Re\mathfrak{St}(\mathbf{N})$); so we have

$$\mathbf{TS}_q \vdash (\tau_1 \leqslant_C^{\sigma} \tau_2)^{\mathfrak{St}(\mathbf{M})} \equiv (\tau_1 \leqslant_C^{\sigma_1} \tau_2)^{\mathfrak{Supp}(\sigma_1) * \mathfrak{St}(\mathbf{M})}.$$

By the faithfulness of arrow 2 and by absoluteness of normal formulas in the support model, we obtain

$$\mathbf{TSS}, \mathrm{TSupp}(\boldsymbol{\sigma}), \mathrm{Supp}(\boldsymbol{\sigma}_1) \vdash (\tau_1 \leqslant_C^{\sigma} \tau_2) \equiv (\tau_1 \leqslant_C^{\sigma_1} \tau_2)^{\mathfrak{Supp}(\sigma_1)} \equiv (\tau_1 \leqslant_C^{\sigma_1} \tau_2),$$

which implies

$$\mathbf{TSS}_q \vdash (\tau_1 \leqslant_C^{\sigma} \tau_2) \equiv (\tau_1 \leqslant_C^{\sigma_1} \tau_2) \qquad \text{(arrow 1 and absoluteness)}.$$

This completes the proof.

Obviously, $\tau_1 \leqslant_D \tau_2$ implies $\tau_1 \leqslant_C \tau_2$ (in each reasonable theory); we are now able to show that the converse in general does not hold.

2.11. LEMMA $(\mathbf{TSS}_1, \mathbf{V} = \mathbf{L})$. *If* $2^{2^{\aleph_0}} \mathrel{\hat{\approx}} \aleph_0$ *then there are* $\tau_1 \subseteq a$ *and* τ_2 *such that* $\tau_2 \leqslant_C \tau_1$ *but* $\tau_2 \nleqslant_D \tau_1$.

Proof. We work in the conservative extension of $\mathbf{TSS}_1, \mathbf{V} = \mathbf{L}$ to set theory obtained by 1.3, i.e. in $\mathbf{TS}, (\exists\sigma^{\square} \subseteq \mathbf{L})(\mathrm{TSupp}^{\square}(\sigma^{\square}))$. Let $\sigma^{\square}$ be a support for $\mathbf{V}$ over $\mathbf{L}$ (cf. [7] 6316); then $\mathbf{V} = \mathbf{L}[\sigma^{\square}]$ by [7] 6311. By 2.6, there is a $\tau_1^{\square} \subseteq \mathbf{L}$ such that $\tau_1^{\square}$ is not a support. Put $M = \mathbf{L}[\sigma_1^{\square}]$; by [7] 6320, there is a support $\tau_2^{\square}$ for M over $\mathbf{L}$, hence $M = \mathbf{L}[\sigma_2^{\square}]$, $\tau_1^{\square} \equiv_C^{\square} \tau_2^{\square}$, $\tau_1^{\square} \leqslant_D^{\square} \tau_2^{\square}$ but necessarily $\tau_2^{\square} \nleqslant_D^{\square} \tau_1^{\square}$ since otherwise $\tau_1^{\square}$ would be a support.

2.12. Remark. (1) I was told another example by B. Balcar: work in TS and suppose that there is a support for $\mathbf{V}$ over $\mathbf{L}$ on the collapsing algebra $(\mathbf{Coll}_0^1)^{\square}$. Hence $\mathbf{V} = \mathbf{L}[\tau_2^{\square}]$ for a support $\tau_2^{\square}$ which is a 1-1 mapping of $\aleph_1^{\mathbf{L}}$ onto $\aleph_2^{\mathbf{L}}$. Then there is a $\tau_1^{\square} \subseteq \omega \times \omega$ which is an ordering of ω of the type $\aleph_1^{\mathbf{L}}$ and such that $\tau_1^{\square} \equiv_C^{\square} \tau_2^{\square}$. But $\tau_1^{\square}$ cannot be a support for $\mathbf{V}$ over $\mathbf{L}$ (hence, $\tau_1^{\square} \equiv_D^{\square} \tau_2^{\square}$ is impossible), since otherwise an analysis of [1] would show that there is a Boolean support for $\mathbf{V}$ over $\mathbf{L}$ which is a [complete ultrafilter on an algebra satisfying the countable chain condition]$^{\square}$, which would imply absoluteness of cardinals, contradiction. Balcar's example is better than ours since its assumption $(2^{\aleph_0} \mathrel{\hat{\approx}} \aleph_0)$ is weaker than ours $(2^{2^{\aleph_0}} \mathrel{\hat{\approx}} \aleph_0)$ but we stress the method of proof rather than the result.

(2) One can formulate a corollary to 2.11 analogously to 2.4 and 2.6.

(3) The following lemma shows that τ_1 in 2.11 cannot be a support, i.e. there can be a semiset whose C-degree is bigger than its D-degree but the C-degree of each support coincides with its D-degree.

2.13. LEMMA (TSS, BS) *If* σ *is a support then* $(\forall\tau)(\tau \leqslant_C \sigma \equiv \tau \leqslant_D \sigma)$.

Proof. In $(\mathbf{TSS}_1)$ it follows by [7] 6315; the provability in $(\mathbf{TSS}_1)$ implies the provability in (TSS, BS) by our definition of $\leqslant_C$.

2.14. Remark. (1) The last theorem shows that if one wants to compare $\leqslant_C$ with $\leqslant_R$ and $\leqslant_D$ with $\leqslant_{\mathrm{red}}$ (cf. 2.7) then one has the following difference: in (TSS, BS) we have

$$(\forall\varrho)(\exists\sigma \geqslant_C \varrho)(\forall\tau)(\tau \leqslant_D \sigma \equiv \tau \leqslant_C \sigma)$$

by 2.13, but in Recursion theory one easily shows

$$(\forall A \geqslant_R \emptyset^*)(\exists B \leqslant_R A)(B \nleqslant_{\mathrm{red}} A).$$

(Put $n \in B \equiv \langle I_n, n\rangle \in \emptyset^* \,\&\, A([I_n](n)) = 0$; denotation as in [5].)

(2) As far as existence of maximal degrees is concerned one sees immediately that in **TSS**$_1$ one has a largest degree (namely the degree of any total support) and that in **TSS**$_2$ there is no maximal degree.

§ 3. Lower and upper bounds. Obviously, each pair of D-degrees has a supremum (the supremum of $\mathrm{dg}_D(\sigma)$ and $\mathrm{dg}_D(\varrho)$ is $\mathrm{dg}_D\big((\{0\}\times\sigma)\cup\cup(\{1\}\times\varrho)\big)$).

Our aim is to prove the following

3.1. THEOREM (**TSS**$_1$). *If $2^{2^{\aleph_0}} \mathrel{\hat{\approx}} \aleph_0$ then there are supports σ, π whose D-degrees have no infimum.*

3.2. COROLLARY (**TSS**). *If there is a support τ such that there is a 1-1 mapping of $2^{2^{\aleph_0}}$ onto $\aleph_0$ dependent on τ then there is a pair of support D-degrees without an infimum. In particular,* (AS) *implies the existence of a pair of D-degrees without infimum.*

Our proof of 3.1 will be a transfer of the proof of the Kleene-Post-Spector theorem in [5] Sect. 9 (pp. 43–45) and will also yield an example of an ascending countable sequence of D-degrees without a supremum.

We need some preliminaries.

3.3. Recall the notion of a separatively ordered set ([7] 2439) and of a complete ultrafilter on a separatively ordered set ([7] 4247). By [7] 2442, 4239 and 4250, separatively ordered sets are just bases of complete Boolean algebras and complete ultrafilters on a separatively ordered set are just restrictions of complete ultrafilters on the corresponding Boolean algebra. Recall the definition [7] 2521 of the product $b_1 \odot b_2$ of two separatively ordered sets and the (slightly different) definition [7] 2515 of the product $\prod_{x\in s} b$ of a system of separatively ordered sets (over the ideal of all finite subsets of s).

3.4. LEMMA (**TSS**). *Let b_1, b_2 be separatively ordered sets with greatest elements; a semiset ϱ is a complete ultrafilter on $b_1 \odot b_2$ iff there are $\varrho_1 \subseteq b_1$ and $\varrho_2 \subseteq b_2$ such that ϱ_i is a complete ultrafilter on b_i $(i = 1, 2)$ and $(\forall\tau)(\tau \leqslant_D \varrho_2\ \&\ \tau$ is dense on $b_1 . \to \tau \cap \varrho_1 \neq 0)$.*

This is a reformulation of Solovay's [6] 2.3; the proof in **TSS** is a routine reformulation of Solovay's proof and is left to the reader.

3.5. In the sequel, b denotes the usual base of the Cantor algebra, i.e.

$$b = \{f;\ \mathrm{Un}(f)\ \&\ \mathbf{D}(f) \subsetneq \omega\ \&\ \mathbf{D}(f) \text{ finite } \&\ \mathrm{W}(f) \subseteq \{0,1\}\}$$

and

$$f \leqslant g \equiv \mathbf{D}(g) \subseteq \mathbf{D}(f)\ \&\ \big(\forall x \in \mathbf{D}(g)\big)(f'x \leqslant g'x)$$

(cf. [7] 6101). (The greatest element of b is the empty function 0.) Let $\dot{b}$ be $b - \{0\}$ and put $b_n = \dot{b}$ for $n \in \omega$. Put $b^n = \prod_n b_i$ $(n > 0)$ and $b^\omega = \prod_\omega b_i$.

(It can be shown that each b^n and b^ω is isomorphic to b.) By [7] 2522, b^{n+1} is isomorphic to $b^n \odot b$. The ordering b has evidently the following homogeneity property:

3.6. LEMMA (**TSS**). *If σ is a complete ultrafilter on b and if $u \in b$ then there is an automorphism p of b such that $p''\sigma \ni u$.*

We prove our 3.1 in the following form:

3.7. LEMMA (**TSS**$_1$). *If $\mathbf{P}\big(\mathbf{P}(\omega)\big) \mathrel{\hat{\approx}} \omega$ then there are supports σ, π on b such that*

$$(*) \qquad \neg(\exists\xi)(\forall\tau)(\tau \leqslant_D \xi \equiv . \tau \leqslant_D \sigma\ \&\ \tau \leqslant_D \pi)\,.$$

Proof. By [7] 5321, there is a complete ultrafilter ϱ^ω on b^ω. Note that (under a suitable representation) each b_n and b^n are substructures of b^ω. Put $\varrho_n = \varrho^\omega \cap b_n$ and $\varrho^n = \varrho^\omega \cap b^n$; then ϱ_n is a complete ultrafilter on b_n, ϱ^n is a complete ultrafilter on b^n and ϱ^{n+1} is isomorphic (and hence similar) to $\varrho^n \times \varrho_n$. Furthermore, we have the following:

$$(**) \qquad i \leqslant n \equiv \varrho^i \leqslant_D \varrho^n \quad (1 \leqslant i, n \leqslant \omega)\,.$$

We construct supports (complete ultrafilters) σ, π on b^ω such that

$$\text{(i)} \qquad (\forall n \in \omega)(\varrho^n \leqslant_D \sigma\ \&\ \varrho^n \leqslant \pi)$$

and

$$\text{(ii)} \qquad (\forall\tau)(\tau \leqslant_D \sigma\ \&\ \tau \leqslant_D \pi . \to (\exists n \in \omega)(\tau \leqslant_D \varrho^n)\,.$$

Evidently, (i) and (ii) imply (*) by (**). Note that the quantifier $(\forall\tau)$ in (ii) can be restricted to subsemisets of $\mathbf{P}(b^\omega)$ by 2.1; evidently, $b^\omega \approx \aleph_0$. Hence we have to satisfy the following conditions:

$$(1_{r_1 r_2}) \qquad \tau = r_1''\sigma = r_2''\pi \to (\exists n \in \omega)(\tau \leqslant_D \varrho^n)$$

(r_1, r_2 antimonotone relations on $\mathbf{P}(b^\omega) \times b^\omega$),

$$(2_q) \qquad \sigma \cap q \neq 0 \neq \pi \cap q$$

(q dense in b^ω).

Let $\{c_s\}_{s\in\omega}$ be a list of all the conditions. We construct σ and π in countably many steps; the sth step will assure c_s. In step s we define automorphisms f_s and g_s of b_s and elements $u_s, v_s \in b^\omega$ (markers) such that putting $\sigma_s = f_s''\varrho_s$, $\pi_s = g_s''\varrho_s$, $\sigma^0 = \pi^0 = \emptyset$, $\sigma^{s+1} = \sigma^s \times \sigma_s$, $\pi^{s+1} = \pi^s \times \times \pi_s$, $\sigma = \bigcup \sigma^s$ and $\pi = \bigcup \pi^s$ then $\{u_s\}_{s\in\omega}$, $\{v_s\}_{s\in\omega}$ are descending sequences of elements of b^ω, $u_s \in \sigma$ and $v_s \in \pi$ for each s and σ, π satisfy our conditions.

We now describe the construction. Let $\mathrm{FIP}(\tau)$ mean that τ has the finite intersection property, i.e. $\tau \subseteq b^\omega$ and for each finite set $e \subseteq \tau$ there is a $u \in b^\omega$ with $(\forall x \in e)(u \leqslant x)$. We first describe the choice of u_s and v_s and then the choice of f_s and g_s.

(i) Definition of u_s and v_s.

(a) c_s is $(1_{r_1 r_2})$.

Case 1. There are $u \leqslant u_{s-1}$ and $v \leqslant v_{s-1}$ such that $\mathrm{FIP}(\sigma^s \cup \{u\})$, $\mathrm{FIP}(\pi^s \cup \{v\})$ and there is an x such that either

$$x \in r_1''\{u\} \;\&\; (\forall w \leqslant v)(x \notin r_2''\{w\})$$

or

$$x \in r_2''\{v\} \;\&\; (\forall w \leqslant u)(x \notin r_1''\{w\}) .$$

(Say, $u_{-1} = v_{-1} = 1_{b^\omega}$.) Then we pick such u and v (using a list of b^ω) and put $u_s = u$ and $v_s = v$.

Case 2. Otherwise. Put $u_s = u_{s-1}$ and $v_s = v_{s-1}$.

(b) c_s is (2_q). Pick a $u_s \leqslant u_{s-1}$ such that $u_s \leqslant w$ for some $w \in q$; pick a v_s analogously.

(ii) Definition of f_s, g_s. Let $u_s \in b^k$ for some $k > s$; then u_s can be represented as $\langle u^0, u^1, u^2\rangle$ where $u^0 \in b^s$, $u^1 \in b_s$ and $u^2 \in b' = b_{s+1} \odot \ldots \odot b_k$. $\mathrm{FIP}(\sigma^s \cup \{u\})$ implies $u^0 \in \sigma^s$. By 3.6, pick an automorphism f_s of b_s such that $f_s''\varrho_s \ni u^1$. If we put $\sigma_s = f_s''\varrho_s$ then we have $\mathrm{FIP}(\sigma^s \times \sigma_s \cup \{u_s\})$. We find g_s similarly. (We use a list of automorphisms of b^ω and hence again use the assumption $\mathbf{P}(\mathbf{P}(\omega)) \hat{\approx} \omega$.)

We have $u_s \in \sigma$ and $v_s \in \pi$ for each s. We show that our conditions are satisfied. First, let c_s be $(1_{r_1 r_2})$ and let $\tau = r_1''\sigma = r_2''\pi$. Then we have Case 2 for the step s. We prove

$$x \in \tau \equiv (\exists u \leqslant u_{s-1})\big(\mathrm{FIP}(\sigma^s \cup \{u\}) \;\&\; x \in r_1''\{u\}\big) .$$

The implication $\rightarrow$ is trivial. Conversely, let $x \in r_1''\{u\}$ for a $u \leqslant u_{s-1}$ such that $\mathrm{FIP}(\sigma^s \cup \{u\})$ and suppose $x \notin r_1''\sigma$. Then $x \notin r_2''\pi$ and, by [7] 4301, there is a $v \in \pi$ such that $(\forall w \leqslant v)(r_2''\{w\} \not\ni x)$. We can suppose $v \leqslant v_{s-1}$ and hence we have Case 1, which is a contradiction. Hence our τ is definable from σ^s by a normal formula and since σ^s is a support we obtain $\tau \leqslant_D \sigma^s \equiv_D \varrho^s$. This shows that c_s is satisfied.

Finally, if c_s is (2_q) then c_s is satisfied since $u_s \in \sigma$ and $u_s \leqslant w$ for some $w \in q$; this means that $q \cap \sigma \neq 0$. Similarly for v_s and π. This completes the proof.

§ 4. Applications to Set theory.

We use the extensibility of $\mathbf{TSS}_1$ to a set theory to obtain some results in **TS** concerning model-classes. Recall that M is a *model-class with* E1 $(\mathrm{Mcl}_{\mathrm{E1}}(M))$ if M is a model-class such that for each $x \in M$ there is a $f \in M$ which is a bijection of x onto some ordinal number. $\mathbf{V}$ is the greatest model-class and $\mathbf{L}$ is the least model-class;

$\mathbf{L}$ is a model-class with E1. (Hence if $\mathbf{V} = \mathbf{L}$ there is only one model-class.) For the sake of simplicity we restrict ourselves to supports over $\mathbf{L}$. Theorem 3.1 has the following

4.1. Corollary (**TS**). *If* $\mathbf{V}$ *has a support* $\varrho^\square$ *over* $\mathbf{L}$ *and if* $\aleph_2^{\mathbf{L}} \approx \aleph_0$ *then there are supports* $\sigma^\square$, $\pi^\square$ *over* $\mathbf{L}$ *such that* $\mathbf{L}[\sigma^\square] \cap \mathbf{L}[\pi^\square]$ *is not a model-class with* E1.

Proof. Let $\sigma^\square$ and $\pi^\square$ be as in 3.1 (where we additionally assume $\mathbf{V} = \mathbf{L}$). Put $M_1 = \mathbf{L}[\sigma^\square]$, $M_2 = \mathbf{L}[\pi^\square]$. If $M_1 \cap M_2$ were a model-class with E1 then, by [7] 6320, $M_1 \cap M_2$ would have a support $\xi^\square$ over $\mathbf{L}$, i.e. $M_1 \cap M_2 = \mathbf{L}[\xi^\square]$ for some $\xi^\square \subseteq \mathbf{L}$. This $\xi^\square$ would be the infimum of $\sigma^\square$ and $\pi^\square$ (since by 2.13 $(\forall \tau^\square)(\tau^\square \leqslant_C^\square \sigma^\square \equiv \tau^\square \leqslant_D^\square \sigma^\square)$ and the same holds for $\pi^\square$ and for $\xi^\square$).

4.2. Remark. Consequently, if **TS** is consistent then one cannot prove in **TS** that the intersection of two model-classes with E1 is a model-class with E1. (Since $(\mathbf{TSS}_1, \mathbf{V} = \mathbf{L}, \aleph_2 \hat{\approx} \aleph_0)$ is consistent relative to **TS** by [7] 5218 and 6123.) One has the following problem: can we prove in **TS** that the intersection of two model-classes is a model-class? In particular, in the situation of 4.1, is $M_1 \cap M_2$ a model-class (necessarily without E1)? I have not succeeded to solve this problem. Nevertheless, an analysis of the situation just mentioned shows that $M_1 \cap M_2$ contains a model-class without E1 as a subclass. (The question is whether it is a proper subclass.) So we obtain as a by-product a demonstration of the independence of the axiom of choice; this independence is known by Cohen's proof and hence the result is by no means new; but the method of proof could be of some interest since we do not use any symmetry arguments. I note here that Theorem 4.4 was independently proved by B. Balcar; Balcar constructed the same model-class using my proof of 3.7.

4.3. Lemma (**TS**). *If* N_n *is a model-class for each* $n \in \omega$ *and if* $n < m$ *implies* $N_n \subset N_m$ *for each* $n, m \in \omega$ *then* $\bigcup_n N_n$ *is not a model-class.*

Proof. Let p_α denote the set of all sets of rank $\leqslant \alpha$. Put $N = \bigcup N_n$. If N were a model-class then, by [7] 3225, we would have $p_\alpha \cap N \in N$ for each α. Then there would be an n such that $p_\alpha \cap N \in N_n$ for almost all α; this would imply $N = N_n$, which is a contradiction.

4.4. Theorem (**TS**). *If* $\mathbf{V}$ *has a support* $\varrho \subset \mathbf{L}$ *over* $\mathbf{L}$ *and if* $\aleph_2^{\mathbf{L}} \approx \aleph_0$ *then there is a model-class* K *without axiom of choice.*

Proof. Let $\sigma, \pi, \sigma_n, \pi_n, \sigma^n, \pi^n$ be as in 3.7 and put $N_n = \mathbf{L}[\sigma^n] = \mathbf{L}[\pi^n]$, $M_1 = \mathbf{L}[\sigma]$ and $M_2 = \mathbf{L}[\pi]$. Then the classes N_n form a strictly increasing sequence of model-classes (with E1) and $N = \bigcup N_n$ is a subclass of $M_1 \cap M_2$. For each n, let $[\sigma_n] = \{\tau; (\exists f \in L)\,(f$ automorphism of $b_n^{\mathbf{L}} \;\&\; \tau = f''\sigma_n)\}$ and similarly for $[\pi_n]$. By [7] 4327, $[\sigma_n] = [\pi_n]$ for each n. Hence $\{[\sigma_n]\}_{n \in \omega} \in M_1 \cap M_2$. Put $K = \mathbf{L}\big[\{[\sigma_n]\}_{n\in\omega}\big]$ (i.e. $K = \mathbf{Cstr}(\{[\sigma_n]\}_{n\in\omega})$;

K is the least model-class containing $\{[\sigma_n]\}_{n\in\omega}$ as element). Then $K\subseteq M_1\cap\cap M_2$ and $N\subseteq K$. We show that $\neg\mathrm{Mcl}_{\mathrm{E1}}(K)$. Suppose the contrary. Then K contains a selector for $\{[\sigma_n]\}_{n\in\omega}$, i.e. there are $\tau_n\in[\sigma_n]$ such that $\{\tau_n\}_{n\in\omega}\in K$. The system $\{\tau_n\}_{n\in\omega}$ is representable as a subset τ of $\mathbf{L}$ (as an exact functor see [7] 1408) and is in $M_1\cap M_2$. Consequently, $\tau\leqslant_D^{\square}\sigma$ and $\tau\leqslant_D^{\square}\pi$ and hence there is an n such that $\tau\in N_n$. But $\{[\sigma_n]\}_{n\in\omega}\leqslant_C\tau$ and hence $K\subseteq N_n$ which contradicts to $N\subseteq K$.

4.5. Remark. (1) The set $\{[\sigma_n]\}_{n\in\omega}$ can be represented as a sequence of disjoint sets of subsets of ω and hence one has a countable disjointed system of sets of reals without a selector.

(2) K is the least model-class with N as a subclass and we have $N\neq K$ by 4.3. Is $K=M_1\cap M_2$? Can one obtain K as a "symmetric submodel of a support extension of $\mathbf{L}$"?

(3) An analysis shows that the assumption "$\mathbf{V}$ has a set support over $\mathbf{L}$" in 4.4 can be weakened to the Gödel's form of the axiom of choice (E2 of [7]).

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MATHEMATICAL INSTITUTE OF THE CZECHOSLOVAK ACADEMY OF SCIENCES
Prague, Czechoslovakia

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Elementary interpretations of negationless arithmetic

by

E. G. K. López-Escobar (College Park, Maryland)

Abstract. Some systems of negationless arithmetic (in the spirit, if not the form of Griss' negationless mathematics) are introduced and their relation to intuitionism are considered.

§ 1. The negationless mathematics of G. F. C. Griss

1.1. Few would disagree with Griss' criterion that all the well-formed parts of a meaningful formula should also be meaningful. Unfortunately there is plenty of room for disagreement on the meaning of "meaningful". Although it may well be impossible to determine what Griss had in mind, two aspects of this interpretation are (more or less) evident: namely that for a formula to have meaning it is necessary that it have a constructive interpretation and that it be satisfiable.

If one accepts such a condition on the notion of meaningful and still adheres to the principle that all the well-formed parts of a meaningful formula should also be meaningful, then one finds that the propositional connectives "or", "if ... then" and "it is not the case that" cause a lot of problems. For example the sentence

$$\mathbf{0}=\mathbf{0}\vee\mathbf{0}=\mathbf{1}$$

could not possibly have any meaning for Griss, since if it had then so would $\mathbf{0}=\mathbf{1}$ and the latter could not have any meaning for him since it is not satisfiable.

The connective "$\rightarrow$" is even more problematic. Already the impredicative aspect of the intuitionistic interpretation of "$\rightarrow$" leaves much to be desired, and if to the intuitionistic interpretation one adds Griss' criterion, the formulae such as

$$\mathbf{0}=\mathbf{1}\rightarrow\mathbf{1}=\mathbf{1}\quad\text{and}\quad\mathbf{0}=\mathbf{0}\rightarrow\mathbf{0}=\mathbf{1}$$

are banished from mathematics. What is worst still is that a sentence of the form $\forall x Ax$ may have meaning and yet $A\mathbf{n}$ may be meaningless for certain numerals $\mathbf{n}$, for example let $Ax\equiv(x=\mathbf{0}\rightarrow x=\mathbf{0})$.