Prioric games and minimal degrees below $0^{(1)}$

by

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Abstract. The purpose of this paper is to present an analysis of one particular type of priority-argument in terms of what we call *prioric games*: one interesting feature of this analysis is that prioric games in no way involve recursion theory. The type of priority-argument referred to is that which involves $\mathbf{0}^{(1)}$-approximation by means of trees, and in full generality the framework which we devise is adequate for all known applications. A side-product of our investigation is a notion of $(C, \boldsymbol{d})$-priorcomeager set, which seems to be more restrictive than the notion of $(C, \boldsymbol{d})$-priorabundant set arising from prioric games. It is nevertheless adequate for classifying many sets: for example, $\{\boldsymbol{b}: \boldsymbol{b}$ is minimal & $\boldsymbol{b}^{(1)} > \boldsymbol{b} \cup \mathbf{0}^{(1)}\}$ is weakly $(C, \mathbf{0}^{(1)})$-priorcomeager for certain C; this yields a very recent result due to Sasso: there is a minimal degree $< \mathbf{0}^{(1)}$ whose jump is $> \mathbf{0}^{(1)}$.

It is easy to translate most of the basic existence-problems in the theory of degrees of unsolvability into questions of the form: is $\mathcal{A}$ non-empty? where $\mathcal{A}$ is a Borel subset of 2^{ω}. It has been known for some time that some of the more elementary questions of this type can be settled by proving that $\mathcal{A}$ is comeager (**) in the usual product topology on 2^{ω}, then using Baire's theorem. More recently, we have shown that far more of the general theory can be developed in this way than had hitherto been thought; this will appear in particular in our forthcoming book [16].

(*) Many of the ideas in this paper grew out of a seminar on the degrees of unsolvability which was held at the University of Colorado when I took sabbatical leave there in 1971/72. I would like to take this opportunity of thanking the members of that seminar for their enthusiasm and encouragement. The ideas were further developed in response to some lectures which I gave at the University of Illinois, Urbana, in May 1972, and I am particularly indebted to Professor Carl Jockusch Jr. for pressing me further towards a game-theoretic approach. I am also grateful to Professor Leonard Sasso Jr. for early notice of his results.

(**) I.e. residual, or complementary to a set of first category.

The wider development is made possible by allowing a variety of topologies other than the standard one and by formulating a constructive notion, that of a $(\boldsymbol{C}, \boldsymbol{d})$-comeager set, where $\boldsymbol{C}$ essentially describes a constructive neighbourhood system formed by adding new perfect sets to the old topology, and $\boldsymbol{d}$ is a degree of unsolvability bounding both that of $\boldsymbol{C}$ and that of the intersection operation in the formation of the comeager set concerned. The relevant existence-theorem then asserts that every $(\boldsymbol{C}, \boldsymbol{d})$-comeager set contains an element of degree $\leqslant \boldsymbol{d}$. As an example of an application of this method, it may be proved that the set of minimal degrees, and more generally the set of degrees $\boldsymbol{b}$ such that $\mathfrak{D}\,(\leqslant \boldsymbol{b})$ is isomorphic to a given finite distributive lattice, is $(\boldsymbol{C}, \boldsymbol{0}^{(2)})$-comeager for certain systems $\boldsymbol{C}$. The existence-theorem then implies the existence of a minimal degree below $\boldsymbol{0}^{(2)}$, or more generally that any finite distributive lattice may be embedded as an initial segment of $\mathfrak{D}\,(\leqslant \boldsymbol{0}^{(2)})$.

An alternative notion is the following: a set $\mathcal{A} \subseteq 2^\omega$ is $(\boldsymbol{C}, \boldsymbol{d})$-abundant if there is a winning strategy of degree $\leqslant \boldsymbol{d}$ for entering $\mathcal{A}$ in $G(\boldsymbol{C}; \mathcal{A})$, which is a straightforward generalization to the topology defined by $\boldsymbol{C}$ of the Banach-Mazur game associated with $\mathcal{A}$. This notion is not important for most of the basic applications because, for most useful $\boldsymbol{C}, \boldsymbol{d}$, it is equivalent to the more elegant notion of a $(\boldsymbol{C}, \boldsymbol{d})$-comeager set. This equivalence is proved by generalising a classical theorem of Banach concerning Banach-Mazur games. Nevertheless, there are some situations in which the game-theoretic notion is more intuitive and others in which it admits of appropriate refinements for which one can extract no obvious parallels from the notion of comeager set. Hence, even in the general theory one concludes that the most appropriate overall framework must be game-theoretic.

Now, the type of problem which is usually of most interest to a degree-theorist consists of constructivising existence-theorems; this invariably requires some sort of priority-argument. There are two types of problem here: (I) reducing degree and (II) improving the quantifier-form. The priority-method was first introduced to deal with a problem of type (II), namely Post's problem: Kleene and Post has exhibited the existence of Δ_2^0 incomparable degrees, and Friedberg and Mučnic independently devised the new method in order to replace Δ_2^0 by Σ_1^0. The first application of type (I) was Sacks' use of a priority-argument [3, 5] to improve Spector's construction of a minimal degree below $\boldsymbol{0}^{(2)}$ by replacing $\boldsymbol{0}^{(2)}$ by $\boldsymbol{0}^{(1)}$. Both of these applications involve priority-arguments which are fairly simple in comparison with much of the more recent work in the field, but they do serve as an appropriate starting point for analysing the priority-method. Our purpose here is to analyse those arguments of type (I) which involve a single approximation, such as in Saks' result mentioned above. There are of course now many results about the degrees below $\boldsymbol{0}^{(1)}$, beginning with our construction [12] of a minimal degree below an arbitrary Σ_1^0 degree, which involve an additional recursive approximation much more complex than those analysed here. We plan to treat these in a later paper. Although we shall not touch on problems of type (II) here, we should mention that finite-injury arguments can be treated in a framework which is similar to and in some ways simpler than the present one; this will be described in [15]. Again, most of the more difficult constructions lie outside the bounds of this framework, in this case because they need infinite-injury arguments. These we also plan to treat in a later paper.

We now turn to describing the layout of the present paper. After the introductory section on tree-systems and operators, we begin work in earnest with the study of the prioric game $PG(\boldsymbol{C}; \mathcal{A})$ in § 2. Then in § 3 we define a set $\mathcal{A} \subseteq 2^\omega$ to be $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant if there is a winning strategy of degree $\leqslant \boldsymbol{d}$ for entering $\mathcal{A}$ in $PG(\boldsymbol{C}; \mathcal{A})$; this is accompanied by a basic existence theorem which asserts that any such set contains an element of degree $\leqslant \boldsymbol{d}$. The next section is devoted to a rather technical discussion of what we call remedial strategies for $PG(\boldsymbol{C}; \mathcal{A})$. This is directed to showing that these strategies are well-behaved and do not forfeit the game in any trivial way; one reason for needing this section is that all the strategies used in applications are remedial. It would be possible at this point to prove that, for example, the set $\mathfrak{M}$ of minimal degrees is $(\boldsymbol{C}, \boldsymbol{0}^{(1)})$-priorabundant for certain $\boldsymbol{C}$. We choose, however, to devote § 5 to a notion of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set, showing in particular that every $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant. The converse of this result, which would provide an analogue of a classical theorem of Banach, does not appear to hold although the matter has not been finally settled. Another disadvantage of the notion of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set is that in order to prove that a given such set $\mathcal{A}$ contains an element of degree $\leqslant \boldsymbol{d}$ it is necessary to use the fact that $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant. Since it appears to be generally just as easy to prove directly that $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant, the latter procedure leads more quickly to existence-theorems in degree theory. For these and other reasons, it would seem that a game-theoretic framework is even more appropriate for priority-arguments than it is for the general theory. Nevertheless, in order to obtain the strongest possible classifications, we go on in § 6 to prove that, in particular, the set $\{\boldsymbol{b}\colon \boldsymbol{b} \text{ is minimal } \& \ \boldsymbol{b} \cup \boldsymbol{0}^{(1)} < \boldsymbol{b}^{(1)}\}$ is weakly $(\boldsymbol{C}, \boldsymbol{0}^{(1)})$-priorcomeager for certain $\boldsymbol{C}$; it follows immediately that there is a minimal degree $< \boldsymbol{0}^{(1)}$ whose jump is $> \boldsymbol{0}^{(1)}$. The trick behind the proof remains the same one used by Sasso [7] to first obtain the latter result, thus answering a question which we raised in [12]. Other applications are mentioned in the concluding § 7.

Finally, we survey the little notation which we need to assume, in addition to some general background knowledge from [2] and [5]. First, we assume that the basis of recursion theory has been set up on the hereditarily finite sets rather than the integers; this follows a trend initiated by Shoenfield [9]. In particular, a dominant role is played by the set $\boldsymbol{S}$ of *strings*, i.e. finite sequences of 0's and 1's. Small Greek letters will be used for denoting strings, with the exception of ω which as usual denotes the set of nonnegative integers. $\emptyset$ will be used to denote the empty string as well as the empty set. 0 and 1 will be used to denote the two strings of length 1 in addition to their normal use. If $\sigma, \tau \in \boldsymbol{S}$ then we write $\sigma \leqslant \tau$ to mean that σ is an initial segment of τ; $\sigma < \tau$ indicates that σ is a *proper* initial segment of τ. We also write $\sigma < B$ where $B \in 2^\omega$ to indicate that σ is an initial segment of B. $|\sigma|$ denotes the *length*, i.e. number of elements, of the string σ; $\sigma(n)$ denotes the $(n+1)$-st element of σ. $\sigma * \tau$ denotes the *concatenation* of σ and τ, i.e. the string formed by adding τ to the end of σ. σ and τ are said to be *compatible* if one is an extension of the other, and *incompatible* otherwise (written $\sigma|\tau$).

Relative recursiveness is treated as in [12], by using a uniformly recursive sequence $(\boldsymbol{F}_e)$ of recursive order-preserving $\boldsymbol{S}$-maps such that X is recursive in Y (written $X \leqslant_T Y$) if and only if $\boldsymbol{F}_e^*(Y) = X$ for some e, where $\boldsymbol{F}_e^*(Y) = \lim(\boldsymbol{F}_e(\sigma)\colon \sigma < Y)$. A string σ is $\boldsymbol{F}_e$-split by σ^0, σ^1 if $\sigma < \sigma^0$, $\sigma < \sigma^1$ and $\boldsymbol{F}_e(\sigma^0)|\boldsymbol{F}_e(\sigma^1)$. Also, we let $(\boldsymbol{R}_x^B)$ be a standard enumeration (Σ_1^0 in B) of the subsets of ω which are Σ_1^0 in B. Then as usual we can define $B^{(1)} = \{x\colon x \in \boldsymbol{R}_x^B\}$.

The degrees of unsolvability are the equivalence classes of 2^ω under $\equiv_T$, where $X \equiv_T Y$ if and only if $X \leqslant_T Y$ and $Y \leqslant_T X$. We denote the *degree* of $X \in 2^\omega$ by $\boldsymbol{X}$. In general, however, small boldface Roman letters are used to denote degrees. They are ordered by the relation $\leqslant$ induced by $\leqslant_T$, the resulting partial ordering being denoted by $\mathfrak{D}$. The *jump* operation is induced by the map $B \to B^{(1)}$; iterations of the jump are defined by setting $\boldsymbol{b}^{(n+1)} = (\boldsymbol{b}^{(n)})^{(1)}$. Initial and final segments of $\mathfrak{D}$ consisting of the degrees $\leqslant$ or $\geqslant$ a given degree $\boldsymbol{b}$ are denoted by $\mathfrak{D}(\leqslant \boldsymbol{b})$ and $\mathfrak{D}(\geqslant \boldsymbol{b})$ respectively. We shall frequently abuse notation by confusing a set of degrees, both with its set of constituent elements of 2^ω, and with the corresponding subordering of $\mathfrak{D}$.

§ 1. Tree systems and operators. There are numerous notions of tree in the literature. For the general theory of $\mathfrak{D}$ it is most natural to limit attention to what we call perfect trees, and these can be defined in many different ways even for the purposes of degree-theory. The following very general notion, however, best serves our purpose in studying priority-arguments involving trees.

1.1. Definition. A *tree* is any subset T of $\boldsymbol{S}$ which

(i) possesses a least element (denoted from hereon by $\mu(T)$),

(ii) is such that every element of T which has a proper extension in T has a pair of incompatible extensions in T.

A tree T' is defined to be a *subtree* of T if $T' \subseteq T$.

We think of a tree as being partially ordered by $\leqslant$. Note that trees may be finite, and that in fact even singleton trees are admitted although they do constitute a special case (see 1.6 below). $\boldsymbol{S}$ itself is of course a tree. Each tree T generates a subset $\mathcal{N}(T)$ of 2^ω, which may be loosely described as the set of all branches of T, but is more precisely defined in 1.3 below.

1.2. Definition. A subset X of a tree T is a *branch* of T if

(i) X is linearly ordered by $\leqslant$,

(ii) X is closed under predecessors in T,

(iii) whenever an element of X has a proper successor in T then it has one in X.

An element B of 2^ω is said to *lie on* T if $B = \lim X$ for some branch X of T.

1.3. Definition. For each tree T we define

$$\mathfrak{N}(T) = \{B\colon B \text{ lies on } T\}$$

and call $\mathfrak{N}(T)$ the *neighbourhood generated by* T.

The basic ordering between trees is $\supseteq$, and this clearly implies the same relationship between the corresponding neighbourhoods.

1.4. Definition. A tree T is *perfect* if every element of T has a pair of incompatible extensions in T.

Note that if T is perfect in the sense of 1.4 then $\mathfrak{N}(T)$ is perfect in the usual sense relative to the standard topology on 2^ω.

1.5. Definition. For each tree T and string $\tau \in T$ we define

$$T \wedge \tau = \{\sigma\colon \sigma \in T \;\&\; \tau \leqslant \sigma\}.$$

We sometimes refer to this tree as T *above* τ.

This brings us to an important concept, that of a tree system. Rather than discuss it in full generality we will restrict ourselves to the most important special case. We shall also introduce an extra aspect $\star(\boldsymbol{C})$ to our systems $\boldsymbol{C}$ which is unnecessary when we are dealing (as in [14] and Part I of [15]) only with perfect trees.

1.6. Definition. A *tree-system* $\boldsymbol{C}$ is a countable set of trees such that

(i) $\boldsymbol{S} \in \boldsymbol{C}$,

(ii) if $T \in \boldsymbol{C}$ and $\tau \in T$ then $T \wedge \tau \in \boldsymbol{C}$.

In addition, there is a distinguished subset $\star(C)$ of C whose only required property is that any singleton trees in C must all belong to $\star(C)$.

In most applications it is sufficient to let $\star(C)$ consist of the singleton trees, but in the last theorem of this paper it will be necessary to expand $\star(C)$ a little, although it will still be restricted to finite trees. The role of $\star(C)$ is to distinguish a subset of C containing trees which are too small to be useful for the application concerned.

In the general definition (used in [15] for example) one weakens condition (i) to assert the existence of a maximal perfect tree in C. This is necessary in particular for proving relativised results about $\mathfrak{D}$ $(\geqslant c)$ for arbitrary c, but we prefer to leave such technicalities to the interested reader since they will only confuse the main issues.

We shall restrict our applications to Σ_1^0 tree-systems, in other words to systems consisting of Σ_1^0 trees; in this case it is advisable to assume also that $\star(C)$ is of degree $\leqslant \mathbf{0}^{(1)}$: by this we mean that the indices of elements of $\star(C)$ form a subset of the set of all indices of Σ_1^0 trees (usually ω of course) which is of degree $\leqslant \mathbf{0}^{(1)}$. It is surprising how many subsets of 2^ω may be classified using arbitrary Σ_1^0 systems, although for various special cases it is necessary to introduce further closure conditions on the system C. There are two obvious extreme cases: the smallest system S^*, which consists of all trees of the form $S\wedge\sigma$ for arbitrary σ, and the maximal system M_1 which consists of *all* Σ_1^0 trees. In both cases, their $\star$ sets are taken to consist simply of their singleton trees and that is the purpose of the subscript in the second case; S^* of course contains no singleton trees and so $\star(S^*)$ is empty — we shall generally ignore $\star(C)$ for any perfect system for exactly this reason. We have elsewhere [14] devoted a fair amount of space to elucidating the interesting inverse relationship which exists between the complexity of an initial segment $\mathfrak{D}$ $(\leqslant b)$ and the size of the systems which can be used to construct b. In the present paper, our classifications will either involve all C or else be limited to fairly large C such as M_1 which satisfy closure conditions additional to 1.6 (ii); in the latter case, S^* will in particular be excluded.

Finally, we turn to defining the operators which will play such an important role in the sequal.

1.7. Definition. Let C be a Σ_1^0 tree-system. A (C, d)-*operator* $\Omega(e_1, \ldots, e_m;\ T^1, \ldots, T^m)$ is a many-valued map from $\omega^m\times C^{<\omega}$ into which has the following property: there is a function v: $\omega^m\times\omega^{<\omega}\to\omega$, which is partial recursive in d and whose domain is of degree $\leqslant d$, such that if $i_1, \ldots, i_n$ are indices of $T^1, \ldots, T^m$ as Σ_1^0 subsets of S then

$$v(e_1, \ldots, e_m;\ i_1, \ldots, i_n)$$

is an index of a value of $\Omega(e_1, \ldots, e_m;\ T^1, \ldots, T^m$..

Our convention is that whenever Ω is used to construct a sequence of trees then we will have fixed indices of $T^1, \ldots, T^m$ in mind (provided by the earlier stages of the inductive construction) so that the value of $\Omega(e_1, \ldots, e_m;\ T^1, \ldots, T^m)$ is also fixed. This seemingly roundabout procedure saves constant reference to indices and the resulting complex notation.

We should also take this opportunity to clarify the operation $\mu(T)$. It is possible to arrange that all applications involve tree-systems for which a partial recursive μ may be provided (this is of course for Σ_1^0 systems). Nevertheless, it is usually sufficient for μ to be of degree $\leqslant \mathbf{0}^{(1)}$, and it is easy to see that such a μ is always available. For the question, as to whether or not a Σ_1^0 subset of S possesses a least element w.r.t. $\leqslant$, is of degree $\leqslant \mathbf{0}^{(1)}$: we first ask whether a given T is nonempty and then, given an affirmative answer, choose an element which has no predecessor: if any of the finitely many strings which are no longer than it and are incompatible with it have an extension in T then there is no least element, but otherwise it *is* the least element.

Finally, we shall frequently need to pick a pair of incompatible elements of a tree T when that tree is not a singleton. Such a pair will be denoted by $\mu_0(T)$, $\mu_1(T)$ and it is easy to see that they may be assumed to be uniformly of degree $\leqslant \mathbf{0}^{(1)}$. This may be generalised in the obvious way to provide $\mu_0(T)$ for any string σ: if $\mu_\sigma(T)$ has been defined and $T\wedge\mu_\sigma(T)$ is not a singleton then just let $\mu_{\sigma*0}(T)$, $\mu_{\sigma*1}(T)$ be $\mu_0\big(T\wedge\mu_\sigma(T)\big)$, $\mu_1\big(T\wedge\mu_\sigma(T)\big)$, where for uniformity we derive the enumeration of $T\wedge\mu_\sigma(T)$ from that of T.

§ 2. The prioric game PG$(C; \mathcal{A})$.

The important notion of (C, d)-prior-abundance will be defined in § 3. As we have already mentioned, this involves the existence of a winning strategy for entering $\mathcal{A}$ in an appropriate prioric game: this game is the subject of the present section, and in no way involves recursion-theory.

2.1. Definition. Let $\mathcal{A}\subseteq 2^\omega$ and let C be a tree-system. We describe a two-person infinite-positional prioric game PG$(C; \mathcal{A})$ in which the *ultimate* purpose (as with the generalised Banach–Mazur game $G(C; \mathcal{A})$ introduced in [14] and Part I of [15]) is for the two players to produce the alternate members of a contracting chain $U_0\supseteq U_1\supseteq\ldots$ of elements of C in such a way that $B=\lim\big(\mu(U_r)\big)$ exists and of course belongs to $\bigcap\mathfrak{N}(U_r)$. We then declare Player 2 to be the winner if $B\in\mathcal{A}$ and Player 1 to be the winner if $B\notin\mathcal{A}$, subject to certain limitations on the manner of play described below.

The difference in the present game is that the two players actually make a succession of attempts at each U_r associated with them, the last of these attempts being U_r itself. More precisely, they produce a chain $T_0\leqslant^\mu T_1\leqslant^\mu\ldots$ (which just means that $\mu(T_0)\leqslant\mu(T_1)\leqslant\ldots$) of elements of C in which each T_k is an attempt at some particular move $U_{R(k)}$ where $R(k)$ is defined inductively by setting $R(0) = 0$ and for $k > 0$:

$$R(k) = \begin{cases} \text{the least } r \leqslant R(k-1) \text{ such that } U_r \text{ requires attention at stage } k, \text{ if such an } r \text{ exists;} \\ R(k-1)+1 \text{ otherwise.} \end{cases}$$

The notion U_r requires attention at stage k is deliberately left undefined here, although it is of course made completely precise in applications; this notion will be pinned-down a little in 3.1 below, when we discuss strategies for $\mathrm{PG}(C;\ \mathcal{A})$. The last stage at which U_r is attempted (if such a stage exists — see the last paragraph of this definition) is denoted by $K(r)$ and we define $U_r = T_{K(r)}$.

Now, there are three stage-by-stage limitations on the conduct of play needed to ensure that $U_0 \supseteq U_1 \supseteq \ldots$ and that $B = \lim(\mu(U_r))$ exists; that they succeed in achieving these aims will be verified after we have completed the definition of the game.

(i) $T_k \subseteq T_{M(k)}$ for all k, where $M(k) = M(R(k)-1, k)$ and $M(r, k)$ is the largest $j \leqslant k$ such that $R(j) \leqslant r$, for all r and k,

(ii) if $T_{M(r,k)} \wedge \mu(T_k) \in \star(C)$ but $T_{M(s,k)} \wedge \mu(T_k) \notin \star(C)$ for all $s < r$ then, assuming that $r \leqslant R(k)$, U_r requires attention at stage $k+1$,

(iii) if $R(k+1) = R(k)+1$ then $\mu(T_k) < \mu(T_{k+1})$.

These limitations are enforced simply by ruling that the first player to transgress them loses the game. In our discussion of instances of $\mathrm{PG}(C;\ \mathcal{A})$ it is obviously easiest to assume that neither player is so stupid as to lose the game by flouting these conditions, and so we shall assume them always to be observed. We lose no generality by doing this.

Finally, we come to the central limitation on the conduct of the game. This limitation is a deeper one which can not be satisfied at a single stage. It asks that the *stability function*

(*) $K(r)$ is defined for all r.

In other words, each player makes only finitely many attempts at each of its moves, hence that the moves only require attention finitely often. This is enforced by ruling that (the stage-by-stage conditions (i)-(iii) being observed) the player associated with the least r such that $K(r)$ is undefined (if there is such an r) loses the game. This completes the definition of $\mathrm{PG}(C;\ \mathcal{A})$.

Let us now look at a play $T_0, T_1, \ldots$, of the game $\mathrm{PG}(C;\ \mathcal{A})$, assuming that all the conditions are observed so that neither player trivially loses the game. Hence, in particular, $K(r)$ is defined for all r. We wish to prove that in this case we do have:

(a) $U_0 \supseteq U_1 \supseteq \ldots$,

(b) $B = \lim(\mu(U_r)) = \lim(\mu(T_k))$ is a well-defined element of 2^ω.

That (a) holds is a consequence of condition 2.1 (i), and follows immediately from the lemma below. Notice first that $R(K(r)) = r$, $R(k) > r$ for all $k > K(r)$; also, it is easy to see that $K(r) < K(r+1)$ for all r.

2.2. Lemma. *If $r < R(k)$ then $T_k \subseteq T_{M(r,k)}$ for all r and k.*

Proof. By induction on k. It follows from 2.1 (i) that $T_k \subseteq T_{M(R(k)-1,k)}$, so that if $r = R(k)-1$ then there is nothing to prove. If $r < R(k)-1$ then $M(r, k) < M(R(k)-1, k) < k$. Also, $r < R(M(R(k)-1, k)) = R(k)-1$ and $M(r, M(R(k)-1, k)) = M(r, k)$ so that it follows from the induction hypothesis that $T_{M(R(k)-1,k)} \subseteq T_{M(r,M(R(k)-1,k))} = T_{M(r,k)}$. We deduce that $T_k \subseteq T_{M(r,k)}$. ■

It follows immediately that if $k \geqslant K(r)$ then $T_k \subseteq T_{K(r)}$, and hence that $T_{K(r+1)} \subseteq T_{K(r)}$, i.e. $U_{r+1} \subseteq U_r$ for all r, thus proving (a).

It is relatively easy to verify (b). It follows from 2.1 (ii) that $T_{K(r)} \wedge \mu(T_k)$ is not a singleton, for all $k \geqslant K(r)$. Since $R(K(r)+1) = R(K(r))+1$, it then follows from 2.1 (iii) that $\mu(T_{K(r)}) < \mu(T_{K(r)+1})$ for all r. Hence (b) is satisfied.

There is one final point which should be cleared up. It might seem rather unfair to rule that in 2.1 (iii) the player moving at stage $k+1$ should lose the game if $\mu(T_k) \not< \mu(T_{k+1})$, since T_k might be a singleton. But in fact if $R(k+1) = R(k)+1$ then $M(R(k), k) = k$ and so $R(k+1) \leqslant R(k)$ by 2.1 (ii) if T_k is a singleton (remember singleton trees $\varepsilon \star (C)$): contradiction. Therefore T_k is not a singleton and this problem does not occur.

To conclude this section some motivation would no doubt be helpful. Although the definition of $\mathrm{PG}(C;\ \mathcal{A})$ involves no reference to recursion-theory, this is the only area where it has as yet been utilised and so we should explain *why* it is useful there. The point of this prioric game is that an element $B = \lim(\mu(U_r))$ constructed in some *ultimate* play $U_0 \supseteq U_1 \supseteq \ldots$ can also be defined to be $\lim(\mu(T_k))$ where the sequence $T_0, T_1, \ldots$, may be arranged to be of lower degree of unsolvability than $U_0, U_1, \ldots$ This will of course be familiar to those recursion-theorists who are intimate with the construction of, for example, a minimal degree below $\mathbf{0}^{(1)}$. A rough explanation is that the less constructive aspects of the sequence $U_0, U_1, \ldots$, have been visited upon the stability function $K(r)$ rather than the sequence $T_0, T_1, \ldots$ The sense in which what is usually called the Priority Method is involved here lies in the definition of $R(k)$ as the index of the move of highest priority (i.e. smallest index) which requires attention at stage k.

§ 3. (C, d)-priorabundant sets. We now turn to a means of classifying subsets of 2^ω, and prove an existence-theorem for any set so classified: every (C, d)-priorabundant set contains an element of degree $\leqslant \boldsymbol{d}$.

First, we need to define exactly what we mean by a strategy for $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$ which is of degree $\leqslant \boldsymbol{d}$.

3.1. DEFINITION. A $(\boldsymbol{C}, \boldsymbol{d})$-strategy $\Omega(e;\ T^1, \ldots, T^n)$ is a $(\boldsymbol{C}, \boldsymbol{d})$-operator (see 1.7) such that:

(i) $\Omega(e;\ T^1, \ldots, T^n) \subseteq T^1$ & $\mu(T^n) \leqslant \mu(\Omega(e;\ T^1, \ldots, T^n))$,

(ii) $\Omega(e;\ T^1)$ is always defined and such that $\mu(T^1) < \mu(\Omega(e;\ T^1))$ if T^1 is not a singleton.

The way in which Player 2 uses a strategy Ω is as follows (the interpretation for Player 1 is exactly analogous but less important). First, Ω notifies Player 2 when a move requires attention: more precisely, if $\Omega(e;\ T_{M(2e,k)}, \ldots, T_{k-1})$ is defined for $2e+1 \leqslant R(k-1)$ then U_{2e+1} requires attention at stage k. Secondly, if $R(k) = 2e+1$ then Ω is actually used to define T_k: if $R(k) = R(k-1)+1$ then $T_k = \Omega(e;\ T_{k-1}) = \Omega(e;\ T_{M(2e,k)})$, and otherwise $T_k = \Omega(e;\ T_{M(2e,k)}, \ldots, T_{k-1})$.

There is no reference to the trees T_h for $h < M(2e, k)$ simply because in practice they play no role. We shall always assume here that $\boldsymbol{d} \geqslant \boldsymbol{0}^{(1)}$.

3.2. DEFINITION. Let Ω be a $(\boldsymbol{C}, \boldsymbol{d})$-strategy. Ω is a *winning strategy for* $\mathcal{A}$ in $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$ if Player 2 always wins when he uses Ω, irrespective of the play of Player 1.

3.3. DEFINITION. A set $\mathcal{A} \subseteq 2^\omega$ is $(\boldsymbol{C}, \boldsymbol{d})$-*priorabundant* if there is a winning $(\boldsymbol{C}, \boldsymbol{d})$-strategy for $\mathcal{A}$ in $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$.

It is obvious that any superset of a $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant set is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant. It is also not difficult to prove that the $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant sets are closed under finite intersections: one just combines the strategies in a rather obvious way, coding them alternately. It is also possible to prove closure under certain rather limited countable intersections but we shall not need any of this in the present paper.

We now prove our simple but absolutely basic Existence Theorem:

3.4. THEOREM. *If $\mathcal{A} \subseteq 2^\omega$ and $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant then $\mathcal{A}$ contains an element of degree $\leqslant \boldsymbol{d}$ and has the cardinality of the continuum.*

Proof. We deal with the two assertions simultaneously. Let X be an arbitrary element of 2^ω. We shall describe a strategy Ω^X for Player 1 which is of degree $\leqslant X \cup \boldsymbol{0}^{(1)}$. Hence, in particular, if $X \leqslant \boldsymbol{d}$ then the resulting play, in which Player 2 uses the winning strategy available for $\mathcal{A}$, must be of degree $\leqslant \boldsymbol{d}$ (taking into account the constant limitation that $\boldsymbol{d} \geqslant \boldsymbol{0}^{(1)}$). On the other hand, the play *is* dependent on X and so we obtain a continuum of different plays all of which produce elements of $\mathcal{A}$.

The simple strategy to be used by Player 1 is as follows: $\Omega^X(e;\ T)$ is defined to be $T \wedge \mu_{X(e)}(T)$, and Ω^X is otherwise undefined. In any particular play, it is easy to see that Ω^X obeys 2.1 (i), (iii) and $(*)$, and it is not much trouble to verify 2.1 (ii); so we leave this to the reader.

Clearly, Ω^X *is* of degree $\leqslant X \cup \boldsymbol{0}^{(1)}$, so that since $\boldsymbol{d}$ is always $\geqslant \boldsymbol{0}^{(1)}$ it follows that if $X \leqslant \boldsymbol{d}$ then Ω^X is a $(\boldsymbol{C}, \boldsymbol{d})$-strategy. Hence, it only remains to prove that if $X \neq Y$ then the resulting plays (T_k^X) and (T_k^Y) result in distinct elements B^X and B^Y of 2^ω. But if e is the least such that $X(e) \neq Y(e)$ then it is clear that $\mathfrak{N}(T^X_{k(2e)})$ and $\mathfrak{N}(T^Y_{k(2e)})$ are disjoint so that B^X and B^Y must be distinct.

The essence of our approach is that, in order to prove that a set $\mathcal{A} \subseteq 2^\omega$ contains an element of degree $\leqslant \boldsymbol{d}$, we no longer need a construction: we simply prove that $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant and then use the theorem which we have just proved. To prove that $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant requires the production of a winning $(\boldsymbol{C}, \boldsymbol{d})$-strategy which complies with the stage-by-stage conditions of $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$. This is the only penalty which we have to pay for the formal notion of $(\boldsymbol{C}, \boldsymbol{d})$-strategy, and it leads us in the next section to describe a class of strategies which are well-behaved in this way.

§ 4. Remedial strategies.

In this section we introduce and examine a class of strategies which comply in a natural way with the stage-by-stage restrictions of $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$. These strategies, which we call remedial, are used in all the applications which we know of in degree-theory. They also turn out to be needed for the concept of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set, discussed in the next section. It is probably worth emphasising that the material in this section is only made necessary through formalizing the notion of $(\boldsymbol{C}, \boldsymbol{d})$-strategy, and so does not correspond to any steps in the usual *ad hoc* constructions which are hereby replaced. It appears, therefore, to be the penalty we pay for the organised framework of $(\boldsymbol{C}, \boldsymbol{d})$-priorabundance.

Before turning to remedial strategies, we define an auxiliary operator which is crucial to their definition.

4.1. DEFINITION. Let Ω be a $(\boldsymbol{C}, \boldsymbol{d})$-strategy. Then $\hat{\Omega}(e;\ T^1, \ldots, T^n)$ is defined as follows. We set $\hat{\Omega}(e;\ T^1) = T^1$ and for $n > 1$ set $\hat{\Omega}(e;\ T^1, \ldots, T^n) = \Omega(e;\ T^1, \ldots, T^m)$ where m is the largest number $< n$ such that $\Omega(e;\ T^1, \ldots, T^m)$ is defined.

It can be seen that $\hat{\Omega}$ is also a $(\boldsymbol{C}, \boldsymbol{d})$-operator and has the advantage of being everywhere-defined.

The notion of remedial strategy arises in a quite obvious way. Suppose that Ω is a $(\boldsymbol{C}, \boldsymbol{d})$-strategy which is used by Player 2 in $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$. Let $T_0, T_1, \ldots,$ be the play produced and let us assume all the notation of 2.1. The following lemma is intuitively fairly clear but we prove it in detail.

4.2. LEMMA. *For all k and all e such that $2e+1 \leqslant R(k)$:*

$$T_{M(2e+1,k)} = \hat{\Omega}(e;\ T_{M(2e,k)}, \ldots, T_k).$$

Proof. We prove the lemma by induction on k, and there are three cases.

Case 1. $2e+1 = R(k) > R(k-1)$. Then $R(k) = R(k-1)+1$ so that $R(k-1) = 2e$ and $M(2e, k) = k-1$. It follows that $T_k = \Omega(e;\ T_{k-1})$ and so $T_{M(2e+1,k)} = T_k = \hat{\Omega}(e;\ T_{k-1}, T_k) = \Omega(e;\ T_{M(2e,k)}, T_k)$.

Case 2. $2e+1 = R(k) \leqslant R(k-1)$. In this case

$$T_k = \Omega(e;\ T_{M(2e,k)}, \ldots, T_{k-1})$$

whence $T_{M(2e+1,k)} = T_k = \hat{\Omega}(e;\ T_{M(2e,k)}, \ldots, T_k)$ as required.

Case 3. $2e+1 < R(k)$. Then $2e+1 \leqslant R(k-1)$ and $k-1 > M(2e, k) = M(2e, k-1)$ so that

$$\hat{\Omega}(e;\ T_{M(2e,k)}, \ldots, T_k) = \hat{\Omega}(e;\ T_{M(2e,k-1)}, \ldots, T_{k-1}) .$$

The R.H.S. may be identified with $T_{M(2e+1,k-1)}$ by the induction hypothesis, and hence with $T_{M(2e+1,k)}$ because $R(k) > 2e+1$. ■

We can now observe that if 2.1 (ii) is satisfied by Ω then, whenever $\hat{\Omega}(e;\ T_{K(2e)}, \ldots, T_k) \wedge \mu(T_k) \in \star(C)$ with $k \geqslant K(2e)$ we must have $\Omega(e;\ T_{K(2e)}, \ldots, T_k)$ defined. This may be analysed more closely with respect to 2.1 (ii), but anyway we have said enough to suggest the following concept:

4.3. Definition. A $(\boldsymbol{C}, \boldsymbol{d})$-strategy Ω is *remedial* if $\Omega(e;\ T^1, \ldots, T^m)$ is defined whenever $\hat{\Omega}(e;\ T^1, \ldots, T^m) \wedge \mu(T^m) \in \star(\boldsymbol{C})$ but $T^1 \wedge \mu(T^m) \notin \star(\boldsymbol{C})$.

We are assuming in this paper that $\star(\boldsymbol{C})$ is restricted to finite trees so that if $\boldsymbol{C}$ consists solely of perfect trees then every $(\boldsymbol{C}, \boldsymbol{d})$-strategy is remedial.

The following result justifies our choice of Definition 4.3:

4.4. Theorem. *If Ω is remedial then Ω complies with* 2.1 (i)-(iii).

Proof. Any $(\boldsymbol{C}, \boldsymbol{d})$-strategy satisfies 2.1 (i) and (iii). To verify 2.1 (ii), note that if $T_{M(2e+1,k)} \wedge \mu(T_k) \in \star(\boldsymbol{C})$ with $2e+1 \leqslant R(k)$ and there is no $s < 2e+1$ such that $T_{M(s,k)} \wedge \mu(T_k) \in \star(\boldsymbol{C})$ then $\Omega(e;\ T_{M(2e,k)}, \ldots, T_k)$ is defined by 4.2 and 4.3. We deduce that $2e+1$ requires attention at stage $k+1$, by 3.1. ■

It should now be clear that the remedial strategies form a class which it is natural to isolate. Their usefulness in applications lies in the result which we have just proved; for, it is usually easy enough to see on inspection that a given strategy is remedial and we know then from 4.4 that it complies with the stage-by-stage conditions of PG$(\boldsymbol{C};\ \mathcal{A})$.

To conclude this section we mention a very common type of remedial strategy, namely those Ω which take the form

$$\Omega(e;\ T^1, \ldots, T^m) = T^1 \wedge \delta(e;\ T^1, \ldots, T^m)$$

where $\delta(e;\ T^1, \ldots, T^m)$ belong to T^1 and is $> \mu(T^1)$. Clearly, $\hat{\Omega}(e;\ T^1, \ldots, T^m)$ is always of the form $T^1 \wedge \delta(e;\ T^1, \ldots, T^m)$ for some $m < n$ and so

$$\hat{\Omega}(e;\ T^1, \ldots, T^m) \wedge \mu(T^m) = T^1 \wedge \mu(T^m) .$$

Therefore, Ω is trivially remedial.

§ 5. $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager sets. The purpose of this section is to introduce a notion of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set and prove that every $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant. We shall in fact also introduce a weaker notion of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set which also has this property but which is less natural; the significance of this weaker notion is that if seems to be needed for a number of applications.

In order to show that a $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set contains an element of degree $\leqslant \boldsymbol{d}$ it does seem necessary to resort to the Existence Theorem of § 3 along with the result mentioned above; certainly the only direct proof which we have produced (we do not do so here) makes implicit use of both of these results. This is an unsatisfactory situation as regards existence-theorems in degree-theory, but we are devoting the rest of this paper to the two notions of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set in order to demonstrate that they are natural and useful concepts.

5.1. Definition. Let Ω be a remedial $(\boldsymbol{C}, \boldsymbol{d})$-strategy. Let e be fixed and let $T^1, T^2, \ldots$, be a *proper* $\leqslant^\mu$ chain over $\boldsymbol{C}$ (i.e. $\mu(T^n) < \mu(T^{n+1})$ for infinitely many n). We say that $T^1, T^2, \ldots$, is Ω, e-*prioric* if for all n:

(a) if $\Omega(e;\ T^1, \ldots, T^{m-1})$ is defined then it is $= T^m$,

(b) $\hat{\Omega}(e;\ T^1, \ldots, T^m) \supseteq T^m$,

(c) $T^1 \wedge \mu(T^m) \notin \star(\boldsymbol{C})$.

Note that (c) is redundant if $\boldsymbol{C}$ is a perfect system. It is easy to prove by induction on n that if $T^1, T^2, \ldots$, is Ω, e-prioric then $T^1 \supseteq T^m$ for all n. It follows that if $B = \lim(T^m) = \lim(\mu(T^m))$ then $B \in \mathfrak{N}(T^1)$.

The next definitions lead up to the notion of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set; the corresponding weaker notions will be defined in parentheses.

5.2. Definition. Let $(\mathcal{A}_e)$ be a sequence of subsets of 2^ω and let Ω be a remedial $(\boldsymbol{C}, \boldsymbol{d})$-operator. We say that Ω is a $(\boldsymbol{C}, \boldsymbol{d})$-*prioric probe for* $(\mathcal{A}_e)$ if, for each e, whenever $T^1, T^2, \ldots$, is a Ω, e-prioric $\leqslant^\mu$ chain then:

(a) $T^{N(e)} = \lim_n \hat{\Omega}(e;\ T^1, \ldots, T^m)$ exists,

(b) $\mathfrak{N}(T^{N(e)}) \subseteq \mathcal{A}_e$.

We say that Ω is a *weak* $(\boldsymbol{C}, \boldsymbol{d})$-prioric probe for $(\mathcal{A}_e)$ if it satisfies (b)^w below instead of the stronger (b):

(b)^w $B = \lim(\mu(T^m))$ belongs to $\mathcal{A}_e$.

5.3. Definition. Let $(\mathcal{A}_e)$ be a sequence of subsets of 2^ω. We say that $(\mathcal{A}_e)$ is (*weakly*) $(\boldsymbol{C}, \boldsymbol{d})$-*priordense* if $(\mathcal{A}_e)$ possesses a (weak) $(\boldsymbol{C}, \boldsymbol{d})$-prioric probe.

5.4. Definition. Let $\mathcal{A} \subseteq 2^{\omega}$. We say that $\mathcal{A}$ is (*weakly*) $(\boldsymbol{C}, \boldsymbol{d})$-*priorcomeager* if $\mathcal{A} \supseteq \bigcap \mathcal{A}_e$ for some (weakly) $(\boldsymbol{C}, \boldsymbol{d})$-priordense $(\mathcal{A}_e)$. Also, we say that $\mathcal{A}$ is (*weakly*) $(\boldsymbol{C}, \boldsymbol{d})$-*priormeager* if $2^{\omega} - \mathcal{A}$ is (weakly) $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager.

Every superset of a (weakly) priorcomeager set is obviously (weakly priorcomeager. Also, as with priorabundance, it is easy to prove that the $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager sets are closed under finite intersections. We again note that this generalises to certain countable intersections, namely those which can be roughly described as being of degree $\leqslant \boldsymbol{d}$.

Our main interest in this section lies in the following result:

5.5. Theorem. *If $\mathcal{A} \subseteq 2^{\omega}$ is weakly $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager then $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant.*

Proof. We claim that if Ω is a weak $(\boldsymbol{C}, \boldsymbol{d})$-prioric probe for $(\mathcal{A}_e)$ then it is also a winning $(\boldsymbol{C}, \boldsymbol{d})$-strategy for $\mathcal{A}$ in $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$. Let us therefore examine a typical instance of the game $\mathrm{PG}(\boldsymbol{C}; \mathcal{A})$ in which Player 2 uses Ω. We use all the standard notation for various aspects of the game introduced in 2.1. Since otherwise there is nothing to prove, let us assume that Player 1 plays a strategy which complies with the conditions 2.1 (i)-(iii) and (*).

We claim that $K(r)$ is defined for all r, an assertion which we now need only prove for odd r. Also we claim that if B is the result of the play then $B \in \mathcal{A}_e$ for all e. Clearly, this will complete the proof of the theorem. We shall be able to justify these claims once we have proved:

Lemma. *For each e, $(T_k)_{k \geqslant K(2e)}$ is Ω, e-prioric.*

Proof. We verify the three clauses in Definition 5.1.

(a) If $\Omega(e;\ T_{K(2e)}, \ldots, T_{k-1})$ is defined for $k > K(2e)$ then $R(k) = 2e+1$ and so $T_k = \Omega(e;\ T_{K(2e)}, \ldots, T_{k-1})$ because $K(2e) = M(2e, k)$.

(b) In order to prove that $\hat{\Omega}(e;\ T_{K(2e)}, \ldots, T_k)\ T_k$ we prove more generally that $\hat{\Omega}(e;\ T_{K(2e)}, \ldots, T_k) \supseteq T_j$ for all j such that $M(2e+1, k) \leqslant j \leqslant k$ and we do this by induction on j. For $j = M(2e+1, k)$ it follows immediately from Lemma 4.2. For $j > M(2e+1, k)$ note first that $R(j) > 2e+1$ so that $M(2e+1, j) = M(2e+1, k)$ and hence $M(2e+1, k) \leqslant M(j) < j$.

But $T_{M(j)} \supseteq T_j$ by 2.1 (i) and so we may then use the induction hypothesis.

(c) $T_{K(2e)} \wedge \mu(T_k) \notin \star(\boldsymbol{C})$ for all $k \geqslant K(2e)$ by 2.1 (ii). ■

Now, it follows from 5.2 (a) that $\lim_k \hat{\Omega}(e;\ T_{K(2e)}, \ldots, T_k)$ exists for each e, and by Lemma 4.2 is equal to $\lim_k T_{M(2e+1,k)} = T_{K(2e+1)}$. It further follows from 5.2 (b) that $B \in \mathcal{A}_e$, for all e, and hence $B \in \mathcal{A}$. This completes the proof of the theorem.

5.6. Corollary. *If $\mathcal{A} \subseteq 2^{\omega}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager then $\mathcal{A}$ is $(\boldsymbol{C}, \boldsymbol{d})$-priorabundant.* ■

This corollary is the true analogue for the present framework of the comparatively trivial observation that every $(\boldsymbol{C}, \boldsymbol{d})$-comeager set is $(\boldsymbol{C}, \boldsymbol{d})$-abundant. As we have already mentioned, it is possible to generalise a theorem of Banach to prove conversely that every $(\boldsymbol{C}, \boldsymbol{d})$-abundant set is $(\boldsymbol{C}, \boldsymbol{d})$-comeager; this is subject to the minor restrictions that $\boldsymbol{C}$ consist of closed trees (i.e. their neighbourhoods are closed in the standard topology) and $\boldsymbol{d} \geqslant \boldsymbol{0}^{(2)}$. (The latter restriction can be lifted when $\boldsymbol{C} = \boldsymbol{S}^*$, which is the case Banach was implicitly dealing with). It is natural to ask whether some converse result like this can be found for the theorem above or its corollary. We feel, however, that in neither case is this likely. It does not need a very detailed examination of the Banach-type theorem which we have just mentioned (and which will appear in [14] and [15]) to appreciate the great obstacles which lie in the way of any such theorem for the present framework. Further, there is good reason to suppose from applications that the three concepts concerned are all distinct. All this provides yet further evidence that the game-theoretic notions are the widest and most natural ones for investigating degree-theory. We believe, however, that the concept of $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set is the true analogue for the present framework of the concept of $(\boldsymbol{C}, \boldsymbol{d})$-comeager set discussed in [14] and [15]. This is supported by the following observations. If $\boldsymbol{C}$ is a perfect tree system (we did not formulate a concept of comeager set for more general systems and indeed can not see how one could) then:

(i) every $(\boldsymbol{C}, \boldsymbol{d})$-comeager set is $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager,

(ii) every $(\boldsymbol{C}, \boldsymbol{d})$-priorcomeager set is $(\boldsymbol{C}, \boldsymbol{d}^{(1)})$-comeager.

§ 6. The set of minimal degrees which do not attain their least possible jump. The concepts studied in the earlier sections of this paper have been introduced as much for their intrinsic interest as for obtaining results in degree-theory. Nevertheless, the framework which has been developed does make it possible to isolate open problems and new results in such a way as to focus attention on the particular difficulty with which we are faced, instead of having to view it in the context of constructions which are already known. We shall give an example of this in the present section. It has recently been proved by Sasso [7] that there is a minimal degree $\boldsymbol{b}$ which does not attain its least possible jump: in other words, $\boldsymbol{b}$ is such that $\boldsymbol{b} \cup \boldsymbol{0}^{(1)} < \boldsymbol{b}^{(1)}$. The basic argument provides a degree $\boldsymbol{b} < \boldsymbol{0}^{(2)}$, but it was observed by Sasso (in company with S. B. Cooper and R. Epstein) that the technique involved can be blended with the construction of a minimal degree below $\boldsymbol{0}^{(1)}$. In our framework this is absolutely transparent.

The task which we face is to classify the set

$$\{B\colon\ \boldsymbol{B} \text{ is minimal } \&\ \boldsymbol{B} \cup \boldsymbol{0}^{(1)} < \boldsymbol{B}^{(1)}\}.$$

This breaks up into two separate problems, namely classifying

$$\mathfrak{M}' = \{B\colon \text{ if } C \leqslant_T B \text{ then either } C \text{ is recursive or } B \leqslant_T C\}\,,$$
$$\mathfrak{J} \;= \{B\colon B^{(1)} \nleq B \cup \mathbf{0}^{(1)}\}\,.$$

Notice that $\mathfrak{J} \subset \mathfrak{R}$, the set of nonrecursive functions. The set $\mathfrak{M}$ of functions of minimal degree is $\mathfrak{M}' \cap \mathfrak{R}$ and if we merely wish to classify $\mathfrak{M}$ then it is enough to show that both $\mathfrak{M}'$ and $\mathfrak{R}$ are $(\boldsymbol{C}, \mathbf{0}^{(1)})$-priorcomeager for some $\boldsymbol{C}$; in fact, $\mathfrak{R}$ is $(\boldsymbol{C}, \mathbf{0}^{(1)})$-priorcomeager for any $\boldsymbol{C}$. These results were first obtained some time ago in [13], which was, however, never published. For this reason, we repeat the classification of $\mathfrak{M}'$ here for the particular case $\boldsymbol{C} = \boldsymbol{I}_2$ defined below. We then show that $\mathfrak{J}$ is weakly $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-priorcomeager, whence the desired set $\mathfrak{M}' \cap \mathfrak{J}$ is weakly $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-priorcomeager through closure under finite intersections. The classification of $\mathfrak{J}$ isolates the essence of Sasso's argument, and of course the classification of $\mathfrak{M}' \cap \mathfrak{J}$ yields his theorem through the main results of § 3 and § 5.

6.1. Definition. $\boldsymbol{I}_2$ is the system which consists of all the images of partial recursive maps $F\colon \boldsymbol{S} \to \boldsymbol{S}$ satisfying:

(i) If one of $F(\sigma * 0)$, $F(\sigma * 1)$ is defined then so is the other and so also is $F(\sigma)$; in addition $F(\emptyset)$ is always defined.

(ii) $F(\sigma * 0)$, $F(\sigma * 1)$ are incompatible if they are defined.

Moreover, $\star(\boldsymbol{I}_2)$ is defined to be the set of ranges of maps F such that $F(\sigma)$ is undefined for some σ of length $\leqslant 2$.

This is not an unfamiliar system apart from its $\star$. For, the system $\boldsymbol{I}_1$ in which length $\leqslant 2$ is replaced by length $\leqslant 1$ has been used many times (implicitly) for the construction of minimal degrees (for example, in [12] and [13]).

Let $T \in \star(\boldsymbol{I}_2)$ be $\operatorname{im}(F)$. We define $\delta(T)$ to be some $F(\sigma)$, with length $\sigma \leqslant 1$, such that $F(\sigma * 0)$, $F(\sigma * 1)$ are undefined but $F(\sigma)$ is defined.

6.2. Theorem $\mathfrak{M}'$ *is* $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-*priorcomeager*.

Proof. It can be seen that $\mathfrak{M}' = \bigcap \mathfrak{M}_e$ where

$$\mathfrak{M}_e = \{B\colon F_e^*(B) \text{ is undefined, recursive or } \equiv_T B\}.$$

We claim that Ω is a $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-prioric probe for $(\mathfrak{M}_e)$, where

$$\Omega(e;\ T) = \operatorname{Sp}_e\big(T \wedge \mu_0(T)\big)\,,$$
$$\Omega(e;\ T^1, \ldots, T^n) = T^1 \wedge \delta\big(\hat{\Omega}(e;\ T^1, \ldots, T^n) \wedge \mu(T^n)\big)\,,$$

if $\hat{\Omega}(e;\ T^1, \ldots, T^n) \wedge \mu(T^n) \in (\boldsymbol{I}_2)$ but $T^1 \wedge \mu(T^n) \notin (\boldsymbol{I}_2)$. Here, Sp_e is an operation for forming an $\boldsymbol{F}_e$-splitting subtree of a given tree (such as in [12] for example). It is clear that Ω is a remedial $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-strategy.

Suppose now that e is arbitrary and $T^1, T^2, \ldots$, is a Ω, e-prioric sequence. Then $T^2 = \Omega(e;\ T^1) = \operatorname{Sp}_e\big(T^1 \wedge \mu_0(T)\big)$. To prove 5.2 (a) we claim that $\Omega(e;\ T^1, \ldots, T^n)$ is defined at most once for $n > 1$. For let p be the least such n if there is one.

Then $\hat{\Omega}(e;\ T^1, \ldots, T^p) = T^2$ and $T^{p+1} = T^1 \wedge \mu(T^{p+1})$. If $n > p$ but $\Omega(e;\ T^1, \ldots, T^m)$ is undefined for $p \leqslant m \leqslant n$ then $T^{p+1} = \hat{\Omega}(e;\ T^1, \ldots, T^m)$ and $T^{p+1} \wedge \mu(T^m) = T^1 \wedge \mu(T^m)$, which $\notin \star(\boldsymbol{I}_2)$ by 5.1 (c). Hence

$$\Omega(e;\ T^1, \ldots, T^n)$$

is undefined.

To prove 5.2 (b) we have to show that either $\mathfrak{N}(T^{p+1}) \subset \mathfrak{M}_e$ if p exists, or $\mathfrak{N}(T^2) \subseteq \mathfrak{M}_e$ otherwise. In the former case, we must have $T^2 \wedge \mu(T^{p+1}) \in \star(\boldsymbol{I}_2)$ and hence $\delta\big(T^2 \wedge \mu(T^{p+1})\big)$ is defined. But $T^2 = \operatorname{Sp}_e\big(T^1 \wedge \mu_0(T^1)\big)$ and so $\delta\big(T^2 \wedge \mu(T^{p+1})\big)$ is not $\boldsymbol{F}_e$-split by any pair of strings in T^1. Since $T^{p+1} \subseteq T^1$ (because $T^1, T^2, \ldots$, is Ω, e-prioric) it follows from the usual well known splitting-lemma that $\mathfrak{N}(T^{p+1}) \subseteq \mathfrak{M}_e$, because if $\mu(T^{p+1}) < B$ then $F_e^*(B)$ is either undefined or recursive, for all $B \in \mathfrak{N}(T^1)$. In the second case, if $B \in \mathfrak{N}(T^2)$ then $B \in \mathfrak{N}\big(\operatorname{Sp}_e(T^1)\big)$ and so by another standard argument it follows that $B \equiv_T F_e^*(B)$; hence again we have $\mathfrak{N}(T^2) \subseteq \mathfrak{M}_e$. ■

Before moving on the second theorem, recall that if $B, D \in 2^\omega$:

$$(B \oplus D)(2e) = B(e)\,, \qquad (B \oplus D)(2e+1) = D(e)\,.$$

The operation $\sigma \oplus \tau$ for strings σ, τ of the same length is defined similarly. As is well known, the degree of $B \oplus D$ is $\boldsymbol{B} \cup \boldsymbol{D}$.

6.3. Theorem. $\mathfrak{J}$ *is weakly* $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-*priorcomeager.*

Proof. First, notice that $\mathfrak{J} = \bigcap \mathfrak{J}_e$ where

$$\mathfrak{J}_e = \{B\colon B^{(1)} \neq F_e^*(B \oplus D)\}$$

and D is a fixed element of degree $\mathbf{0}^{(1)}$. Our aim then will be to provide a weak $(\boldsymbol{I}_2, \mathbf{0}^{(1)})$-prioric probe for $(\mathfrak{J}_e)$, but first we need a subsidiary function. For each $T \in \boldsymbol{I}_2$ define $x(T)$ to be an x such that if F is the partial recursive $\boldsymbol{S}$-map behind T then

$$R_x^B = \begin{cases} \omega & \text{if} \quad (\exists \tau)\big(F(\tau * 0 * 0) < B\big)\,, \\ \varphi & \text{otherwise}\,. \end{cases}$$

Clearly, $x(T)$ can be computed from an index of T or F. The important property of this function is:

$$B^{(1)}\big(x(T)\big) = 0 \leftrightarrow x(T) \in R^B_{x(T)}(\exists \tau)\big(F(\tau * 0 * 0) < B\big)\,.$$

Now we turn to the definition of Ω.

First we define $\Omega(e;\ T)$ to be in (G) where:

$$G(\varphi) = F(0)\,,$$
$$G(\tau * i) = F(\tau_0 * 1 * i) \quad \text{if} \quad G(\tau) = F(\tau_0)\,.$$

For $n > 1$ there are two cases.

Case A. If $\big(F_e(\mu(T^m)\oplus D[|\mu(T^m)|])x(T^1)\big) = 1$ then we set:

$$\Omega(e;\ T^1, \ldots, T^m) = T^1\wedge F^1(\tau * 0 * 0)\,,$$

where $\mathrm{im}(F^1) = T^1$ and $\mu(T^m) = F^1(\tau)$, *as long as we have not already done this for* $T^1, \ldots, T^m$ *with* $m < n$.

Case B. If $\hat{\Omega}(e;\ T^1, \ldots, T^m)\wedge\mu(T^m) \in \star(I_2)$ but $T^1\wedge\mu(T^m) \notin \star(I_2)$ then we set

$$\Omega(e;\ T^1, \ldots, T^m) = T^1\wedge\delta\big(\hat{\Omega}(e;\ T^1, \ldots, T^m)\wedge\mu(T^m)\big)\,.$$

Otherwise, Ω is undefined. It is easy to see that Ω is a remedial $(I_2, \mathbf{0}^{(1)})$-strategy.

Now suppose that e is arbitrary and $T^1, T^2, \ldots$, is a Ω, e-prioric sequence. First, it is clear from the definition of Ω that $\Omega(e;\ T^1, \ldots, T^m)$ is defined at most once for $n > 1$ through Case A. We claim next that $\Omega(e;\ T^1, \ldots, T^m)$ can never be defined through Case B. For, suppose that n is such that $\Omega(e;\ T^1, \ldots, T^m)$ has not been defined through case B for any $m < n$. If $\Omega(e;\ T^1, \ldots, T^m)$ has also not been defined through Case A for any $m < n$ then $\hat{\Omega}(e;\ T^1, \ldots, T^m) = T^2 = \Omega(e;\ T^1)$. Hence, $\Omega(e;\ T^1)\wedge\mu(T^m) \in \star(I_2)$. But if $\Omega(e;\ T^1, \ldots, T^m)$ is defined through Case B then $T^{m+1} = T^1\wedge\delta\big(\Omega(e;\ T^1)\wedge\mu(T^m)\big)$ and so $T^1\wedge\delta\big(\Omega(e;\ T')\wedge\mu(T^m)\big) = T^1\wedge\mu(T^{m+1}) \notin \star(I_2)$. It can then be seen, because of the way in which δ is defined, that $\Omega(e;\ T^1)\wedge\mu(T^m) \notin \star(I_2)$: contradiction. If on the other hand $\Omega(e;\ T^1, \ldots, T^m)$ has been defined through Case A for some $m < n$ then $\hat{\Omega}(e;\ T^1, \ldots, T^m) = T^{m+1} = T^1\wedge\sigma$ for some $\sigma \geqslant \mu(T^m)$. But then

$$\hat{\Omega}(e;\ T^1, \ldots, T^m)\wedge\mu(T^m) = T^1\wedge\mu(T^m) \notin \star(I_2)\,,$$

and so $\Omega(e;\ T^1, \ldots, T^m)$ is not defined through Case B. This proves 5.2 (a).

The proof of $5.2\,(\mathrm{b})^w$ divides into two cases, depending on whether or not $\Omega(e;\ T^1, \ldots, T^m)$ is ever defined through Case A.

If Case A does indeed occur, at stage N say, then it is clear that $F_e^*(B\oplus D)\big(x(T^1)\big) = 1$ if defined. Also, if $B = \lim\big(\mu(T^m)\big)$ then $B \in \mathfrak{N}\big(\hat{\Omega}(e;\ T^1, \ldots, T^m)\big)$ for all $n > N$ so that if $F^1(\tau) = \mu(T^N)$, where F^1 is the $\boldsymbol{S}$-map behind T^1, then $F^1(\tau * 0 * 0) < B$ and hence $B^{(1)}\big(x(T^1)\big) = 0$.

If, on the other hand, Case A does not occur then B belongs to $\mathfrak{N}\big(\Omega(e;\ T^1)\big)$, and it is easy to see that if $F'(\sigma) < B$ then σ is either 0 or of the form $\tau * 1 * i$: theorefore σ is certainly not of the form $\tau * 0 * 0$ and hence $B^{(1)}\big(x(T^1)\big) = 1$. It can also be seen that $F_e^*(B\oplus D)\big(x(T^1)\big)$ is either undefined or $= 0$; for, if $F_e^*(B\oplus D)\big(x(T^1)\big) = 1$ then

$$\Big(F_e\big(\mu(T^m)\oplus D[|\mu(T^m)|]\big)\Big)\big(x(T^1)\big) = 1$$

for some large enough n, contradicting our assumption that Case A never occurs.

All this shows that $B \in \mathfrak{I}_e$ and hence completes the proof of the theorem. ∎

We can first deduce the result of Sasso announced in [7]:

6.4. Corollary. *There is a minimal degree* $\boldsymbol{b} < \mathbf{0}^{(1)}$ *such that* $\boldsymbol{b}^{(1)} > \mathbf{0}^{(1)}$.

Proof. It follows from Theorems 6.2 and 6.3, along with closure under finite intersections, that $\mathfrak{M}' \cap \mathfrak{I}$ is weakly $(I_2, \mathbf{0}^{(1)})$-priorcomeager. Hence $\mathfrak{M}' \cap \mathfrak{I}$ is $(I_2, \mathbf{0}^{(1)})$-priorabundant by 5.5 and contains an element of degree $\leqslant \mathbf{0}^{(1)}$ by 3.4. Clearly, the degree of such an element satisfies this corollary. ∎

This answers a question which we raised in [12] after proving that there is a minimal degree with jump $\mathbf{0}^{(1)}$.

Another question, which we asked in [13], is whether $2^\omega - \mathfrak{I}$ is $(\boldsymbol{C}, \mathbf{0}^{(1)})$-abundant for all $\boldsymbol{C}$, since it can be shown to be $(\boldsymbol{S}^*, \mathbf{0}^{(1)})$-abundant (this was first pointed out to the author by Carl Jockusch, and is essentially a companion result to Sacks' theorem that $2^\omega - \mathfrak{I}$ has measure 1, mentioned and used in Stillwell [10]). This question is answered negatively by framing Sasso's original result in [7] (that there is a minimal degree $< \mathbf{0}^{(2)}$ which does not attain its least possible jump) in the language of $(\boldsymbol{C}, \boldsymbol{d})$-comeager sets: $\mathfrak{I}$ is $(I_0, \mathbf{0}^{(2)})$-comeager, where I_0 is the perfect tree-system consisting of the ranges of all recursive $\boldsymbol{S}$-isomorphisms, and so $2^\omega - \mathfrak{I}$ cannot be $(I_0, \boldsymbol{d})$-comeager for any $\boldsymbol{d}$.

Yet another related question raised in [13] was whether every $(\boldsymbol{C}, \mathbf{0}^{(1)})$-priorcomeager set contains an element with jump $\mathbf{0}^{(1)}$. We conjectured a negative answer and, at least for weakly $(\boldsymbol{C}, \mathbf{0}^{(1)})$-priorcomeager sets, this follows from 6.3: $\mathfrak{I}$ certainly contains no element with jump $\mathbf{0}^{(1)}$. We remain certain that our slightly stronger conjecture is true, but we do not see, for example, how to prove that $\mathfrak{I}$ is $(I_2, \mathbf{0}^{(1)})$-priorcomeager.

Finally, this last observation serves to explain another phenomenon. One of the corollaries which we noted in [12] was that there is a minimal degree whose jump is $\mathbf{0}^{(1)}$; the proof of this involved recursive approximation but we wondered whether this was necessary. A proof by $\mathbf{0}^{(1)}$-approximation would by analogy with other results of this kind, be virtually certain to be convertible in our framework into a proof that every $(\boldsymbol{C}, \mathbf{0}^{(1)})$-priorabundant set contains an element with jump $\mathbf{0}^{(1)}$: hence the necessity of the more complex argument. It is perhaps worth noting that Cooper [1] has used recursive approximation to prove that every degree $\geqslant \mathbf{0}^{(1)}$ is the jump of a minimal degree, and has also proved there that $\mathbf{0}^{(2)}$ is *not* the jump of a minimal degree $< \mathbf{0}^{(1)}$. Jockusch has conjectured that (i) every minimal degree $< \mathbf{0}^{(1)}$ has *double* jump $\mathbf{0}^{(2)}$ (this would strengthen Cooper's second result) and (ii) every degree $\geqslant \mathbf{0}^{(1)}$, which is Σ_2^0 and has jump $\mathbf{0}^{(2)}$, is the jump of a minimal degree $< \mathbf{0}^{(1)}$. This plausible conjecture would (if verified) nicely characterise the jumps of the minimal degrees $<\mathbf{0}^{(1)}$. We note that 6.4 can be strengthened to assert also that $\boldsymbol{b}^{(2)} = \mathbf{0}^{(2)}$, using an observation of Jockusch. In our framework this is proved by showing that *any* $(I_2, \mathbf{0}^{(1)})$-priorabundant set contains an element with double jump $\mathbf{0}^{(2)}$.

§ 7. Conclusion. There are a number of other applications of the ideas in this paper, although they need slight additional complications of format. For example, it is possible to prove that, for any $\boldsymbol{d}$ such that $\mathbf{0}^{(1)} \leqslant \boldsymbol{d} \leqslant \mathbf{0}^{(2)}$, the set $\{B: \boldsymbol{d} \leqslant \boldsymbol{B}^{(1)}\}$ is $\big((C_e), \mathbf{0}^{(1)}\big)$-priorcomeager for certain sequential systems (C_e). We will not go into the details here, but this yields one of the original finite-injury arguments: Shoenfield's proof [8] that if $\mathbf{0}^{(1)} \leqslant \boldsymbol{d} \leqslant \mathbf{0}^{(2)}$ and $\boldsymbol{d}$ is Σ_1^0 in $\mathbf{0}^{(1)}$ then there is a degree $\boldsymbol{b} \leqslant \mathbf{0}^{(1)}$ such that $\boldsymbol{b}^{(1)} = \boldsymbol{d}$. (Shoenfield's result was of course later strengthened by Sacks [4, 5] who used an infinite-injury argument to show that $\boldsymbol{b}$ could be made Σ_1^0.) The point of the new proof is that it is easier to understand with what other techniques the proof may be combined.

Other examples are Shoenfield's proof [9] that if $\mathbf{0} < \boldsymbol{a} < \mathbf{0}^{(1)}$ then there is a minimal degree $\boldsymbol{b} < \mathbf{0}^{(1)}$ such that $\boldsymbol{b} \mid \boldsymbol{a}$; one now simply proves that $\mathfrak{D}(\not\leqslant \boldsymbol{a})$ is $(C, \mathbf{0}^{(1)})$-priorabundant in a rather weaker sense than used in this paper. In fact, in this latter result one may replace $\boldsymbol{a}$ by the elements of a $\mathbf{0}^{(1)}$-uniform sequence of degrees which lie between $\mathbf{0}$ and $\mathbf{0}^{(1)}$: this yields our theorem [11] that there is a degree $<\mathbf{0}^{(1)}$ which is incomparable with all the Σ_1^0 degrees between $\mathbf{0}$ and $\mathbf{0}^{(1)}$. In combination with the classification of $\mathfrak{M}'$ obtain in 6.2 above, this move subtle classification leads to Sasso's strengthening [6] of the results just mentioned; there is a minimal degree $<\mathbf{0}^{(1)}$ which is incomparable with all the Σ_1^0 degrees between $\mathbf{0}$ and $\mathbf{0}^{(1)}$. These classifications were first obtained in [13], and in conjunction with the present paper they show that the entire extant theory, of $\mathbf{0}^{(1)}$-approximating priority-arguments which yield results about $\mathfrak{D}$ $(\leqslant \mathbf{0}^{(1)})$, can be developed in a framework only slightly more complicated than the one which we have set up here. This entire theory will appear in [15].

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