On expansions of β-models

by

J. Bílý and **L. Bukovský** (Košice)

Abstract. We prove the following generalization of a theorem by P. Zbierski: *a minimal β-model of the second order arithmetic (with the axiom of choice) cannot be expanded to a model of the third order arithmetic (without the axiom of choice)*. In the proof, a generic extension of the β-model is used.

P. Zbierski [5] has shown that a minimal β-model of the second order arithmetic (with the axiom of choice) cannot be expanded to a model of third order arithmetic (with the axiom of choice). In this note, we show that the main obstruction, which does not allow to expand such a model, is the comprehension scheme. Main tool, we shall use for the proof, is a generic extension, which is closely related to a construction given by A. Mostowski [2]. Before proving the main result of this note we shall briefly discuss the notion of a generic extension in the "semiset" style, as it is done for the set theory by P. Vopěnka and P. Hájek [4].

§ 1. Several systems of higher order arithmetics. The language $\mathfrak{L}_n$ $(n \geqslant 2)$ consists of n sorts of variables $x_1, y_1, z_1, u_1, \ldots;$ $x_2, y_2, z_2, u_2, \ldots;$ $x_n, y_n, z_n, u_n, \ldots$, two binary predicates $=$ (equality), ε (membership relation), two operations over the first sort $+$, $\cdot$ and two constants $0, 1$ of the first sort. The basis system BS_n of the nth order arithmetic contains the Peano's axioms (for the first sort), the extensionality axioms

$$(\forall x_i)(x_i \varepsilon x_{i+1} \equiv x_i \varepsilon y_{i+1}) \equiv x_{i+1} = y_{i+1}, \qquad i = 1, 2, \ldots, n-1,$$

and the axioms $(i, j \leqslant n, i \neq j+1)$:

$$\neg x_j \varepsilon x_i,$$

(compare [3]).

For every $j, k \leqslant n$. we can define the pairing function $I_{j,k}(x_j, y_k) = z_l$, $l = \max\{j, k\}$ (more precisely: we can find a formula which defines the pairing function) in the usual way:

$$I_{1,1}(x_1, y_1) = z_1 \quad \text{as} \quad 2^{x_1}\cdot(2y_1+1) = z_1+1\,,$$

$$I_{1,2}(x_1, y_2) = z_2 \quad \text{as} \quad (\forall y_1)\big(y_1 \varepsilon z_2 \equiv (\exists z_1)(z_1 \varepsilon y_2 \;\&\; y_1 = I_{1,1}(x_1, z_1))\big)\,,$$

etc.

The operation $\{x_l\}$ is defined by the formula

$$(\forall y_l)(y_l \varepsilon \{x_l\} \equiv y_l = x_l)\,.$$

If $l, k \leqslant n$, $h = \max\{l, k\}$, we denote by $\mathrm{Im}_{l,k}(z_h, x_l) = y_k$ the formula

$$(\forall y_{k-1})\big(y_{k-1} \varepsilon y_k \equiv (\exists x_{l-1})(x_{l-1} \varepsilon x_l \;\&\; I_{l-1,k-1}(x_{l-1}, y_{k-1}) \varepsilon z_h)\big)\,.$$

($\mathrm{Im}_{l,k}(z_h, x_l)$ is the image of x_l by the binary relation z_h).

Thus, e.g. $\mathrm{Im}_{2,2}(z_2, \{x_1\})$ is $z_2^{(x_1)}$ in the usual notation (compare [2], [3]).

An l, k-formula ($l, k \leqslant n$) is a formula which contains only variables of the sorts less than $l+1$ and every bounded variable is of the sort $j \leqslant k$. We shall need the following axioms:

$\mathrm{Comp}_{l,k}$ (scheme of l, k-comprehension): if φ is an l, k-formula, $j \leqslant l$, $j < n$, x_{j+1} does not occur in φ, then the following formula is an axiom

$$(\exists x_{j+1})(\forall z_j)(z_j \varepsilon x_{j+1} \equiv \varphi)\,.$$

$\mathrm{AC}_{l,k}$ (scheme of l, k-choice, $k \geqslant 2$, $l < n$, $k \leqslant n$): for any formula φ, $h = \max\{l+1, k\}$, the following formula is an axiom

$$(\forall x_l)(\exists y_k)\varphi(x_l, y_k, \ldots) \rightarrow (\exists z_h)(\forall x_l)\varphi\big(x_z, \mathrm{Im}_{l+1,k}(z_h, \{x_l\}), \ldots\big)\,.$$

Let $\mathrm{Bord}_j(x_j)$, $1 < j \leqslant n$ denote the formula saying that "x_j is a well-ordering relation" (Bord_2 is defined in [3], p. 85). We denote by W_j the formula

$$(\exists x_{j+1})\Big(\mathrm{Bord}_{j+1}(x_{j+1}) \;\&\; (\forall x_j)(\exists y_j)\big(I_{j,j}(x_j, y_j) \varepsilon x_{j+1} \vee I_{j,j}(y_j, x_j) \varepsilon x_{j+1}\big)\Big)\,.$$

Thus, W_j says that "the jth universe can be well ordered" ($j < n$!).

The system A_n introduced by Zbierski [5] is equivalent to $\mathrm{BS}_n + \mathrm{Comp}_{n,n} + \mathrm{AC}_{n-1,n}$. A_n^- will denote the system $\mathrm{BS}_n + \mathrm{Comp}_{n,n}$.

Let us remark some connections between the axioms of choice:

(1) $\quad A_n^- \vdash W_k \rightarrow \mathrm{AC}_{l,k} \qquad (l, k < n)\,,$

(2) $\quad A_n^- \vdash \mathrm{AC}_{l-1,l} \rightarrow W_{l-1} \qquad (l < n)\,,$

(3) $\quad A_n^- \vdash \mathrm{AC}_{l-1,l} \rightarrow \mathrm{AC}_{k,l-1} \qquad (l, k < n)\,.$

(1) is evident (using $\mathrm{Comp}_{n,n}$). The implication in (2) can be proved in the same way as the Zermelo theorem in the set theory (the proof given in Bourbaki [1], III. 2.3, may be literally translated for A_n^-). (3) follows from (1) and (2).

We shall consider only ω-models of BS_n (or other systems). Thus, if $\mathfrak{M} = \langle U_1, \ldots, U_n, =, \epsilon, +, \cdot, 0, 1\rangle$ is a model of BS_n, then U_1 is the set of natural numbers N, $U_2 \subseteq P(N)$ ($P(x)$ is the power set of x), $U_3 \subseteq P(U_2)$, ..., "ϵ" is the real membership relation and $=, +, \cdot, 0, 1$ have the usual arithmetical meaning.

A set $X \subseteq U_n$ is definable in $\mathfrak{M}$, if there is a formula φ and parameters $a_1 \epsilon U_1, \ldots, a_n \epsilon U_n$ such that $X = \{a;\ \mathfrak{M} \vDash \varphi(a, a_1, \ldots, a_n)\}$.

We denote by $\mathrm{Def}_{\mathfrak{M}}$ the set of all definable subsets of U_n.

It is well known (proof by induction) that

(4) if $\mathfrak{M} = \langle U_1, \ldots, U_n\rangle$ is a model of A_n^-, then the expansion $\mathfrak{M}^* = \langle U_1, \ldots, U_n, \mathrm{Def}_{\mathfrak{M}}\rangle$ is a model of $\mathrm{BS}_{n+1} + \mathrm{Comp}_{n+1,n}$.

If $\mathfrak{M} = \langle U_1, \ldots, U_n\rangle$ is an ω-model of BS_n, then a function $f\colon U_j \times U_k \rightarrow U_l$ corresponds to the "function" $I_{j,k}$, $l = \max\{j, k\}$. We shall denote this function f by the same letter $I_{j,k}$. Similarly for $\mathrm{Im}_{j,k}$. Moreover, the function $\mathrm{Im}_{j,k}$ may be defined also for $X \subseteq U_{j-1}$, $X \notin U_j$ as

$$\mathrm{Im}_{j,k}(Y, X) = \{Z \epsilon U_{k-1};\ I_{l-1,k-1}(T, Z) \epsilon Y \text{ for some } T \epsilon X\}\,.$$

§ 2. Generic extensions. Generic extensions of the higher order arithmetics have been studied by several authors. Our aim is to present this extensions in the "semiset" style, which seems to us more natural and simple. We shall investigate a special case, which is needed for the generalization of Zbierski theorem. A general investigation will be published eventually elsewhere.

Let $\mathfrak{M} = \langle U_1, \ldots, U_n\rangle$ be an ω-model of BS_n. We say that $X \subseteq U_i$ is $\mathfrak{M}$-dependent on $Y \subseteq U_j$ ($\mathrm{Dep}_{\mathfrak{M}}(X, Y)$) iff there exists a set $Z \epsilon U_{l+1}$, $l = \max\{i, j\}$, such that $X = \mathrm{Im}_{j+1,i+1}(Z, Y)$, i.e.

$$X = \{a \epsilon U_i;\ \text{there exists a } b \epsilon Y \text{ such that } I_{j,i}(b, a) \epsilon Z\}\,.$$

A set $X \subseteq U_i$ is said to be an $\mathfrak{M}$-*support* iff there exist a formula ψ and elements $a_1 \epsilon U_1, \ldots, a_n \epsilon U_n$ such that for every $l = 1, \ldots, n$ the following holds true:

(5) if $Y = \mathrm{Im}_{i+1,l+1}(Z, X)$, $Y \subseteq U_l$, then $U_l - Y = \mathrm{Im}_{i+1,l+1}(Z', X)$, where $Z' = \{a;\ \mathfrak{M} \vDash \psi(a, Z, a_1, \ldots, a_n)\}$.

An $\mathfrak{M}$-support $X \subseteq U_i$ is said to be *closed* iff for every $Y \epsilon U_{i+1}$, $\mathrm{Im}_{i+1,i}(Y, X) \epsilon U_i$. Thus, roughly speaking, if X does not produce new sets of the type i.

If $X \subseteq U_i$ is an $\mathfrak{M}$-support, one can construct the generic extension $\mathfrak{M}[X]$. However, the construction is generally rather complicated. We shall investigate a special case only: $i = n-1$ and X is closed. In this case, the generic extension $\mathfrak{M}[X] = \langle U_1[X], \ldots, U_n[X]\rangle$ is defined as follows:

a) $U_i[X] = U_i$ for $i < n$,

b) $U_n[X] = \{Y \subseteq U_{n-1};$ there exists a $Z \in U_n$ such that

$$Y = \mathrm{Im}_{n,n}(Z, X)\}\,.$$

Now, we can formulate a result necessary for our investigation:

(6) Let $\mathfrak{M}$ be an ω-model of A_n^-. Let $X \subseteq U_{n-1}$ be a closed $\mathfrak{M}$-support. Then $\mathfrak{M}[X]$ is an ω-model of A_n^-.

It is easy to see that $\mathfrak{M}[X]$ is a model of the basic system BS_n. We show that the comprehension scheme $\mathrm{Comp}_{n,n}$ holds true in $\mathfrak{M}[X]$.

We shall say that a set $Z \in U_j$ codes the set $Y \in U_j[X]$, written $Z = \overline{Y}$ iff either $j < n$ and $Z = Y$ or $j = n$ and $Y = \mathrm{Im}_{n,n}(Z, X)$.

Let $\psi, a_1, \ldots, a_n$ be those of the definition of an $\mathfrak{M}$-support. For every formula $\varphi(a, A, B, \ldots)$ we construct a formula

$$\overline{\varphi}(a, a_1, \ldots, a_n, \overline{A}, \overline{B}, \ldots)$$

such that

(7) if $Y = \{a \in U_{j-1}[X];\ \mathfrak{M}[X] \models \varphi(a, A, B, \ldots)\}$ then $\{a \in U_{j-1};\ \mathfrak{M} \models \overline{\varphi}(a, a_1, \ldots, a_n, \overline{A}, \overline{B}, \ldots)\}$ codes Y.

The construction of $\overline{\varphi}$ is given by the induction:

(i) φ is an atomic formula, i.e. φ is $a \varepsilon x_i$, $a = x_i$ or $x_i \varepsilon a$. We set:

$\overline{a \varepsilon x_i}$ is $a \varepsilon x_i$,

$\overline{a = x_i}$ is $a = x_i$ for $i < n-1$,

$\overline{a = x_{n-1}}$ is $(\exists y_{n-1})\big(a = I_{n-1,n-1}(y_{n-1}, x_{n-1})\big)$,

$\overline{x_i \varepsilon a}$ is $x_i \varepsilon a$ for $i \leqslant n-3$,

$\overline{x_{n-2} \varepsilon a}$ is $(\exists x_{n-1})(\exists y_{n-1})\big(a = I_{n-1,n-1}(x_{n-1}, y_{n-1})\ \&\ x_{n-2} \varepsilon y_{n-1}\big)$.

(ii) $\overline{\varphi_1 \vee \varphi_2}$ is $\overline{\varphi_1} \vee \overline{\varphi_2}$,

(iii) $\overline{(\exists x_i)\varphi}$ is $(\exists x_i)\overline{\varphi}$,

(iv) $\overline{\neg\varphi}$ is $\neg\overline{\varphi}$ for $j < n$ (i.e. if a is a variable of a sort less than $n-1$). Finally,

$\overline{\neg\varphi(x_{n-1}, A, B, \ldots)}$ is

$(\forall x_n)\big[(\forall y_{n-1})\big(y_{n-1} \varepsilon x_n \equiv \overline{\varphi}(y_{n-1}, \overline{A}, \overline{B}, \ldots)\big) \rightarrow \psi(x_{n-1}, x_n, a_1, \ldots, a_n)\big]$.

Now, one can easily show that (6) holds true. Since $\mathfrak{M}$ is a model of $\mathrm{Comp}_{n,n}$ we have

$$\{a \in U_{j-1};\ \mathfrak{M} \models \overline{\varphi}(a, a_1, \ldots, a_n, \overline{A}, \overline{B}, \ldots)\} \in U_j \quad \text{for} \quad \overline{A}, \overline{B}, \ldots \in U_l\,.$$

Since this set codes Y, we have $Y \in U_j[X]$.

§ 3. Generalization of Zbierski theorem. The promised result is expressed by the

THEOREM. *If $\mathfrak{M}$ is a minimal β-model of A_n, then there exists no $U \subseteq P(U_n)$ such that $\overline{\mathfrak{M}} = \langle U_1, \ldots, U_n, U\rangle$ is a model of A_{n+1}^-.*

Proof of the theorem is by reductio ad absurdum. Assume, that there exists a set $U \subseteq P(U_n)$ such that $\overline{\mathfrak{M}} = \langle U_1, \ldots, U_n, U\rangle$ is a model of A_{n+1}^-.

Evidently $\overline{\mathfrak{M}} \models W_{n-1}$ (by (2)).

Let $\mathrm{Cond} \subseteq U_n$ be set of those non-empty elements of U_n, which code a well ordering (in a natural way) of a subset of U_n (for $n = 2$, the exact definition is given by A. Mostowski [2], p. 225). Cond is ordered by the relation "$x < y$" which says that "the well ordering x is an initial segment of the well ordering y". Since Cond and $<$ are $\mathfrak{M}$-definable, we have $\mathrm{Cond}, < \in U$. Let $X \subseteq \mathrm{Cond}$ be an $\overline{\mathfrak{M}}$-generic set (for $n = 2$, the exact definition is given by A. Mostowski [2], p. 227). Then we have

1) X is an $\overline{\mathfrak{M}}$-support,

2) X is closed,

3) $\overline{\mathfrak{M}}[X] \models W_n$.

For $n = 2$, the assertions 2) and 3) have been proved in [2]. We shall not repeat the proofs for $n > 2$.

We show 1). If we set a_{n+1} to be $<$ then the formula $\psi(a, Z, a_1, \ldots, a_{n+1})$ from the definition of an $\overline{\mathfrak{M}}$-support reads as follows:

$$(\exists x_n)(\exists y_n)\Big(a = I_{n,n}(x_n, y_n)\ \&\ (\forall z_n)\big(I_{n,n}(z_n, y_n) \in Z \rightarrow \neg(\exists u_n)(u_n > z_n\ \&\ u_n > x_n)\big)\Big)\,.$$

If $Y = \mathrm{Im}_{n+1,n+1}(Z, X)$, then for every $b \in U_n$ the set

$$A = \{a;\ I_{n,n}(a, b) \in Z \vee (\forall x_n)\big(I_{n,n}(x_n, b) \in Z \rightarrow \neg(\exists y_n)(y_n > x_n\ \&\ y_n > a)\big)\}$$

is dense. Therefore $A \cap X \neq \emptyset$. Hence, there exists an $a \in A$ such that $a \in X$. Now, either $I_{n,n}(a, b) \in Z$ and $b \in \mathrm{Im}_{n+1,n+1}(Z, X)$, or

$$\overline{\mathfrak{M}} \models \psi(I_{n,n}(a, b), Z, a_{n+1})$$

and

$$b \in U_n - \mathrm{Im}_{n+1,n+1}(Z, X) = \mathrm{Im}_{n+1,n+1}(Z', X)$$

where

$$Z' = \{c;\ \overline{\mathfrak{M}} \models \psi(c, Z, a_{n+1})\}\,.$$

P. Zbierski [5] uses a form of downward Skolem-Lövenheim theorem for A_{n+1}. It is easy to see that the same results holds true for the system $\mathrm{BS}_{n+1} + \mathrm{Comp}_{n+1,n+1} + W_n$. The strong comprehension scheme $\mathrm{Comp}_{n+1,n+1}$ is needed for "the definition of truth". Using W_n one can define Skolem functions and prove (for notation see Zbierski [5] p. 561):

(8) $$\mathrm{BS}_{n+1} + \mathrm{Comp}_{n+1,n+1} + W_n \vdash (\forall Y)$$
$$(Y \text{ is a model of } A_n \rightarrow (\exists a)(a \in Y \ \&\ a \prec Y))\,.$$

Now, the result follows directly: applying the Skolem-Lövenheim theorem (8) in $\overline{\mathfrak{M}}[X]$ we obtain a $b \in U_n$, $b \prec U_n$. Thus b codes a β-model — a contradiction with the minimality of $\mathfrak{M}$.

Remark. By (4), $\mathfrak{M}^* = \langle U_1, \ldots, U_n, \mathrm{Def}_{\mathfrak{M}}\rangle$ is a model of $\mathrm{BS}_{n+1} + \mathrm{Comp}_{n+1,n} + W_{n-1}$. The generic extension

$$\mathfrak{M}^*[X] = \langle U_1, \ldots, U_n, \mathrm{Def}_{\mathfrak{M}}[X]\rangle$$

is a model of $\mathrm{BS}_{n+1} + \mathrm{Comp}_{n+1,n} + W_n$. Thus, we may conclude, that the Skolem-Lövenheim theorem (8) is not provable in $\mathrm{BS}_{n+1} + W_n + \mathrm{Comp}_{n+1,n}$. By further analysis, we may conclude that "the truth cannot be defined" in this system.

References

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Некоторые простые следствия аксиомы конструктивности

С. Р. Когаловский (Иваново)

Abstract. Addison's lemma on Δ_2^1 well orderability of $P(w)$ is generalized (in the supposition V = L) to any infinite structures (Theorem 2.10) that gives a possibility of transferring a lot of well-known results referring to the standard model of arithemetics to any infinite structures. In the article there are considered some "direct" applications of Theorem 2.10. So A. Tarsky's question on definability of the definability notion is settled for any infinite structure. There is found a relation between semantic completeness and cathegoricity for sentences of the 2nd higher orders. There is proved isomorphism of denumerable elementary structures of the finite signature satisfying the same sentences of the class V_2^1.

В этой статье рассмотрения проводятся в рамках ZF + V = L.

Порядком структуры $\mathfrak{S}$ мы называем ординал $\mathrm{Od}\,\mathfrak{S} = \min\{\mathrm{Od}'\mathfrak{S}' : \mathfrak{S}' \simeq \mathfrak{S}\}$, а порядком отношения a^τ (типа τ) в $\mathfrak{S}$ — ординал $\mathrm{Od}\,\mathfrak{S}(a^\tau)$, где $\mathfrak{S}(a^\tau)$ — структура, образованная из $\mathfrak{S}$ добавлением a^τ к множеству определяющих отношений $\mathfrak{S}$. Мы доказываем, что для всякой бесконечной элементарной структуры $\mathfrak{S}$ и всякого типа $\tau = (0, \ldots 0)$ отношение $\mathrm{Od}\,\mathfrak{S}(a^\tau) < \mathrm{Od}\,\mathfrak{S}(b^\tau)$ определимо в $\mathfrak{S}$ как формулой из V_2^1, так и формулой из Δ_2^1. Этот результат обобщается на случай произвольных бесконечных структур и произвольных типов τ (теорема 2.10), что создает возможность перенесения на произвольные бесконечные структуры известных результатов, относящихся к стандартной модели арифметики, в частности, результатов Аддисона [1], связанных со свойствами проективных иерархий, результатов, связанных с вопросом Тарского [11] об определимости понятия определимости, и т. д. В статье рассмотрены лишь некоторые из „непосредственных" теоретико модельных приложений теоремы 2.10, именно-следующие.

1. Пользуясь тем, что отношение квазипорядка $\mathrm{Od}\,\mathfrak{S}(a^\tau) < \mathrm{Od}\,\mathfrak{S}(b^\tau)$ в бесконечной структуре $\mathfrak{S}$ индуцирует отношение вполне упорядочения на множестве $\mathrm{In}_\tau(\mathfrak{S})$ отношений типа τ, инвариантных относительно автомор-