Combinatory logic and the ω-rule

by

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Abstract. The ω-rule for combinatory logic is: $MZ = NZ$ for all closed $Z \Rightarrow M = N$. The following results are proved: Combinatory logic together with the ω-rule is consistent. For a large class of terms, the non-universal generators, the ω-rule is valid, but the ω-rule is not valid in general.

Introduction. The type free theory of combinators is axiomatized by a set of universal axioms. Hence for models of this theory, the interior, i.e. in the language of universal algebra the set of algebraical constants, is again a model.

Adding the axiom of extensionality,

$$\forall x\, Mx = Nx \rightarrow M = N\,,$$

changes the situation. The interior of an extensional model is not a priori extensional. The situation is the following:

i) There exists an extensional model with a non extensional interior [8].

ii) There exists a model with an extensional interior [2].

iii) The free extensional model (free in the category of extensional models with surjective homomorphisms as morphisms) has a non extensional interior [10].

These results will be presented in a proof theoretic context. Extensionality of the interior can be expressed (in an infinitary language) as

$$(*) \qquad \bigwedge_{Z \text{ closed term}} MZ = NZ \rightarrow M = N\,.$$

The corresponding rule

$$(\omega) \qquad MZ = NZ\,, \text{ for all closed } Z \Rightarrow M = N\,,$$

is called the *ω-rule*. (As it is the case with the ω-rule for arithmetic, a universal conclusion ($\forall x\, Mx = Nx$) is drawn from all instances with closed terms). We have

It will be shown that

1. Combinatory logic with the ω-rule is consistent (§ 1), [2], § 2.2.
2. For a large class of combinators, the non universal generators, the ω-rule is valid in the extensional theory (§ 2), [2], § 2.5 (¹).
3. The ω-rule is not generally valid in the extensional theory (§ 3), [10].

By taking termmodels 1 implies ii) and 3 implies iii) and i). Result 2 shows why the construction for 3 had to be complicated.

Remarks. The main part of this paper, § 1 and § 2, is a published version of [2]. We thank Dr. Plotkin for permission to present (in § 3) his interesting result [10]. In [11] a different proof of the consistency of the ω-rule is given. There it is proved that Scott's lattice theoretic models satisfy (∗).

We thank professor Myhill for an improvement in the proof of 2.18.

Notations. $FV(M)$ is the set of free variables of M.

Terms that are equal up to a change of bound variables are identified. $\equiv$ denotes syntactic equality. $[x/N]M$ denotes the result of substituting N for the free occurrences of x in M. In this case we assume that the bound variables of M differ from the free ones in N to prevent confusion of variables.

We assume familiarity with combinatory logic (CL) and the λ-calculus as axiomatized e. g. in [4]. By the standard translations between CL and the λ-calculus the results can be proved in either of the systems. So the most appropriate one is chosen. For result 1, this is CL. Results 2 and 3 are proved in the λ-calculus.

§ 1. The consistency of the ω-rule. Let CLω denote the theory CL together with the ω-rule. By an ordinal analysis the consistency of CLω will be proved. The main step is that if $\mathrm{CL}\omega \vdash M = I$, then $\mathrm{CL} \vdash MI\ldots I = I$, where $I\ldots I$ is some finite string of I's. From this, information about CL+extensionality can be obtained, since this theory is contained in CLω. For example it follows immediately that solvability in CL is the same as in CL+ext (or CLω) and there fore in the λ-calculus (cf. [3], 1.5).

The results about CLω are proved by introducing a conservative extension CLω′ in which it is possible to formalize the ordinal length of a proof in CLω.

(¹) Added in proof: However Parikh's ω-rule [9] is valid in general for this theory (see 2.27).

1.1. Definition. CLω′ has the following language:

$\mathrm{Alphabet}_{\mathrm{CL}\omega'} = \mathrm{Alphabet}_{\mathrm{CL}} \cup \{\approx_\alpha | \alpha \text{ countable}\} \cup \{\sim_\alpha | \alpha \text{ countable}\} \cup \{=_\alpha | \alpha \text{ countable}\}$.

The terms are those of CL.

Formulas. If M, N are terms, then $M \geqslant N$, $M = N$, $M \approx_\alpha N$, $M \sim_\alpha N$ and $M =_\alpha N$ are formulas.

CLω′ is defined by the following axiomschemes and rules:

I. 1. $IM \geqslant M$,
2. $KMN \geqslant M$,
3. $SMNL \geqslant ML(NL)$.

II. 1. $M \geqslant M$,
2. $M \geqslant N,\ N \geqslant L \Rightarrow M \geqslant L$,
3. $M \geqslant M' \Rightarrow ZM \geqslant ZM',\ MZ \geqslant M'Z$.

III. 1. $M \geqslant M' \Rightarrow M \approx_0 M'$,
2. $M \approx_\alpha M' \Rightarrow M \sim_\alpha M'$,
3. $M \sim_\alpha M' \Rightarrow M =_\alpha M'$,
4. $M =_\alpha M' \Rightarrow M = M'$.

IV. 1. $M =_\alpha M,\ M \approx_\alpha M,\ M \sim_\alpha M$,
2. $M =_\alpha N \Rightarrow N =_\alpha M,\ M \approx_\alpha N \Rightarrow N \approx_\alpha M,\ M \sim_\alpha N \Rightarrow N \sim_\alpha M$,
3. $M = N,\ N = L \Rightarrow M = L$,
4. $M = M' \Rightarrow ZM = ZM',\ MZ = M'Z$,
$M \sim_\alpha M' \Rightarrow ZM \sim_\alpha ZM',\ MZ \sim_\alpha M'Z$,
5. $M =_\alpha M',\ \alpha \leqslant \beta \Rightarrow M =_\beta M'$.

V. $\forall Z$ closed $\exists \beta < \alpha\ MZ = NZ \Rightarrow M \approx_\alpha N$.

In the above M, M', N, L and Z denote arbitrary terms, and α, β arbitrary countable ordinals.

The intuitive interpretation of

$M =_\alpha N$ is: $M = N$ is provable using the ω-rule at most α times.

$M \sim_\alpha N$ is: $M =_\alpha N$ is provable without use of transitivity.

$M \approx_\alpha N$ is: $M =_\alpha N$ follows directly form the ω-rule (or is provable in CL in case $\alpha = 0$).

The following properties follow inductively from the definitions.

1.2. Lemma.

i) $\mathrm{CL} \vdash M \geqslant N \Leftrightarrow \mathrm{CL}\omega \vdash M \geqslant N \Leftrightarrow \mathrm{CL}\omega' \vdash M \geqslant N$.

ii) $\mathrm{CL}\omega' \vdash M = N \Leftrightarrow \exists \alpha\ \mathrm{CL}\omega' \vdash M =_\alpha N$.

iii) $\mathrm{CL}\omega \vdash M = N \Leftrightarrow \mathrm{CL}\omega' \vdash M = N$.

iv) $\mathrm{CL} \vdash M = N \Leftrightarrow \mathrm{CL}\omega' \vdash M =_0 N$.

v) $\mathrm{CL}\omega' \vdash M =_\alpha N \Leftrightarrow \exists N_1 \ldots N_k\ \exists \beta_1 \ldots \beta_k \leqslant \alpha$, $[\mathrm{CL}\omega' \vdash M \sim_{\beta_1} N_1 \sim_{\beta_2} \ldots \sim_{\beta_k} N_k \equiv N]$.

vi) $\mathrm{CL}\omega' \vdash M \approx_\alpha N,\ \alpha \neq 0 \Leftrightarrow \forall Z$ *closed* $\exists \beta < \alpha\ \mathrm{CL}\omega' \vdash MZ =_\beta NZ$.

1.3. Lemma.

$$\mathrm{CL}\omega' \vdash M \sim_a N,\ M, N \text{ closed} \Rightarrow$$

$$\exists P, Q, Z \text{ closed } [\mathrm{CL}\omega' \vdash ZP =_0 M,\ \mathrm{CL}\omega' \vdash ZQ =_0 N \text{ and } \mathrm{CL}\omega' \vdash P \approx_a Q].$$

Proof. $\sim_a$ is the monotone closure of $\approx_a$. So if $M \sim_a N$, then for a certain context $\mathcal{C}[\ ]$ and certain P, Q $M \equiv \mathcal{C}[P]$, $N \equiv \mathcal{C}[Q]$ and $P \approx_a Q$. Then take $Z \equiv \lambda x\, \mathcal{C}[x]$ and use 1.2 iv). ■

1.4. Main Lemma. *Suppose $a \neq 0$ and M, N and Z are closed, then*

$$[\mathrm{CL}\omega' \vdash ZM \geqslant I \wedge \mathrm{CL}\omega' \vdash M \approx_a N] \Rightarrow \exists \beta < a [\mathrm{CL}\omega' \vdash ZNI =_\beta I].$$

The proof of this fact occupies 1.5-1.12. An auxiliary theory $\underline{\mathrm{CL}}$, which is a conservative extension of CL, will be introduced. $\underline{\mathrm{CL}}$ examines the behaviour of a subterm and its "residuals" by underlining them. Attention is paid to see if in the reduction of ZM the occurrences of M become "active" (in a context (MA)) or "passive" (in a context (AM)).

1.5. Definition. $\underline{\mathrm{CL}}$ has the following language.

$\mathrm{Alphabet}_{\underline{\mathrm{CL}}} = \mathrm{Alphabet}_{\mathrm{CL}} \cup \{\geqslant_1, \_\}$.

Simple terms of $\underline{\mathrm{CL}}$ are the terms of CL.

Terms are defined inductively by

1) Any simple term is a term.

2) If M is a simple term, then $\underline{M}$ is a term.

3) If M, N are terms, then (MN) is a term.

Formulas. If M, N are terms, then $M \geqslant_1 N$, $M \geqslant N$ and $M = N$ are formulas.

For a $\underline{\mathrm{CL}}$ term, we define $|M|$ to be M itself without any underlining. If M, N are $\underline{\mathrm{CL}}$ terms, then we define $M \simeq N \Leftrightarrow |M| \equiv |N|$. $\underline{\mathrm{CL}}$ is defined by the following axiomschemes and rules:

I. 1. $IM \geqslant_1 M$,
   2. $KMN \geqslant_1 M$,
   3. $SMNL \geqslant_1 ML(NL)$.

II. 1. $M \geqslant_1 M$,
   2. $M \geqslant_1 M' \Rightarrow ZM \geqslant_1 ZM'$, $MZ \geqslant_1 M'Z$,
   3. $M \geqslant_1 M' \Rightarrow \underline{M} \geqslant_1 \underline{M'}$,
   4. $M \geqslant_1 M' \Rightarrow M \geqslant M'$.

III. 1. $M \geqslant N$, $N \geqslant L \Rightarrow M \geqslant L$,
   2. $M \geqslant N \Rightarrow M = N$.

IV. 1. $M = N \Rightarrow N = M$,
   2. $M = N$, $N = L \Rightarrow M = L$.

V. $\underline{M}N \geqslant_1 MN$.

In the above M, M', N, L and Z denote arbitrary terms except in II. 3 and V where M, M' denote simple terms.

The following properties follow inductively from the definitions.

1.6. Lemma.

i) $\underline{\mathrm{CL}} \vdash M \geqslant M' \Leftrightarrow \exists N_1 \ldots N_k\ \underline{\mathrm{CL}} \vdash M \equiv N_1 \geqslant_1 \ldots \geqslant_1 N_k \equiv M'$.

ii) $\underline{\mathrm{CL}} \vdash M \geqslant M' \Leftrightarrow \mathrm{CL} \vdash M \geqslant M'$ *for simple terms* M, M'.

iii) $\underline{\mathrm{CL}} \vdash M \geqslant M' \Leftrightarrow \underline{\mathrm{CL}} \vdash \underline{M} \geqslant \underline{M'}$ *for simple terms* M, M'.

iv) $[\underline{\mathrm{CL}} \vdash M \geqslant_1 M'$ *and* $\underline{N'}$ sub $M'] \Rightarrow \exists N [\underline{N}$ sub M *and* $\mathrm{CL} \vdash N \geqslant_1 N']$.

1.7. Lemma. *Let Z be simple, then*

$$[\underline{\mathrm{CL}} \vdash Z\underline{M} \geqslant M' \text{ and } \underline{N} \text{ sub } M'] \Rightarrow \mathrm{CL} \vdash M \geqslant N.$$

Proof. By 1.6 i) and iv). ■

1.8. Lemma. *Let M, M' be simple terms, then* (see fig. 1)

$$[\underline{\mathrm{CL}} \vdash M \geqslant M' \text{ and } M \simeq N] \Rightarrow \exists N' [\underline{\mathrm{CL}} \vdash N \geqslant N' \text{ and } M' \simeq N'].$$

Proof. By induction on the length of proof of $M \geqslant M'$. ■

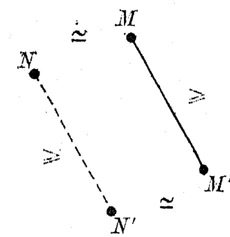

Fig. 1

1.9. Definition. Let A be a CL term. φ_A is them apping $\underline{\mathrm{CL}} \to \mathrm{CL}$ defined as follows:

$\varphi_A(c) = c$ if c is a constant or variable,

$\varphi_A(MN) = \varphi_A(M)\varphi_A(N)$, $\varphi_A(\underline{M}) = A$.

1.10. Lemma. *If* $\underline{\mathrm{CL}} - \{V\} \vdash M \geqslant_1 M'$, *then* $\mathrm{CL} \vdash \varphi_N(M) \geqslant \varphi_N(M')$ *for simple terms* N.

Proof. Immediate. ■

1.11. Lemma. *Let Z, M and M' be closed $\underline{\mathrm{CL}}$ terms such that Z and M are simple. Then*

$$[\underline{\mathrm{CL}} \vdash Z\underline{M} \geqslant M' \text{ and } \mathrm{CL}\omega' \vdash M \approx_a N] \Rightarrow \exists \beta < a\ \mathrm{CL}\omega' \vdash \varphi_N(Z\underline{M}) =_\beta \varphi_N(M').$$

Proof. Suppose $\underline{\mathrm{CL}} \vdash Z\underline{M} \geqslant M'$ and $\mathrm{CL}\omega' \vdash M \approx_a N$. Then

$$\exists N_1 \ldots N_k\ \underline{\mathrm{CL}} \vdash Z\underline{M} \equiv N_1 \geqslant_1 \ldots \geqslant_1 N_k \equiv M'.$$

By induction on the length of proof of $N_i \geqslant_1 N_{i+1}$ we show that

$$(*) \qquad \exists \beta_i < a\ \mathrm{CL}\omega' \vdash \varphi_N(N_i) =_{\beta_i} \varphi_N(N_{i+1}).$$

If $N_i \geqslant_1 N_{i+1}$ is an instance of an axiomscheme not being V, $(*)$ follows from 1.10 and 1.2 iv) by taking $\beta_i = 0$. If $N_i \geqslant_1 N_{i+1}$ is an instance of axiom V, say $\underline{M_1}M_2 \geqslant_1 M_1M_2$, then we have to show that

$$(**) \qquad \exists \beta < \alpha\ \mathrm{CL}\omega' \vdash N\varphi_N(M_2) = M_1\varphi_N(M_2)$$

because M_1 is simple and hence $\varphi_N(M_1) \equiv M_1$. Since $\underline{\mathrm{CL}} \vdash Z\underline{M} \geqslant \underline{M_1}M_2$ it follows from 1.7 that $\mathrm{CL} \vdash M \geqslant M_1$. Hence since $\mathrm{CL}\omega' \vdash M \approx_\alpha N$ it follows from 1.2 vi) and 1.2 iv) that

$$\exists \beta < \alpha\ \mathrm{CL}\omega' \vdash N\varphi_N(M_2) =_\beta M\varphi_N(M_2) =_0 M_1\varphi_N(M_2)\,.$$

This implies $(**)$.

In the other cases $(*)$ follows easily from the induction hypothesis. Now we have established $(*)$. Let $\beta = \mathrm{Max}\{\beta_0, \ldots, \beta_k\}$ then $\beta < \alpha$ and

$$\mathrm{CL}\omega' \vdash \varphi_N(Z\underline{M}) =_\beta \varphi_N(N_1) =_\beta \ldots =_\beta \varphi_N(N_k) =_\beta \varphi_N(M')\,. \ \blacksquare$$

1.12. Proof of 1.4. Let $\mathrm{CL}\omega' \vdash ZM \geqslant I$, $\mathrm{CL}\omega' \vdash M \approx_\alpha N$, $\alpha \neq 0$ where M, N, Z are closed.

By 1.2 i) $\mathrm{CL} \vdash ZM \geqslant I$, hence by 1.6 ii) $\underline{\mathrm{CL}} \vdash ZM \geqslant I$ and therefore by 1.8 $\underline{\mathrm{CL}} \vdash Z\underline{M} \geqslant I'$ with $I' \equiv I$ or $I' \equiv \underline{I}$.

Case 1. $I' \equiv I$. By 1.11 it follows that $\exists \beta < \alpha\ \mathrm{CL}\omega' \vdash ZN =_\beta I$, hence a fortiori $\exists \beta < \alpha [\mathrm{CL}\omega' \vdash ZNI =_\beta II =_0 I]$.

Case 2. $I' \equiv \underline{I}$. Then $\mathrm{CL} \vdash Z\underline{M} \geqslant \underline{I}$, hence by 1.7,

$$(1) \qquad \mathrm{CL} \vdash M \geqslant I\,,$$

and by 1.11 $\exists \beta < \alpha\ \mathrm{CL}\omega' \vdash ZN =_\beta N$. Therefore

$$(2) \qquad \exists \beta < \alpha\ \mathrm{CL}\omega' \vdash ZNI =_\beta NI\,.$$

Since $\mathrm{CL}\omega' \vdash M \approx_\alpha N$, it follows from 1.2 vi) that

$$(3) \qquad \exists \beta' < \alpha\ \mathrm{CL}\omega' \vdash NI =_{\beta'} MI\,.$$

From (2), (3) and (1) we have

$$\exists \beta, \beta' < \alpha\ \mathrm{CL}\omega' \vdash ZNI =_\beta NI =_{\beta'} MI =_0 II\,.$$

Hence for $\beta'' = \max\{\beta, \beta'\}$

$$\mathrm{CL}\omega' \vdash ZNI =_{\beta''} I \quad \text{and} \quad \beta'' < \alpha\,. \ \blacksquare$$

1.13. Notation. MI_n stands for $M\underbrace{I \ldots I}_{n\,\text{times}}$. Note that I_n is not a term, because $MI \ldots I$ stands for $(\ldots((MI)) \ldots I)$.

1.14. Lemma. *Let M be closed. Then*

$$(*) \qquad \mathrm{CL}\omega' \vdash M =_\alpha I \Rightarrow \exists n \in \omega\ \mathrm{CL} \vdash MI_n \geqslant I\,.$$

Proof. Induction on α. Because we will make use of a double induction the induction hypotheses with respect to this induction is called the α-ind. hyp.

Case 1. $\alpha = 0$. Then $\mathrm{CL}\omega' \vdash M =_\alpha I$ implies that $\mathrm{CL} \vdash M = I$ by 1.2 iv), hence $\mathrm{CL} \vdash M \geqslant I$ by the well known Church–Rosser CR property for CL.

Case 2. $\alpha > 0$. From 2.2.7 it follows that

$$\mathrm{CL}\omega' \vdash M =_\alpha I \Leftrightarrow \exists M_1 \ldots M_k\ \exists \beta_1, \ldots, \beta_k \leqslant \alpha\,,$$

$$(**) \qquad \mathrm{CL}\omega' \vdash M \sim_{\beta_1} M_1 \sim_{\beta_2} M_2 \ldots \sim_{\beta_k} M_k \geqslant I\,.$$

We can suppose that the M_i, $i = 1, \ldots, k$ are all closed. We show by induction on k that $(**) \Rightarrow (*)$.

The induction hypothesis w.r.t. this induction is called the k-ind. hyp.

If $k = 0$ then there is nothing to prove, so suppose that $k > 0$.

Subcase 2.1. $\beta_k < \alpha$. Then $\mathrm{CL}\omega' \vdash M_{k-1} \sim_\beta M_k \geqslant I$ with $\beta = \beta_k$ therefore $\mathrm{CL}\omega' \vdash M_{k-1} =_\beta I$. Hence by the α-ind. hyp. $\exists n \in \omega\ \mathrm{CL} \vdash M_{k-1}I_n \geqslant I$, because we assumed that M_{k-1} is closed. Thus

$$\exists n \in \omega\ \mathrm{CL}\omega' \vdash MI_n \sim_{\beta_1} M_1I_n \sim \ldots \sim_{\beta_{k-1}} M_{k-1}I_n \geqslant I\,,$$

hence by the k-ind. hyp.

$$\exists n, n' \in \omega\ \mathrm{CL} \vdash MI_nI_{n'} \geqslant I\,,$$

which is

$$\exists n, n' \in \omega\ \mathrm{CL} \vdash MI_{(n+n')} \geqslant I\,.$$

Subcase 2.2. $\beta_k = \alpha$. Then $\mathrm{CL}\omega' \vdash M_{k-1} \sim_\alpha M_k \geqslant I$. By 1.3 it follows that there are M'_{k-1}, M'_k, Z such that

$$\mathrm{CL}\omega' \vdash ZM'_{k-1} =_0 M_{k-1}\,, \qquad \mathrm{CL}\omega' \vdash ZM'_k =_0 M_k\,,$$

and

$$\mathrm{CL}\omega' \vdash M'_{k-1} \approx_\alpha M'_k\,.$$

Hence by 1.2. iv) and the CR property it follows that $\mathrm{CL}\omega' \vdash ZM'_k \geqslant I$. By 1.4 it follows that $\exists \beta < \alpha [\mathrm{CL}\omega' \vdash ZM'_{k-1}I =_\beta I]$, thus

$$\exists \beta < \alpha [\mathrm{CL}\omega' \vdash M_{k-1}I =_\beta I]\,.$$

Hence by the α-ind. hyp.

$$\exists n \in \omega\ \mathrm{CL} \vdash M_{k-1}II_n \geqslant I\,,$$

thus

$$\exists n \in \omega\ \mathrm{CL}\omega' \vdash MII_n \sim_{\beta_1} M_1II_n \sim \ldots \sim_{\beta_{k-1}} M_{k-1}II_n \geqslant I\,.$$

Therefore by the k-ind. hyp. we have

$$\exists n, n' \in \omega\ \mathrm{CL} \vdash MII_nI_{n'} \geqslant I$$

i.e.

$$\exists n, n' \in \omega\ \mathrm{CL} \vdash MI_{(n+n'+1)} \geqslant I. \ \blacksquare$$

1.15. THEOREM. *Let M be closed. Then* $\mathrm{CL}\omega \vdash M = I \Rightarrow \exists n \in \omega\ \mathrm{CL} \vdash MI_n = I$.

Proof. Immediate from 1.2 ii) and 1.14. ■

1.16. COROLLARY. $\mathrm{CL}\omega$ *is consistent.*

Proof. Let $\Omega_2 \equiv (\lambda x \cdot xx)(\lambda x \cdot xx)$. If $\mathrm{CL}_\omega \vdash \Omega_2 = I$, then by 1.15 $\mathrm{CL} \vdash \Omega_2 I_n = I$ for some n.

By the Church–Rosser property for CL it follows that $\mathrm{CL} \vdash \Omega_2 I_n \geqslant I$ which is absurd. ■

Let $\lambda\omega$ be the λ-calculus together with the ω-rule. By the standard translations between CL and the λ-calculus we have

1.17. COROLLARY. i) *Let M be closed. Then*

$$\lambda\omega \vdash M = I \Rightarrow \exists n \in \omega \lambda \vdash MI_n = I .$$

ii) *$\lambda\omega$ is consistent.*

§ 2. The validity of the ω-rule for non-universal generators. In this section it is proved that for a large class of terms the ω-rule is valid. The details are carried out in the λ-calculus because there extensionality can be axiomatized by so called *η-reduction*

$$(\eta) \qquad \lambda x \cdot Mx \geqslant M , \quad \text{where} \quad x \notin \mathrm{FV}(M):$$

if $Mx = Nx$, $x \notin \mathrm{FV}(MN)$, then $M = \lambda x \cdot Mx = \lambda x \cdot Nx = N$.

The λ-calculus extended with η-reduction will be denoted by $\lambda\eta$.

2.1. DEFINITION i) The *family* $\mathcal{F}(M)$ of a term M is the following set of terms $\mathcal{F}(M) = \{N |\ \exists M' \lambda\eta \vdash M \geqslant M' \text{ and } N \operatorname{sub} N'\}$.

ii) A term M is an *universal generator* (u.g.) if $\mathcal{F}(M)$ consists of all closed terms.

We will show that if M and N are no u.g.'s, then $\lambda\eta \vdash MZ = NZ$ for all closed $Z \Rightarrow \lambda\eta \vdash M = N$. By the same method it can be proved that $\lambda \vdash MZ = NZ$ for all closed $Z \Rightarrow \lambda \vdash Mx = Nx$, for all x, with the same restrictions on M and N.

The idea is to use a closed term of order zero like $\Omega_2 \equiv (\lambda x \cdot xx)(\lambda x \cdot xx)$: If $MZ = NZ$ for all closed Z, then also $M\Omega_2 = N\Omega_2$. Because Ω_2 behaves almost like a variable we would like to substitute for Ω_2 a new variable x, obtaining $Mx = Nx$ and hence by extensionality $M = N$. However there is a difference between a variable and Ω_2: in a reduction a new variable can never be generated, whereas Ω_2 can. Therefore we should use a different term of order zero Ω, which will not be generated by M or N. If M or N is a u.g. there is no such Ω. Therefore with this method, the validity of the ω-rule can be proved only for non universal generators. First it will be shown that u.g.'s do exist. This fact facilates the proof of the main theorem.

2.2. LEMMA. *There exists a universal generator.*

Proof. (Modification by D. Scott of our original construction [1]). Let $\underline{0}, \underline{1}, \underline{2}, \ldots$ be the numerals in the λ-calculus as defined in [5]. Then there exists a term E such that $\forall Z$ closed $\exists n\ \lambda \vdash E\underline{n} = Z$, see [7], p. 254. Examining a proof of this fact one can actually show that $\exists E\ \forall Z$ closed $\exists n\ \lambda \vdash E\underline{n} \geqslant Z$, see [1].

By the fixed point theorem one can define a B such that $Bx \geqslant Bx^{+}(Ex)$, $\lambda x \cdot x^{+}$ represents the successor function. Then $B\underline{0}$ is a u.g. since

$$B\underline{0} \geqslant B\underline{1}(E\underline{0}) \geqslant B\underline{2}(E\underline{1})(E\underline{0}) \geqslant \ldots \ \blacksquare$$

2.3. DEFINITION. A λ-term Z is *of order* 0 if there is no term P such that $\lambda \vdash Z \geqslant \lambda x \cdot P$.

2.4. EXAMPLES. Any variable is of order 0. $\Omega_2 \equiv (\lambda x \cdot xx)(\lambda x \cdot xx)$ is of order 0.

2.5. LEMMA. *Let Z be of order* 0, *then:*

i) *For no term P we have $\lambda\eta \vdash Z \geqslant \lambda x \cdot P$.*

ii) *If $\lambda\eta \vdash Z \geqslant Z'$, then Z' is of order* 0.

iii) *If $\lambda\eta \vdash ZM \geqslant N$, then there exist terms Z', M' such that $N \equiv Z'M'$, $\lambda\eta \vdash Z \geqslant Z'$ and $\lambda\eta \vdash M \geqslant M'$.*

iv) *For all terms M, ZM is of order* 0.

Proof. For this proof let us call a term of the first kind if it is a variable, of the second kind if it is of the form (MN) and of the third kind if it is of the form $(\lambda x \cdot M)$.

i) Suppose $\lambda\eta \vdash Z \geqslant \lambda x \cdot P$ for some P. By [6], Ch. 4D, Theorem 2, p. 132 it follows that there exists a term Z' such that $\lambda \vdash Z \geqslant Z'$ and $\lambda\eta^{-} \vdash Z' \geqslant (\lambda x \cdot P)$ (where $\lambda\eta^{-}$ is $\lambda\eta$ without the axiom scheme $(\lambda x \cdot M)N \geqslant [x/N]\, M$ (β-reduction)). Because Z is of order 0, Z' is of the first or of the second kind. Z' cannot be a variable since $\lambda\eta \vdash Z' \geqslant \lambda x \cdot P$. Hence Z' is of the second kind. By induction on the length of proof in $\lambda\eta^{-}$ of a reduction $M \geqslant N$ we can show that if M is of the second kind, then N is of the second kind. This would imply that $(\lambda x \cdot P)$ is of the second kind, contradiction.

ii) Immediate, using i).

iii) By induction on the length of proof of $ZM \geqslant N$ using ii).

iv) By iii) it follows that if $\lambda\eta \vdash ZM \geqslant N$, then N is of the second kind. Hence ZM is of order 0. ■

As in the previous section we need an auxiliary theory which is a conservative extension of $\lambda\eta$.

2.6. Definition. $\underline{\lambda\eta}$ is a theory formulated in the following language $\mathrm{Alphabet}_{\underline{\lambda\eta}} = \mathrm{Alphabet}_{\lambda} \cup \{\geqslant_1, \_\}$.
Simple terms of the theory $\underline{\lambda\eta}$ are exactly the terms of the λ-calculus.
Terms are defined inductively by
i) Any simple term is a term.
ii) If M is a simple term and $\mathrm{FV}(M) = \emptyset$, then $\underline{M}$ is a term.
iii) If M, N are terms, then (MN) is a term.
iv) If M is a term, then $(\lambda x \cdot M)$ is a term (x is an arbitrary variable).
Formulas. If M, N are terms, then $M \geqslant_1 N$, $M \geqslant N$, and $M = N$ are formulas.

A term of the theory $\underline{\lambda\eta}$ is called $\underline{\lambda}$-term.

The operation BV, FV and $[x/N]$ can be extended to $\underline{\lambda}$-terms in the obvious way.

(Note that: $\mathrm{BV}(\underline{M}) = \mathrm{BV}(\underline{M})$, $\mathrm{FV}(\underline{M}) = \emptyset$ and $[x/N]\,\underline{M} = \underline{M}$.)

2.7. Definition. i) The relation "... is a subterm of ..." is defined in such a way that only $\underline{M}$ is a subterm of $\underline{M}$. To be explicit:

$$\begin{aligned}
\mathrm{Sub}(x) &= \{x\} \quad \text{for any variable } x, \\
\mathrm{Sub}(MN) &= \mathrm{Sub}(M) \cup \mathrm{Sub}(N) \cup \{MN\}, \\
\mathrm{Sub}(\lambda x \cdot M) &= \mathrm{Sub}(M) \cup \{\lambda x \cdot M\}, \\
\mathrm{Sub}(\underline{M}) &= \{\underline{M}\}, \\
N \operatorname{sub} M &\Leftrightarrow N \in \mathrm{Sub}(M).
\end{aligned}$$

ii) $M \simeq N$ is defined as in 1.5.

2.8. Definition. $\underline{\lambda\eta}$ is defined by the following axioms and rules
I. 1. $(\lambda x M)N \geqslant_1 [x/N]\,M$,
2. $\lambda x \cdot Mx \geqslant_1 M$, if $x \notin \mathrm{FV}(M)$.
II. Same as in 1.5, together with $M \geqslant_1 M' \Rightarrow \lambda x \cdot M \geqslant_1 \lambda x \cdot M'$.
III. Same as in 1.5.
IV. Same as in 1.5.

2.9. Lemma i) $\underline{\lambda\eta} \vdash M \geqslant M' \Leftrightarrow \exists N_1 \;\ldots\; N_k \; \underline{\lambda\eta} \vdash M \equiv N_1 \geqslant_1 \ldots \geqslant_1 N_k \equiv M'$.
ii) $\underline{\lambda\eta} \vdash M \geqslant M' \Leftrightarrow \lambda\eta \vdash M \geqslant M'$, *for simple terms* M, M'.
iii) $\underline{\lambda\eta} \vdash M \geqslant M' \Leftrightarrow \underline{\lambda\eta} \vdash \underline{M} \geqslant \underline{M'}$, *for simple terms* M, M'.
iv) $[\underline{\lambda\eta} \vdash M \geqslant_1 M'$ *and* $\underline{N'} \operatorname{sub} M'] \Rightarrow \exists N [N \operatorname{sub} M$ *and* $\lambda\eta \vdash N \geqslant N']$.

Proof. By induction on the length of proof. For iv) use

$$\underline{N} \operatorname{sub} [x/Q]P \Rightarrow \underline{N} \operatorname{sub} P \text{ or } \underline{N} \operatorname{sub} Q. \;\blacksquare$$

2.10. Lemma $[\underline{\lambda\eta} \vdash M\underline{A} \geqslant M'$ *and* $\underline{A'} \operatorname{sub} M'] \Rightarrow \lambda\eta \vdash A \geqslant A'$.

Proof. By 2.9 i) and iv). ■

2.11. Lemma. *Suppose* M, Z *and* L *are* λ-*terms and* Z *is of order* 0. *Then* $\lambda\eta \vdash MZ \geqslant L \Rightarrow \exists L' [\underline{\lambda\eta} \vdash M\underline{Z} \geqslant L', L \simeq L'$ *and* $\underline{Z'} \operatorname{sub} L' \Rightarrow \lambda\eta \vdash Z \geqslant Z']$.

Proof. A $\underline{\lambda}$-term P is called proper if $\underline{Z} \operatorname{sub} P \Rightarrow Z$ is of order 0. Show that

$$[\underline{\lambda\eta} \vdash M \geqslant N,\; M \simeq M' \text{ and } M' \text{ proper}] \Rightarrow \exists N' [\underline{\lambda\eta} \vdash M' \geqslant N',\; N \simeq N' \text{ and } N' \text{ proper}],$$

by induction on the length of proof of $M \geqslant N$. Then the theorem follows by 2.10. ■

2.12. Definition. Let x be any variable. A mapping φ_x: $\underline{\lambda}$-terms $\rightarrow \lambda$-terms is defined as follows:

$$\begin{aligned}
\varphi_x(y) &= y, \\
\varphi_x(MN) &= \varphi_x(M)\varphi_x(N), \\
\varphi_x(\lambda y \cdot M) &= \lambda y \cdot \varphi_x(M), \\
\varphi_x(\underline{M}) &= x.
\end{aligned}$$

2.13. Lemma. *If* $\underline{\lambda\eta} \vdash M \geqslant N$ *and if* x *is a variable not occurring in this proof, then* $\lambda\eta \vdash \varphi_x(M) \geqslant \varphi_x(N)$.

Proof. Induction on the length of proof of $M \geqslant N$, using the following sublemma. If $z \neq y$, then $\varphi_z([y/N]M) = [y/\varphi_z(N)]\varphi_z(M)$. The proof of the sublemma proceeds by induction on the structure of M. ■

2.14. Lemma. *Let* M, N *be simple and* $x \notin \mathrm{FV}(M)$. *If* $\lambda\eta \vdash Mx \geqslant N$, *then* $\exists M'$ *simple* $[x \notin \mathrm{FV}(M')$, $\lambda\eta \vdash M \geqslant M'$ *and* $\underline{\lambda\eta} \vdash M'x \geqslant_1 N]$.

Proof. Because $\lambda\eta \vdash Mx \geqslant N$ we have by 2.9 ii) and i) that

$$\exists N_1 \ldots N_k \; \underline{\lambda\eta} \vdash Mx \equiv N_1 \geqslant_1 \ldots \geqslant_1 N_k \equiv N.$$

If all N_i, $i < k$ are of the form Px with $x \notin \mathrm{FV}(P)$, then we are done. Otherwise let N_{i+1} be the first term not of the form Px with $x \notin \mathrm{FV}(P)$. Then N_i is of the form $(\lambda z \cdot N_i')x$. By a change of bound variable this is $(\lambda x \cdot [z/x]N_i')x \equiv (\lambda x \cdot N_{i+1})x$. Hence

$$\underline{\lambda\eta} \vdash Mx \geqslant (\lambda x \cdot N_{i+1})x \geqslant (\lambda x \cdot N)x \geqslant_1 N.$$

So we can take $M' \equiv \lambda x \cdot N$. ■

2.15. Lemma. *Suppose* M *is a* λ-*term, then*

$$[\underline{\lambda\eta} \vdash M\underline{A} \geqslant L,\; Z \operatorname{sub} L \text{ and } Z \text{ simple}] \Rightarrow Z \in \mathcal{F}(M).$$

Proof. Without loss of generality we may assume that if $\underline{A'} \operatorname{sub} L$, then $\underline{A'} \equiv \underline{A}$. Let x not occur in the proof of $M\underline{A} \geqslant L$. By 2.13 it follows that $\lambda\eta \vdash Mx \geqslant \varphi_x(L)$. Hence by 2.14 there exists a λ-term M' such that $\lambda\eta \vdash M \geqslant M'$ and $\underline{\lambda\eta} \vdash M'x \geqslant_1 \varphi_x(L)$. Hence $\underline{\lambda\eta} \vdash M'\underline{A} \geqslant_1 [x/\underline{A}]\varphi_x(L) \equiv L$. By distinguishing the different possibilities for the proof of $M'\underline{A} \geqslant_1 L$ we can conclude that $Z \operatorname{sub} M'$, hence $Z \in \mathcal{F}(M)$. ■

2.16. Definition.

i) A term M is called an *Ω_2-term* if M is of the form $\Omega_2 M'$.

ii) A subterm occurrence Z of M is called *non-Ω_2 in M* if Z has no Ω_2 subterm and Z is not a subterm of an Ω_2 subterm of M.

iii) A term U is called a *hereditarily non-Ω_2 universal generator* if U is a closed u.g. and for U' with $\lambda\eta \vdash U \geqslant U'$, there is a subterm occurrence Z of U' which is a u.g. and which occurs non-Ω_2 in U'.

Example. Only the second occurrence of Z in the term $x(\Omega_2(MZ))Z$ is non-Ω_2 (if Z does not have an Ω_2 subterm).

2.17. Lemma. *If U is a hereditarily non-Ω_2 u.g. and if $\lambda\eta \vdash U \geqslant U'$, then U' is a u.g. which is not an Ω_2-term.*

Proof. Immediate. ■

2.18. Lemma. *There exists a closed hereditarily non-Ω_2 universal generator.*

Proof. Let E be as in 2.2. Define $F \equiv xIIE'(xII)x$, where E' is a normal form such that $\lambda \vdash E'I \geqslant E$ (see [4], 2.12). Then F is in normal form and $\lambda \vdash [x/\underline{n}]F \geqslant E\underline{n}$, since $\lambda \vdash \underline{n}II \geqslant I$. Let $\overline{x+}$ be the normal form of x^+, i.e. $\lambda bc\cdot b(xbc)$. Define

$$A \equiv \lambda bx\cdot bb\underline{0}(bb\overline{x+})F \quad \text{and} \quad B \equiv AA.$$

Then

$$\lambda \vdash B\underline{n} \geqslant B\underline{0}(B\underline{n+1})(E\underline{n}).$$

Hence, as in 2.2, $B\underline{0}$ is a universal generator. We claim that it is hereditarily non-Ω_2. Define

$$P \geqslant_k Q \Leftrightarrow \exists N_1 \ldots N_k \lambda\eta \vdash P \equiv N_1 \geqslant_1 \ldots \geqslant_1 N_k \equiv Q.$$

Suppose now $\lambda\eta \vdash B\underline{0} \geqslant U$. Then for some k, $B\underline{0} \geqslant_k U$. Since A is in normal form, U is of the form $B\underline{0}$, $(\lambda x\cdot U'PQ)\underline{0}$ or $U'PQ$, where $B\underline{0} \geqslant_r U'$ for some $r < k$.

Hence it follows by induction on k that U has a subterm which is a u.g. and non-Ω_2 in U. ■

2.19. Definition. A closed term Ξ is called *variable like* if $\Xi \equiv \Omega_2 U$, where U is a hereditarily non-Ω_2 universal generator.

2.20. Definition. Let L, L' be $\underline{\lambda}$-terms such that L is simple and $L \simeq L'$. Then L and L' are equal except for the underlining and we can give the following informal definitions:

i) If Z' is a subterm occurrence of L', then there is a unique subterm occurrence Z of L which *corresponds to* Z', such that $Z \simeq Z'$.

Instead of giving a formal definition we illustrate this concept with an example: Let $L \equiv S(KS)(SKK)$ and $L' \equiv S(\underline{KS})(\underline{SKK})$, then $L \simeq L'$. S corresponds to S, KS corresponds to $\underline{KS}$ and (SKK) corresponds to $(\underline{SKK})$.

ii) Let L'' be another $\underline{\lambda}$-term with $L \simeq L''$. Then we say that L'' *has more line than* L', notation $L' \subset L''$, if for all subterm occurrences $\underline{Z}$ of $\underline{L}'$ there is a subterm occurrence $\underline{Z}''$ of L'' such that Z' sub Z'' where Z', Z'' are the subterm occurrences of L corresponding to $\underline{Z}'$, $\underline{Z}''$ respectively.

For example, let $L'' \equiv S(\underline{KS})(\underline{SKK})$ then $L' \subset L''$ where L' is as in the above example.

iii) Let Z be a subterm occurrence of L. Z is *exactly underlined* in L' if $\underline{Z}$ is a subterm occurrence of L' and Z corresponds to $\underline{Z}$.

iv) Let Z be a subterm occurrence of L. Z is *underlined in* L' if Z is a subterm of Z_1 (sub$L$) which is exactly underlined in L'.

For instance the first occurrence of K in L of the above example is underlined in L'.

v) Let Z be a subterm occurrence of L. Z *has some line in* L' if Z is underlined in L' or if there is a subterm occurrence Z_1 of Z which is exactly underlined in L'.

For instance SKK sub$L$ has some line in L' in the above example.

2.21. Lemma. *Let L, L', L'' be $\underline{\lambda}$-terms such that L is simple and $L' \simeq L \simeq L''$.*

i) *If $L' \subset L''$ and $L'' \subset L'$, then $L' \equiv L''$.*

ii) *If for all subterm occurrences $\underline{Z}$ of L', the corresponding subterm occurrence Z of L is underlined in L'', then $L' \subset L''$.*

iii) *If Z is a subterm occurrence of L such that the corresponding subterm occurrence Z' of L' is not simple, then Z has some line in L'.*

Proof. Immediate. ■

2.22. Lemma. *Let L, L' be $\underline{\lambda}$-terms such that L is simple and $L \simeq L'$. Let Ξ be a variable like λ-term.*

Suppose that

i) *If Z is a subterm occurrence of L which is exactly underlined in L', then Z is an Ω_2-term.*

ii) *If Z is a subterm occurrence of L which is a u.g., then Z has some line in L'.*

Suppose further that $\lambda\eta \vdash \Xi \geqslant \Xi'$ and Ξ' is a subterm occurrence of L. Then Ξ' is underlined in L'.

Proof. Ξ is variable like, hence $\Xi \equiv \Omega_2 U$, where U is a hereditarily non-Ω_2 universal generator. Since Ω_2 is of order 0 it follows from 2.5 iii) that $\Xi' \equiv \Omega_2 U'$, where $\lambda\eta \vdash U \geqslant U'$. Since U is a hereditarily non-Ω_2 u.g. there is a subterm occurrence Z of U' which is a u.g. and a non-Ω_2 subterm occurrence of U' (see Fig. 2). By our assumption ii), Z has some line in L'. The possibility that some subterm occurrence Z_1 of Z is exactly underlined in L' is excluded, since by i) then Z_1 would be an Ω_2-term where as Z is a non-Ω_2 subterm occurrence of L. Therefore Z is underlined in L', i.e. there is a subterm occurrence Z_2 of L which corresponds to $\underline{Z_2}\operatorname{sub}L'$ and such that $Z\operatorname{sub}Z_2$. We claim that $\Omega_2U'\operatorname{sub}Z_2$ (see Fig. 2).

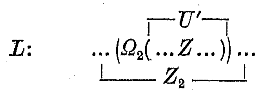

Fig. 2

First note that, since $\underline{Z_2}\operatorname{sub}L'$, it follows from i) that Z_2 is an Ω_2 term. Hence since Z is a non-Ω_2 subterm occurrence of U', Z_2 is not a subterm of U'. Therefore U' is a proper subterm of Z_2, since subterms are either disjoint or comparable with respect to the relation sub.

Hence indeed $\Omega_2U'\operatorname{sub}Z_2$. Therefore $\Xi'\equiv\Omega_2U'$ is underlined in L'. ∎

2.23. THEOREM. *Let M, N be λ-terms which are not universal generators and let Ξ be a variable like λ-term. If $\lambda\eta\vdash M\Xi=N\Xi$, then*

$$\lambda\eta\vdash Mx=Nx \quad \textit{for} \quad x\notin \mathrm{FV}(MN)\,.$$

Proof. It follows from the Church-Rosser theorem for $\lambda\eta$ and the assumption $\lambda\eta\vdash M\Xi=N\Xi$, that there exists a term L such that

$$\lambda\eta\vdash M\Xi\geqslant L \quad \text{and} \quad \lambda\eta\vdash N\Xi\geqslant L\,.$$

Since $\Xi\equiv\Omega_2U$ it follows from 2.4 and 2.5 iv) that Ξ is of order 0. Hence from 2.11 it follows that there are terms L', L'' such that $\underline{\lambda\eta}\vdash M\underline{\Xi}\geqslant L'$, $\underline{\lambda\eta}\vdash N\underline{\Xi}\geqslant L''$ and $L'\simeq L\simeq L''$ (see Fig. 3).

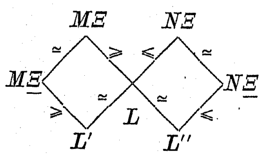

Fig. 3

Now we claim that $L'\equiv L''$. In order to prove this, it is sufficient to show that $L'\subset L''$, since by symmetry argument then also $L''\subset L'$ and hence by 2.21 i) $L'\equiv L''$.

We will show that for every subterm occurrence $\underline{Z'}$ of L', Z' is underlined in L'', where Z' is the subterm occurrence of L corresponding to $\bar{Z}'$. Then it follows by 2.21 ii) that $L'\subset L''$. Suppose therefore that $\underline{Z'}$ is a subterm occurrence of L'. By 2.10 it follows that $\lambda\eta\vdash\Xi\geqslant Z'$.

We verify the conditions i) and ii) of 2.22 for L, L''.

i) If Z is a subterm occurrence of L which is exactly underlined in L'', then $\underline{Z}$ sub L'', hence it follows by 2.10 that $\lambda\eta\vdash\Xi\geqslant Z$, hence Z is an Ω_2-term.

ii) If Z is a subterm occurrence of L which is a u.g. then $Z\notin\mathcal{F}(N)$ (otherwise N would be a u.g.). Hence by 2.15 Z is not the corresponding subterm occurrence of a simple subterm of L''. Therefore Z has some line in L', by 2.21 iii).

Now it follows from 2.22 that Z' is underlined in L''. Hence indeed $L'\equiv L''$.

Let x be a variable not occurring in the reductions represented in Fig. 3. Then it follows from 2.13 that

$$\lambda\eta\vdash Mx=\varphi_x(M)\geqslant\varphi_x(L') \quad \text{and} \quad \lambda\eta\vdash Nx=\varphi_x(N)\geqslant\varphi_x(L'')\,.$$

Hence $\lambda\eta\vdash Mx=Nx$, since $\varphi_x(L')\equiv\varphi_x(L'')$. ∎

By the same method we have

2.24. COROLLARY. *Let M, N be λ-terms which are not u.g.'s and let Ξ be a variable like λ-term. Let $x\notin\mathrm{FV}(MN)$, then*

$$\lambda\vdash M\Xi=N\Xi \Rightarrow \lambda\vdash Mx=Nx\,.$$

2.25. THEOREM. *Let M, N be λ-terms which are not universal generators. Then the ω-rule for M and N is derivable in the λ-calculus with extensionality* (1).

Proof. If $\lambda\eta\vdash MZ=NZ$ for all closed Z, then $\lambda\eta\vdash M\Xi=N\Xi$ for variable like terms Ξ. Therefore by 2.23 and extensionality $\lambda\eta\vdash M=N$. ∎

2.26. DEFINITION. Let $\lambda\eta\vdash_k M=N$ mean that $M=N$ is *provable* in $\lambda\eta$ in less then k steps.

Parikh's ω-rule is (cf. [9]):

If $\exists k\in\omega\forall Z$ closed $\lambda\eta\vdash_k MZ=NZ$, then $\lambda\eta\vdash M=N$.

2.27. COROLLARY. *Parikh's ω-rule is derivable for arbitrary terms.*

Proof. Let $\forall Z$ closed $\lambda\eta\vdash_k MZ=NZ$. Take in the proof of 2.23 $\Xi\equiv\Omega_2U$ such that if $\Omega_2U\in\mathcal{F}(M)$ or $\Omega_2U\in\mathcal{F}(N)$, the necessary reduction takes more then k steps.

§ 3. The ω-rule is not generally valid. In this section a counter example to the ω-rule due to Plotkin will be given. Two terms Ξ and Θ will be defined such that $\lambda\eta\vdash\Xi Z=\Theta Z$ for all closed Z, but not $\lambda\eta\vdash\Xi=\Theta$. It follows by the result of § 2, that the terms Ξ and Θ have to be complicated.

3.1. LEMMA (Double fixed point theorem). $\forall A, B\,\exists P, Q\,\lambda\vdash APQ=P$ *and* $\lambda\vdash BPQ=Q$.

(1) Added in proof. In correspondence Mr. Plotkin pointed out, using two variable like terms, that the ω-rule is valid for M and N assuming only that at least one of them is not a universal generator.

Proof. Let FP be the fixed point operator, i.e. $\forall M \lambda \vdash M(\mathrm{FP}M) = \mathrm{FP}M$. Define

$$P_Q \equiv \mathrm{FP}(\lambda P \cdot APQ), \quad Q_0 \equiv \mathrm{FP}(\lambda Q \cdot BP_QQ) \quad \text{and} \quad P_0 \equiv P_{Q_0}.$$

Then

$$\lambda \vdash AP_0Q_0 = P_0 \quad \text{and} \quad \lambda \vdash BP_0Q_0 = Q_0 .$$

Alternative proof. Let $\lambda xy \cdot [x, y]$ be a pair function with inverses $([x, y])_0 = x$ and $([x, y])_1 = y$, [4], 2.8. Define

$$X = \mathrm{FP}(\lambda x\ [A(x)_0(x)_1, B(x)_0(x)_1]) .$$

Then we can set $P = (X)_0$ and $Q = (X)_1$. ■

3.2. Lemma. *There exist terms F and G such that*
3.2.1. $\lambda \vdash Fxvw = Fx[F(x^+)(G(x^+))wv](Ex)$ *and*
3.2.2. $\lambda \vdash Gx = F(x^+)(G(x^+))(E(x^+))(Gx)$,
where E and $\lambda x \cdot x^+$ are as in 2.2.

Proof. Define

$$A \equiv \lambda fgxvw \cdot fx[f(x^+)(g(x^+))wv](Ex)$$

and

$$B \equiv \lambda fgx \cdot f(x^+)(g(x^+))(E(x^+))(gx) .$$

Then apply 3.1. ■

3.3. Definition. $\Xi \equiv F\underline{0}(G\underline{0})$, $\Theta \equiv \lambda x \cdot \Xi(E\underline{0})$.

3.4. Lemma. $\forall m \forall n\ \lambda \vdash F\underline{n}(G\underline{n})(E\underline{n}) = F\underline{n}(G\underline{n})(E\underline{n+m})$

Proof. Induction on m. If $m = 0$, we are done. Now we show the result for $m+1$:

$$\lambda \vdash F\underline{n}(G\underline{n})(E\underline{n}) = F\underline{n}[F\underline{n+1}(G\underline{n+1})(E\underline{n+1})(G\underline{n})](E\underline{n}) \text{ by 3.2.2.}$$
$$= F\underline{n}[F\underline{n+1}(G\underline{n+1})(E\underline{n+1+m})(G\underline{n})](E\underline{n}) \text{ by the induction hypothesis}$$
$$= F\underline{n}(G\underline{n})(E\underline{n+m+1}) \text{ by 3.2.1.}$$ ■

3.5. Corollary. $\forall n\ \lambda \vdash \Xi(E\underline{0}) = \Xi(E\underline{n})$.

3.6. Theorem (Plotkin). *The ω-rule is not valid for Ξ and Θ.*

Proof. Let Z be a closed term. Then for some n, $\lambda \vdash Z = E\underline{n}$ and therefore $\lambda \vdash \Xi Z = \Xi(E\underline{n}) = \Xi(E\underline{0}) = \Theta Z$. Hence $\lambda \not\vdash \Xi Z = \Theta Z$ for all closed Z. It is not difficult to show that $\lambda\eta \not\vdash \Xi = \Theta$. The idea is that in a reduction Ξw cannot get rid of the free variable w, whereas $\lambda\eta \vdash \Theta w \geqslant \Xi(E\underline{0})$. See for details [10]. ■

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Reçu par la Rédaction le 23. 8. 1973