On expansions of $\beta$-models
by
J. Bílý and L. Bukovský (Košice)

Abstract. We prove the following generalization of a theorem by P. Zbierski: a minimal $\beta$-model of the second order arithmetic (with the axiom of choice) cannot be expanded to a model of the third order arithmetic (without the axiom of choice). In the proof, a generic extension of the $\beta$-model is used.

P. Zbierski [5] has shown that a minimal $\beta$-model of the second order arithmetic (with the axiom of choice) cannot be expanded to a model of third order arithmetic (with the axiom of choice). In this note, we show that the main obstruction, which does not allow to expand such a model, is the comprehension scheme. Main tool, we shall use for the proof, is a generic extension, which is closely related to a construction given by A. Mostowski [2]. Before proving the main result of this note we shall briefly discuss the notion of a generic extension in the "semisets" style, as it is done for the set theory by P. Vopenka and P. Hájek [4].

§ 1. Several systems of higher order arithmetics. The language $\mathcal{L}_n (n \geq 2)$ consists of $n$ sorts of variables $x_1, y_1, z_1, u_1, \ldots; x_2, y_2, z_2, u_2, \ldots; x_n, y_n, z_n, u_n, \ldots,$ two binary predicates $=$ (equality), $\in$ (membership relation), two operations over the first sort $\ast, \cdot$ and two constants $0, 1$ of the first sort. The basis system $\mathcal{BS}_n$ of the $n$th order arithmetic contains the Peano's axioms (for the first sort), the extensionality axioms

$$(\forall i)(x_i \in \mathcal{A} \iff x_i \in \mathcal{A}'),$$

and the axioms ($i, j \leq n, i \neq j + 1$):

$$\neg \exists ! x_i \exists x_1,$$

(compare [3]).

For every $j, k \leq n$, we can define the pairing function $I_{jk}(x_j, y_k) = x_j$, $l = \max (j, k)$ (more precisely, we can find a formula which defines

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the pairing function) in the usual way:

$$I_{a1}(a_1, a_2) = a_1$$ as \(2^a \cdot (2^b + 1) = a + 1\),

$$I_{a1}(a_1, a_2) = a_1$$ as \(\forall y_1, x_2 (x_1, y_1 = I_{a1}(a_1, a_2))\),

etc.

The operation \(a_3\) is defined by the formula

$$\forall y_1 (y_1 \in a_3) = y_2 = a_2$$.

If \(i, k < n, h = max (i, k)\), we denote by \(I_{i,k}(a_i, a_k) = y_k\) the formula

$$\forall y_k (y_k \in a_k) = y_1 \in a_1$$.

The image of \(a_j\) by the binary relation \(a_k\).

Thus, e.g. \(I_{a1}(a_1, a_1) = \varphi\) in the usual notation (compare [2], [3]).

An \(i, k\)-formula \(i, k < n\) is a formula which contains only variables of the sorts less than \(i + 1\) and ever bounded variable is of the sort \(j < k\).

We shall need the following axioms:

- \text{Comp}_{a_2} (scheme of \(i, k\)-comprehension): if \(\varphi\) is an \(i, k\)-formula, \(j < i, j < n, a_{i+1}\) does not occur in \(\varphi\), then the following formula is an axiom
  $$\exists a_{i+1} (\forall y_i) (y_i \in a_{i+1} \rightarrow \varphi).$$

- \text{AC}_{a_2} (scheme of \(i, k\)-choice, \(k \geq 2, i < n, k < n\); for any formula \(\varphi, h = max (i + 1, k)\), the following formula is an axiom
  $$\exists a_{i+1} (\forall y_i) (y_i \in a_{i+1} 
  \forall (y_j, y_k) (y_j \in a_{i+1} \rightarrow \varphi)),$$

Let \(\text{Bord}_a(a_i, a_j)\), \(1 \leq j < i\) denote the formula saying that "\(a_j\) is a well-ordering relation" (Bord is defined in [3], p. 85). We denote by \(W_i\) the formula

$$\exists a_{i+1} (\forall y_i) (\forall y_j, y_k) (y_j \in a_{i+1} \rightarrow (y_j \in a_{i+1} \rightarrow ((y_k \in a_{i+1} \rightarrow y_k = y_j))).$$

Thus, \(W_i\) says that "the \(j\)th universe can be well ordered" \((j < n)\).

The system \(A_n\) introduced by Zbierski [2] is equivalent to \(B_{n-k} + \text{Comp}_{a_2} + \text{AC}_{a_2}\). \(A_n\) will denote the system \(B_{n-k} + \text{Comp}_{a_2}\).

Let us remark some connections between the axioms of choice:

1. \(A_{n} + W_{k} \rightarrow \text{AC}_{a_2}\)
2. \(A_{n} + \text{AC}_{a_2} \rightarrow W_{i+1}\)
3. \(A_{n} + \text{AC}_{a_2} \rightarrow \text{AC}_{a_2}\)

(1) is evident (using \(\text{Comp}_{a_2}\)). The implication in (2) can be proved in the same way as the Zermelo theorem in the set theory (the proof given in Bourbaki [1], III, 2.3, may be literally translated for \(A_{n}\)). (3) follows from (1) and (2).

We shall consider only \(\omega\)-models of \(B_{n}\) (or other systems). Thus, if \(\mathfrak{M} = \langle U_1, \ldots, U_n, =, \in \rangle\) is a model of \(B_{n}\), then \(U_i\) is the set of natural numbers \(N_i\), \(U_i \subseteq P(\mathbb{N})\) \(P(\mathbb{N})\) is the power set of \(\mathbb{N}\), \(U_i \subseteq P(U_i), \ldots, "a" \) is the real membership relation and \(\ldots, =, \in \) have the usual arithmetical meaning.

A set \(X \subseteq U_i\) is definable in \(\mathfrak{M}\) if there is a formula \(\varphi\) and parameters \(a_i \in U_i\) such that \(X = \{a_i \in U_i \mid \varphi(a_i, a_1, \ldots, a_n)\}\).

We denote by \(\text{Def}_{\mathfrak{M}}\) the set of all definable subsets of \(U_i\).

It is well known (proof by induction) that

(4) if \(\mathfrak{M} = \langle U_1, \ldots, U_n \rangle\) is a model of \(A_{n}\), then the expansion \(\mathfrak{M}' = \langle U_1, \ldots, U_n, \text{Def}_{\mathfrak{M}} \rangle\) is a model of \(B_{n+1} + \text{Comp}_{a_2}\).

If \(\mathfrak{M} = \langle U_1, \ldots, U_n \rangle\) is an \(\omega\)-model of \(B_{n}\), then a function \(f: U_i \times \bigcup U_i \rightarrow U_{k+1}\) corresponds to the "function" \(I_{k+1}, l = max (j, k)\). We shall denote this function \(f\) by the same letter \(f_{k, l}\). Similarly for \(f_{n, k}\). Moreover, the function \(f_{k, l}\) may be defined also for \(X \subseteq U_{k+1}, X \notin U_i\) as

$$f_{k, l}(X, X) = \{z \in U_{k+1}, f_{k, l}(X, Z) \in X \text{ for some } Z \in X\}.$$

§ 2. Generic extensions. Generic extensions of the higher order arithmetics have been studied by several authors. Our aim is to present this extensions in the "semist" style, which seems to us more natural and simple. We shall investigate a special case, which is needed for the generalization of Zbierski theorem. A general investigation will be published eventually elsewhere.

Let \(\mathfrak{M} = \langle U_1, \ldots, U_n \rangle\) be an \(\omega\)-model of \(B_{n}\). We say that \(X \subseteq U_i\) is \(\mathfrak{M}\)-dependent on \(Y \subseteq U_i\) \(\text{Def}_{\mathfrak{M}}(X, Y)\) iff there exists a set \(Z \in U_{l+1}\), \(l = max (i, j)\), such that \(X = \text{Im}_{l+1} \text{Def}_{\mathfrak{M}}(Z, X)\), i.e.

$$X = \{a \in U_i \mid \text{there is a } b \in Y \text{ such that } f_{l+1}(a, b) \in Z\}.$$
If \( X \subseteq U_i \) is an \( \mathcal{M} \)-support, one can construct the generic extension \( \mathcal{M}[X] \). However, the construction is generally rather complicated. We shall investigate a special case only: \( i = n-1 \) and \( X \) is closed. In this case, the generic extension \( \mathcal{M}[X] = \langle U_i[X], \ldots, U_n[X] \rangle \) is defined as follows:

- \( U_i[X] = U_i \) for \( i < n \),
- \( U_n[X] = \{ Y \subseteq U_{n-1} \mid \text{there exists a } Z \in U_n \text{ such that} \}
- \( Y = \text{Im}_{n,a}(Z, X) \).

Now, we can formulate a result necessary for our investigation:

(6) Let \( \mathcal{M} \) be an \( \omega \)-model of \( A^*_n \). Let \( X \subseteq U_{n-1} \) be a closed \( \mathcal{M} \)-support. Then \( \mathcal{M}[X] \) is an \( \omega \)-model of \( A^*_n \).

It is easy to see that \( \mathcal{M}[X] \) is a model of the basic system \( \text{BS}_n \). We show that the comprehension scheme \( \text{Comp}_{n,a} \) holds true in \( \mathcal{M}[X] \).

We shall say that a set \( Z \subseteq U_i \) codes the set \( Y \subseteq U_i[X] \), written \( Z = Y \), iff either \( j < n \) and \( Z = Y \) or \( j = n \) and \( Y = \text{Im}_{n,a}(Z, X) \).

Let \((a, a_1, ..., a_n, A, B, ...)\) be those of the definition of an \( \mathcal{M} \)-support. For every formula \( \varphi(a, A, B, ...) \) we construct a formula

\[
\varphi(a, a_1, ..., a_n, A, B, ...)
\]

such that

(7) if \( Y = \{ a \in U_{n-1}[X] \mid \mathcal{M}[X] \models \varphi(a, A, B, ...) \} \) then \( \{ a \in U_{n-1} \mid \mathcal{M} \models \varphi(a, a_1, ..., a_n, A, B, ...) \} \) codes \( Y \).

The construction of \( \varphi \) is given by the induction:

(i) \( \varphi \) is an atomic formula, i.e. \( \varphi \) is \( a \equiv a_1 \), \( a = x \), \( a \in \varphi \). We set:

- \( a \in a_1 \) is \( a = a_1 \),
- \( a = a_i \) is \( a = a_i \) for \( i < n-1 \),
- \( a = a_{n-1} \) is \( a = a_{n-1} \) for \( i < n-1 \),
- \( a = a_{n-1} \) is \( a = a_{n-1} \) for \( i < n-1 \),
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- \( a = a_{n-1} \) is \( a = a_{n-1} \) for \( i < n-1 \).

(ii) \( \varphi \lor \varphi \) is \( \varphi \lor \varphi \).

(iii) \( (\exists a \varphi) \) is \( (\exists a \varphi) \).

(iv) \( \varphi \) is \( \neg \varphi \) for \( j < n \) (i.e. if \( a \) is a variable of a sort less than \( n-1 \)).

Finally,

\[
\neg \varphi(a_{n-1}, A, B, ...) \text{ is } \neg \varphi(a_{n-1}, A, B, ...)
\]

\[
(V_{a_n})[(\forall y_n)(y_n \in a_{n-1} \equiv \varphi(y_n, A, B, ...)] \rightarrow \varphi(a_{n-1}, a_n, a_1, ..., a_n).
\]

Now, one can easily show that (6) holds true. Since \( \mathcal{M} \) is a model of \( \text{Comp}_{n,a} \) we have

\[
\{ a \in U_{n-1} \mid \mathcal{M} \models \varphi(a, a_1, ..., a_n, A, B, ...) \} \subseteq U_i \text{ for } A, B, ..., \in U_i.
\]

Since this set codes \( Y \), we have \( Y \in U_i[X] \).

§ 3. Generalization of Zbierski theorem. The promised result is expressed by the theorem:

Theorem. If \( \mathcal{M} \) is a minimal \( \beta \)-model of \( A_n \), then there exists no \( U \subseteq P(U_n) \) such that \( \mathcal{M} = \langle U_1, ..., U_n, U \rangle \) is a model of \( A_{n+1}^* \).

Proof of the theorem is by reductio ad absurdum. Assume, that there exists a set \( U \subseteq P(U_n) \) such that \( \mathcal{M} = \langle U_1, ..., U_n, U \rangle \) is a model of \( A_{n+1}^* \).

Evidently \( \mathcal{M} \models W_{n-1} \) by (2)).

Let \( \text{Cond} \subseteq U_n \) be set of those non-empty elements of \( U_n \), which code a well ordering (in a natural way) of a subset of \( U_n \) (for \( n = 2 \), the exact definition is given by A. Mostowski [2], p. 225). \( \text{Cond} \) is ordered by the relation \( "x < y" \) which says that "the well ordering \( x \) is an initial segment of the well ordering \( y" \). Since \( \text{Cond} \) and \( < \) are \( \mathcal{M} \)-definable, we have \( \text{Cond} \subseteq U \). Let \( X \subseteq \text{Cond} \) be an \( \mathcal{M} \)-generic set for \( n = 2 \), the exact definition is given by A. Mostowski [2], p. 227). Then we have

1) \( X \) is an \( \mathcal{M} \)-support,
2) \( X \) is closed,
3) \( \mathcal{M}[X] \models W_n \).

For \( n = 2 \), the assertions 2) and 3) have been proved in [2]. We shall not repeat the proofs for \( n > 2 \).

We show 1). If we set \( a_{n+1} \) to be \( a \) then the formula \( \varphi(a, Z, a_1, ..., a_{n+1}) \) from the definition of an \( \mathcal{M} \)-support reads as follows:

\[
\mathcal{M} \models \exists a \in a_{n+1} \forall y_n \forall y_{n+1} \left( a = \text{Im}_{n,a}(x_n, y_n) \& \forall z_n \left( \text{Im}_{n,a}(x_n, y_n) \& a \in Z \rightarrow \neg (\exists a \in a_{n+1} \forall y_n \forall y_{n+1} \left( a = \text{Im}_{n,a}(x_n, y_n) \& a \in Z \rightarrow \neg (\exists y_{n+1} (y_n > a) \& a \in a_{n+1}) \right) \right) \right)
\]
and
\[ b \in U_n - \text{Im}_{n+1, n+2}(Z, X) = \text{Im}_{n+1, n+2}(Z', X) \]
where
\[ Z' = \{ c : \exists Z \in c, a \in Z \} \].

P. Zbierski [5] uses a form of downward Skolem-Löwenheim theorem for \( A_{n+1} \). It is easy to see that the same results holds true for the system \( \text{BS}_{n+1} + \text{Comp}_{n+1, n+2} + W_n \). The strong comprehension scheme \( \text{Comp}_{n+1, n+2} \) is needed for "the definition of truth". Using \( W_n \) one can define Skolem functions and prove (for notation see Zbierski [5] p. 561):

\[ \text{BS}_{n+1} + \text{Comp}_{n+1, n+2} + W_n \vdash (\forall Y) \]
[\( Y \) is a model of \( A_n \rightarrow (\forall a)(a \in Y \& a < Y) \)].

Now, the result follows directly: applying the Skolem-Löwenheim theorem (8) in \( \mathfrak{M}[X] \) we obtain a \( b \in U_n, b < U_n \). Thus \( b \) codes a \( \beta \)-model — a contradiction with the minimality of \( U_n \).

Remark. By (4), \( \mathfrak{M} = \langle U_1, ..., U_n, \text{Deff} \rangle \) is a model of \( \text{BS}_{n+1} + \text{Comp}_{n+1, n+2} + W_{n-1} \). The generic extension
\[ \mathfrak{M}^* = \langle U_1, ..., U_n, \text{Defm} \rangle \]
is a model of \( \text{BS}_{n+1} + \text{Comp}_{n+1, n+2} + W_n \). Thus, we may conclude, that the Skolem-Löwenheim theorem (8) is not provable in \( \text{BS}_{n+1} + \text{Comp}_{n+1, n+2} + \text{Comp}_{n+1, n} \). By further analysis, we may conclude that "the truth cannot be defined" in this system.

References


Reçu par la Redaction le 29. 9. 1973

Некоторые простые следствия аксиомы конструирования

C. Р. Когаловский (Иваново)

Abstract. Addison's lemma on \( A \) well orderability of \( P(a) \) is generalized (in the supposition \( V = L \)) to any infinite structures (Theorem 2.10) that gives a possibility of transferring a lot of well-known results referring to the standard model of arithmetics to any infinite structures. In the article there are considered some "direct" applications of Theorem 2.10. So A. Tarsky's question on definability of the definability notion is settled for any infinite structure. There is found a relation between semantical completeness and categoricity for sentences of the 2nd higher order. There is proved isomorphism of denumerable elementary structures of the finite signature satisfying the same sentences of the class \( V_2 \).

В этой статье рассмотрения проводятся в рамках \( ZF + V = L \).

Порядковые структуры \( \mathfrak{S} \) мы называем ординалом \( \text{Ord} \mathfrak{S} = \min \text{Ord} \mathfrak{S}' \); \( \mathfrak{S}' \equiv \mathfrak{S} \), а порядком отношения \( a' \) (типа \( t \)) в \( \mathfrak{S} \) — ординал \( \text{Ord} \mathfrak{S}(a') \), где \( \mathfrak{S}(a') \) — структура, образованная из \( \mathfrak{S} \) добавлением \( a' \) к множеству определяющих отношений \( \mathfrak{S} \). Мы доказываем, что для всякой бесконечной элементарной структуры \( \mathfrak{S} \) и всякого типа \( t = \langle 0, ..., 0 \rangle \) отношение \( \text{Ord} \mathfrak{S}(a') < \text{Ord} \mathfrak{S}(b') \) определяемо в \( \mathfrak{S} \) как формулой из \( V_2 \), так и формулой из \( A_2 \). Этот результат обобщается на случай произвольных бесконечных структур и произвольных типов \( t \) (теорема 2.10), что создает возможность переопределения на произвольные бесконечные структуры известных результатов, относящихся к стандартной модели архимедовости, в частности, результатов Аддисона [3], связанных со свойствами пространств исчислений, результатов, связанных с вопросом Тарского [11] об определённости понятия определённости и т. д. В статье рассматриваются некоторые из "непосредственных" теорем моделей приложений теоремы 2.10, именно следующие.

1. Пусть, что отношение квазипорядка \( \text{Ord} \mathfrak{S}(a') < \text{Ord} \mathfrak{S}(b') \) в бесконечной структуре \( \mathfrak{S} \) монотонно отношение вполне упорядочено на множестве \( \text{Im}_{a}(\mathfrak{S}) \) отношений типа \( t \), инвариантных относительно автомор-