A model for *HAS*

A topological interpretation of the theory of species of natural numbers

by

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Abstract. Using continuous mappings of the Baire-spaces into the Cantor space, a model for second order intuitionistic arithmetic with set-variables is constructed. The interpretation of sets is based on Kripke's schema. The main result is the validity of a $\forall X \exists!\, x$-uniformity principle.

In this short note we will provide an interpretation of the species-version of second order intuitionistic arithmetic along the lines of Scott's interpretations of (real) intuitionistic analysis [5] and [6].

Model theory for ***HAS*** has entirely been lacking, in contrast to a model theory for theories of choice sequences where some modelling was done by Troelstra [8], Van Dalen and Troelstra [1] and quite recently by Joan Moschovakis, who presented a topological interpretation of a version of Kleene-Vesley's theory of choice sequences [2].

Since the theory of species is a relatively underdeveloped area it seems highly desirable to have a manageable testing ground available for a certain amount of experimenting.

For a proof-theoretic approach one should consult Troelstra [10].

The most direct modelling of ***HAS*** that, recalling Scott's models, comes to mind is the representation of species by characteristic functions, which then can be handled like choice sequences, cf. [2].

Unfortunately, from the intuitionistic point of view, not all species (say, of natural numbers) have characteristic functions. For a characteristic function χ we have $\chi(n) = 0 \vee \chi(n) = 1$ for all natural numbers n. So for the corresponding species we have $n \in X \vee n \notin X$, for all n, i.e. X is *decidable*. Hence only the decidable species would be treated in the suggested approach.

A more comprehensive characterization of species is obtained by employing Kripke's schema.

We determine a species X uniquely by a function $f\colon N \to \{0, 1\}$ such that

$$n \in X \Leftrightarrow \exists m\big(f(\langle m, n\rangle) = 1\big)\ (*)$$

($\lambda mn \cdot \langle m, n\rangle$ is a standard coding of N^2 onto N).

The existence of such an f is claimed by Kripke's schema (strong form, cf. [7], p. 96). Let us call f the *Kripke-function* of X. The following topological interpretation is based on Kripke-functions.

1. The language of ***HAS*** contains individual variables $x, y, z, \ldots$ and species variables $X^{(n)}, Y^{(n)}, Z^{(n)}, \ldots$ $(n \geqslant 1)$. $X^{(n)}$ is an n-argument-relation-variable. Instead of $X^{(1)}$ we write X and $x \in X$ stands for $X(x)$. We will assume the language to contain constants for numbers and various primitive recursive functions. The formation of terms and formulas is routine. We will not distinguish notationally a species and its name, both will be denoted by $\xi, \eta, \ldots$

2. We will freely use notations and facts from topological interpretations and from pseudo-Boolean algebra.

In particular $\mathrm{Int}(S)$ is the interior of a set S,

$$S \Rightarrow T := \mathrm{Int}(\overline{S} \cup T), \qquad S \Box T := (S \Rightarrow T) \cap (T \Rightarrow S).$$

The topological space considered here is N^N (the Baire space) with the familiar topology. **1** and **0** denote resp. N^N and $\emptyset$ (the empty set). We will use the quantifier $\bigvee$ as a shorthand in the metalanguage.

If $\beta \in N^N$ then $\overline{\beta}m$ denotes the open set $\{\gamma \mid \gamma 0 = \beta 0 \wedge \ldots \wedge \gamma(m-1) = \beta(m-1)\}$.

Number and species variables are interpreted as numbers and continuous mappings from N^N into $(2^N)^n$.

Truth values are inductively defined by

(i)

$$\|A\| = \begin{cases} \mathbf{1} & \text{if } A \text{ holds} \\ \mathbf{0} & \text{else} \end{cases} \qquad \text{for } A \text{ a numerical atom},$$

(ii)

$$\|n \in \xi\| = \{\beta \in N^N \mid \bigvee m\, \xi(\beta)\langle m, n\rangle = 1\}$$
$$= \bigcup_m \{\beta \mid \xi(\beta)\langle m, n\rangle = 1\}$$

(*) The first to point out this representation, I think, was J. J. de Iongh (oral communications).

and

$$\|\xi^{(n)}(k_1, \ldots, k_n)\| = \{\beta \mid \bigvee m\, \xi(\beta)\langle k_1, \ldots, k_n, m\rangle = 1\}$$
$$= \bigcup_m \{\beta \mid \xi(\beta)\langle k_1, \ldots, k_n, m\rangle = 1\}.$$

(iii) For composite formulas the truthvalues are defined by the corresponding pseudo-Boolean operations (cf. [2], [3], [5]).

3. The validity of the axioms and rules of the two-sorted predicate calculus is a routine matter. See e.g. [3]. Note that the details are simpler here than in the case of a theory of choice sequences, since the functors here are precisely those of first order arithmetic, cf. [2], § 2.

Since for all sentences of the first-order fragment the truthvalue is either **0** or **1**, the axioms of first order arithmetic (with the possible exception of the induction-schema) are valid. The validity of the induction-schema is shown by an easy induction (in the metalanguage). The only remaining, mathematically interesting axiom is the comprehension-schema. For convenience we only treat the case with one free variable, the general case follows accordingly.

Let $A(x)$ be a formula of ***HAS*** with x as its only free variable. Define a function ξ, which will interprete the species of all n such that $A(n)$ holds, i.e., the abstraction of $A(x)$. Define

$$\xi(\beta)\langle m, n\rangle = \begin{cases} 1 & \text{if } \quad \overline{\beta}m \subseteq \|A(n)\|, \\ 0 & \text{else}. \end{cases}$$

(i) ξ is continuous. Let $\xi(\beta) = \delta$. It suffices to consider individual values of δ:

$$\delta\langle m, n\rangle = 1 \Leftrightarrow \overline{\beta}m \subseteq \|A(n)\|,$$
$$\delta\langle m, n\rangle = 0 \Leftrightarrow \overline{\beta}m \not\subseteq \|A(n)\|.$$

Hence for all $\alpha \in \overline{\beta}m$

$$\xi(\alpha)\langle m, n\rangle = 0.$$

In both cases the neighbourhood $\overline{\beta}m$ of β is mapped into the neighbourhood $\{\gamma \mid \gamma\langle m, n\rangle = \delta\langle m, n\rangle\}$ of δ.

(ii) ξ satisfies the comprehension-schema for $A(x)$:

$$\|n \in \xi\| = \bigcup_m \{\beta \mid \xi(\beta)\langle m, n\rangle = 1\}$$
$$= \bigcup_m \{\beta \mid \overline{\beta}m \subseteq \|A(n)\|\} \subseteq \|A(n)\|.$$

Conversely, if $\beta \in \|A(n)\|$ then there exists an m such that $\overline{\beta}m \subseteq \|A(n)\|$. Therefore

$$\beta \in \{\beta \mid \xi(\beta)\langle m, n\rangle = 1\} \subseteq \|n \in \xi\|.$$

Hence

$$\|n \in \xi\| = \|A(n)\| \quad \text{and} \quad \|\forall x(x \in \xi \leftrightarrow A(x))\| = \mathbf{1}.$$

4. So far we have not considered equality for species. The (topological interpretations of the) Kripke-functions stand more or less for the intensional versions of species. The production of the values 0 and 1 of a Kripke-function is intuitively thought of as the result of the activity of the creative subject (cf. [7]). Therefore it is not at all quite evident how the Kripke-functions will behave under extensional equality.

DEFINITION.

$$\|\xi = \eta\| := \|\forall x(x \in \xi \leftrightarrow x \in \eta)\| = \mathrm{Int}\bigcap_n (\|n \in \xi\| \,\square\, \|n \in \eta\|)$$

(analogously for n-ary relations).

We verify the axioms of equality.

1. $\|\xi = \xi\| = 1$ — immediate,
2. $\|\xi = \eta\| = \|\eta = \xi\|$ — immediate,
3. $\|\xi = \eta\| \cap \|\eta = \xi\| \leqslant \|\xi = \zeta\|$, apply $(a \,\square\, b) \cap (b \,\square\, c) \leqslant a \,\square\, c$,
4. $\|\xi = \eta\| \cap \|A(\xi)\| \leqslant \|A(\eta)\|$.

By Kleene, Introduction to Metamathematics 73, p. 399, it is sufficient to check 4 for atoms. The only interesting case is $A(\xi) = n \in \xi$

$$\|\xi = \eta\| \cap \|n \in \xi\| = \mathrm{Int}\bigcap_m (\|m \in \xi\| \,\square\, \|m \in \eta\|) \cap \|n \in \xi\| \leqslant \|n \in \eta\|.$$

5. Not surprisingly a version of Kripke's schema in the language of ***HAS*** holds in the model (we first put it in!).

We show $\|\exists X(\exists x(x \in X) \leftrightarrow A)\| = \mathbf{1}$. Let A be given and define

$$\xi(\beta)\langle m, n\rangle = \begin{cases} 1 & \text{if} \quad \bar{\beta}n \subseteq \|A\|, \\ 0 & \text{else}. \end{cases}$$

Evidently ξ satisfies $\|\exists x(x \in \xi)\| = \|A\|$ and $\|\forall x(x \in \xi \vee x \notin \xi)\| = \mathbf{1}$.

6. We will show that the model satisfies a modification of *Troelstra's uniformity principle*

UP! $\quad \forall X \exists! x A(X, x) \rightarrow \exists! x \forall X A(X, x).$

A more general version was put forward by Troelstra in [9] and [10] (with $\exists$ instead of $\exists!$), where he proved its consistency with ***HAS*** by means of realizability.

The principle is plausible for intuitionists as may be seen from the following informal considerations. Suppose $\forall X \exists! x A(X, x)$ and le[t] $A(X, n)$, $A(Y, m)$ hold where $n \neq m$ and neither $A(X, m)$ nor $A(Y, n)$ holds. Now consider a species Z which coincides either with X or with Y, depending on some unsolved problem. Say $A(Z, p)$ holds. By comparing p with m and n we can decide whether Z coincides with X, Y or with none of them. This means that we can decide the problem mentioned above positively, negatively or "open", which is not in accordance with intuitionistic principles.

Note that the principle testifies the richness of the power species of N, e.g. if one only considers decidable species then the principle fails, as the following formula shows:

$$A(X, x) := (x = 0 \wedge 0 \in X) \vee (x = 1 \wedge 0 \notin X).$$

There is an alternative explanation of the testimony of the principle, namely it stresses the strength of $\forall X \exists! x$ (the existential requirement) when applied to species in an extensional context. This situation is comparable to Rice's theorem ([4], p. 34). The verification of UP! in the model is based on the second alternative.

The proof below uses ideas of Joan Rand Moschovakis' [2], in particular Lemma 5 and *27.2!

We first prove an auxiliary result.

(a) *If* $\xi(\beta) = \eta(\beta)$ *and* $\|A(\zeta)\|$ *is clopen for every* ζ *then* $\beta \in \|A(\xi)\| \Leftrightarrow \beta \in \|A(\eta)\|$.

Proof. (i) Let $\beta \in \mathrm{Cl}\|\xi = \zeta\|$ (i.e. the closure of $\|\xi = \xi\|$) then

$$\beta \in \|A(\xi)\| \Leftrightarrow \beta \in \|A(\zeta)\|.$$

For if $\beta_1, \beta_2, \beta_3, \ldots \in \|\xi = \zeta\|$ and the sequence converges to β, then $\beta \in \|A(\xi)\| \Leftrightarrow \beta \in \|A(\zeta)\|$, since

$$\beta_i \in \|A(\xi)\| \Leftrightarrow \beta_i \in \|A(\zeta)\| \quad \text{(by 4.4)}$$

and both $\|A(\xi)\|$ and $\|A(\zeta)\|$ are clopen.

(ii) Now we exhibit a suitable ζ such that

$$\beta \in \mathrm{Cl}\|\xi = \zeta\| \quad \text{and} \quad \beta \in \mathrm{Cl}\|\zeta = \eta\|.$$

Define

$$\zeta(\gamma) := \begin{cases} \xi(\gamma) = \eta(\gamma) & \text{if} \quad \gamma = \beta, \\ \xi(\gamma) & \text{if} \quad \bigvee z[\bar{\gamma}(2z) = \bar{\beta}(2z) \wedge \gamma(2z) \neq \beta(2z)], \\ \eta(\gamma) & \text{if} \quad \bigvee z[\bar{\gamma}(2z+1) = \bar{\beta}(2z+1) \wedge \gamma(2z+1) \neq \beta(2z+1)]. \end{cases}$$

Although in [2] the result $\beta \in \mathrm{Cl}\|\xi = \zeta\|$ is fairly evident, because identity is in that case perfectly natural, we will give the details here. Note that the "intensional" equality-case, in terms of Kripke-functions, is evident, so the extensional equality-cause is expected to hold. Define ϱ_i by

$$\bar{\varrho}_i(2i) = \bar{\beta}(2i)\,,$$
$$\varrho_i(2i+j) = \beta(2i+j)+1\,, \quad j \geqslant 0\,.$$

and σ_i by

$$\bar{\sigma}_i(2i+1) = \bar{\beta}(2i+1)\,,$$
$$\sigma_i(2i+j+1) = \beta(2i+j+1)+1\,, \quad j \geqslant 0\,.$$

Clearly $\beta = \lim_{i\to\infty} \varrho_i = \lim_{i\to\infty} \sigma_i$. We will show $\varrho_i \in \|\xi = \zeta\|$. Let $U_i = \bar{\varrho}_i(2i+1)$.

Claim:

$$U_i \subseteq \|\xi = \zeta\| \quad \text{or} \quad U_i \subseteq \operatorname{Int}\bigcap_n (\|n \in \xi\| \square \|n \in \zeta\|)\,.$$

It suffices to show

$$U_i \subseteq \|n \in \xi\| \Rightarrow \|n \in \zeta\| \quad \text{or} \quad U_i \cap \|n \in \xi\| \subseteq \|n \in \zeta\|\,.$$

$$U_i \cap \|n \in \xi\| = U_i \cap \bigcup_m \{\delta|\ \xi(\delta)\langle m, n\rangle = 1\}$$
$$= \bigcup_m (U_i \cap \{\delta|\ \xi(\delta)\langle m, n\rangle = 1\})$$
$$= \bigcup_m (U_i \cap \{\delta|\ \zeta(\delta)\langle m, n\rangle = 1\})$$

since $\xi \upharpoonright U_i = \zeta \upharpoonright U_i$. So

$$U_i \cap \|n \in \xi\| \subseteq \|n \in \zeta\|\,.$$

Likewise $\sigma_i \in \|\xi = \eta\|$ is shown. Now we immediately conclude

$$\beta \in \operatorname{Cl}\|\xi = \zeta\| \quad \text{and} \quad \beta \in \operatorname{Cl}\|\xi = \eta\|\,.$$

(a) now follows from (i) and (ii).

(b) Let $\beta \in \|\forall X\, \exists! x\, A(X, x)\|$, then for each ξ, for some m and a unique n $\beta \in \bar{\beta}m \subseteq \|A(\xi, n)\|$. Moreover for each η, k $\|A(\eta, k)\|$ is clopen relative to $\bar{\beta}m$. Choose $\gamma \in \bar{\beta}m$ and put $\alpha = \xi(\gamma)$. Define $\xi_\alpha(\delta) = \alpha$ for all δ. Then, by a relativized version of (a) we have

$$\gamma \in \|A(\xi, n\| \quad \text{iff} \quad \gamma \in \|A(\xi_\alpha, n)\|\,.$$

Define

$$\varphi(\gamma, \alpha) := \begin{cases} n+1 & \text{if} \quad \gamma \in \bar{\beta}m \cap \|A(\xi_\alpha, n)\|\,, \\ 0 & \text{else}\,. \end{cases}$$

As in [2] φ is continuous. Now let ε be an interpretation of the empty species, e.g. $\varepsilon(\beta)(i) = 0$ for all β and i.

Say $\bar{\beta}m \subseteq \|A(\varepsilon, n_0)\|$, then for all $\gamma \in \bar{\beta}m$ we have

$$\varepsilon(\gamma) = \boldsymbol{0} \ (\boldsymbol{0} = \lambda k \cdot 0) \quad \text{and} \quad \varphi(\gamma, \boldsymbol{0}) = n_0+1 \text{ for } \gamma \in \beta m\,.$$

By the continuity of φ there is a k such that for all $\delta \in \bar{\gamma}k$ and $\sigma \in \bar{\boldsymbol{0}}k$

$$\varphi(\delta, \sigma) = n_0+1\,.$$

Let us now consider an arbitrary ξ, we will look for an extensionally equal η with suitable properties. Define η by

$$\eta(\delta)\langle p, q\rangle := \begin{cases} 0 & \text{if} \quad p < k\,, \\ \xi(\delta)\langle p-k, q\rangle & \text{else}\,. \end{cases}$$

Evidently $\eta(\delta) \in \bar{\boldsymbol{0}}k$ and $\|\xi = \eta\| = \mathbf{1}$.

For $\delta \in \bar{\gamma}k$ and $\eta(\delta) \in \bar{\boldsymbol{0}}k$ we have

$$\varphi(\delta, \eta(\delta)) = n_0+1\,.$$

By definition of φ this is equivalent to

$$\delta \in \|A(\xi_{\eta(\delta)}, n_0)\| \text{ or}$$
$$\delta \in \|A(\eta, n_0)\| \quad (\text{by (a)}) \text{ or}$$
$$\delta \in \|A(\xi, n_0)\| \quad (\text{by } \|\xi = \eta\| = \mathbf{1})\,.$$

We can find a sequence $\delta_1, \delta_2, \delta_3, \ldots$, converging to γ such that $\delta_i \in \|A(\xi, n_0)\|$. Hence, by the clopeness of $\|A(\xi, n_0)\|$, we also have $\gamma \in \|A(\xi, n_0)\|$.

This implies $\bar{\beta}m \subseteq \|A(\xi, n_0)\|$ and therefore also

$$\bar{\beta}m \subseteq \bigcap_\xi \|A(\xi, n_0)\|\,.$$

So

$$\beta \in \operatorname{Int}\bigcap_\xi \|A(\xi, n_0)\| = \|\forall X\, A(X, n_0)\|$$

and by the uniqueness of n_0

$$\beta \in \|\exists! x \forall X\, A(X, x)\|\,.$$

This leads to the desired result

$$\|\forall X\, \exists! x\, A(X, x) \to \exists! x \forall X\, A(X, x)\| = 1\,.$$

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Stable sets, a characterization of β_2-models of full second order arithmetic and some related facts*

by

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Abstract. We study here stable sets i.e. transitive sets with Σ_1-reflection property. As a result we get the following characterization of β_2-models of A_2: M is a β_2-model for A_2 iff there is stable transitive model N of ZFC$^-$ such that $M = N \cap \wp(\omega)$. We get a generalization of both theorems of Kripke and Platek on stability of L_{δ_2} and Lévy on stability of HC.

Zbierski (in [16]) gives the following characterization of β-models for full second order arithmetic A_2 (i.e. arithmetic with the scheme of choice):

M is a β-model of A_2 iff $M = N \cap \wp(\omega)$ for some transitive model N of ZFC$^-$.

We give similar characterization of β_n-models of A_2. The characterization is especially nice in case of β_2-models. Namely we prove:

M is a β_2-model of A_2 iff $M = N \cap \wp(\omega)$ for some transitive model N of ZFC$^-$ *such that* $N \prec_1 \mathrm{V}$.

The proof of these and related facts (for instance we prove that the sets $\mathrm{Th}(\wp(\omega))$ and $\mathrm{Th}(\mathrm{HC})$ are recursively isomorphic) takes first two paragraphs of the paper.

In the third paragraph we prove theorem of Kripke and Platek about stability of δ_2. We generalize this theorem getting result generalizing both aforenamed theorem and theorem of Lévy.

Paragraph four is devoted to the study of levels of constructible hierarchy from the point of definability. As shown in [10] pointwise definability of levels is related with gaps, one of important means while studying fine structure of constructible universe. We show that wide class of stable ordinals gives pointwise definable levels. We finally prove a result complementary to the one of Friedman, Jensen and Saks on characterization of countable admissible ordinals as ω_1^A for $A \subseteq \omega$.

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