Table des matières du tome LXXXI, fascicule 2

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On absolutely measurable sets

by

Jens Erik Fenstad and **Dag Normann** (Blindern–Oslo)

Abstract. Every provable Δ_2^1-set is absolutely (uniformly) measurable. The proof is by forcing and is obtained by analyzing an argument of R. Solovay. In most cases there are more absolutely measurable sets than provable Δ_2^1-sets. The results are next lifted to arbitrary Polish and analytic measure spaces, e.g. it is shown that if every Π_n^1 subset of $\boldsymbol{R}$ is absolutely measurable, then so are also the Π_n subsets of an analytic measure space.

Let E be an *analytic space*, i.e. E is a Hausdorff space which is the continuous image of Baire space N^N. Let $\mathcal{E}$ be the class of *absolutely measurable* sets in E, i.e. the sets which are μ-measurable with respect to every σ-finite, complete, and regular Borel measure μ on E. The definition of absolutely measurable set is highly impredicative and it has been an important problem to obtain a more constructive description of the class $\mathcal{E}$.

A set can be "described" by its definability characteristics, hence one precise version of the problem of how to describe the class $\mathcal{E}$ is: Which sets in the projective hierarchy over E are absolutely measurable.

It turns out that the answer to this question depends on the underlying set theoretic axioms. Classically, i.e. on the basis of ordinary Zermelo-Fraenkel set theory, one can show that every analytic set is absolutely measurable.

And this is as far as one can go as the following basic consistency results due to K. Gödel and R. Solovay show:

1. In 1938 Gödel [1] showed that if one adds the (consistent) assumption that every set is constructible to ZF, then there are Δ_2^1 subsets of $\boldsymbol{R}$ which are not Lebesgue-measurable.

2. In the opposite direction R. Solovay [8] proved the following result: Assume that there is a standard model for ZF plus the assumption that there exists a strongly inaccessible cardinal number. Then there is a standard model for ZF (with the axiom of choice weakened to the axiom of dependent choice) in which every subset of $\boldsymbol{R}$ is Lebesgue measurable.

But consistency results alone do not entirely answer the question. If one e.g. assume that there is a real universe of sets, then every individual set of reals is Lebesgue measurable or not. What the consistency results show is that the current attempts to describe this universe through the ZF axioms have only been partially successful.

Various axiomatic extensions of ZF have been studied which have a bearing on our problem (e.g. constructibility, measurable cardinals, determinateness). But it seems fair to say that there is no universally accepted extension of ZF which is powerful enough to give an answer.

This is the background for our concern in this paper which is to use some recent results in set theory to push the classical results a bit further. First, we shall introduce the notions of *absolute* and *provable* Δ_1^2 sets of reals and show that every such set is absolutely measurable. Second, we shall lift various results about measurability to arbitrary analytic measure spaces. Finally, we shall add some remarks on how complex an absolutely measurable set of reals can be. Much of this material appeared in the cand. real. thesis of D. Normann [6].

Remark on background and terminology. We shall need various results from set theory, topology, and measure theory. In set theory we assume that the reader is familiar with the forcing technique. A convenient reference is Jech [3], and we shall mostly adhere to his terminology. In topology and measure theory we shall use a few basic facts about Polish and analytic spaces, a comprehensive reference is Hoffmann Jörgensen [2].

1. Absolute Δ_2^1 sets. We introduce the basic definitions.

Definition. A set $A \subseteq N^N$ is *provably* Δ_2^1 if there are Σ_2^1 and Π_2^1 formulas Ψ and Φ, respectively, and a parameter $y \in N^N$ such that

(i) $x \in A$ iff $\Psi(x, y)$ iff $\Phi(x, y)$,

(ii) $\mathrm{ZF} \vdash \forall x \forall y \big(\Psi(x, y) \leftrightarrow \Phi(x, y)\big)$.

A set $A \subseteq N^N$ is *absolute* Δ_2^1 if there are Σ_2^1 and Π_2^1 formulas Ψ and Φ, respectively, and a parameter $y \in N^N$ such that

(iii) $x \in A$ iff $\Psi(x, y)$ iff $\Phi(x, y)$,

and such that for all (countable) standard models M of ZF such that $x, y \in M$

(iv) $\Phi(x, y)$ iff $M \models \Phi(x, y)$ and $\Psi(x, y)$ iff $M \models \Psi(x, y)$.

Remark. By an absoluteness argument one immediately sees that if A is provably Δ_2^1 then A is absolute Δ_2^1.

Lemma. a. *The class of absolute Δ_2^1 sets is a σ-algebra.*

b. *There are absolute Δ_2^1 sets which are not in the σ-algebra generated by the Π_1^1 sets.*

The proof of a follows immediately from the fact that arbitrary countable unions can be coded by a single parameter from N^N. For the proof of b let $\sigma(\Pi_1^1)$ denote the σ-algebra generated by the Π_1^1 sets. The elements of $\sigma(\Pi_1^1)$ can be coded in the following way. First observe that there is a Π_1^1 set $A \subset N \times N^N \times N^N$ such that every $B \in \Pi_1^1$ can be written in the form

$$B = \{x;\ \langle n, y, x\rangle \in A\},$$

for some $n \in N$ and $y \in N^N$. We now set

(i) For all $x \in N^N$, $n \in N$ $y = \langle 1, \langle n, x\rangle\rangle$ is a code and

$$B_y = \{z;\ \langle n, x, z\rangle \in A\}.$$

(ii) If x is a code, then $y = \langle 2, x\rangle$ is a code and

$$B_y = N^N \setminus B_x.$$

(iii) Let $x = \langle x_i\rangle_{i \in N}$ and assume that x_i is a code for all i, then $y = \langle 3, x\rangle$ is a code, and

$$B_y = \bigcup_{i \in N} B_{x_i}.$$

We can now prove in ZF that the relations "y is a code for a $\sigma(\Pi_1^1)$-set" and "$x \in B_y$" both are Δ_2^1, hence the set $\{x;\ x \notin B_x\}$ gives the required counterexample for b. (We omit the somewhat messy details of the proof.)

Remark. The notion of an absolute Δ_2^1 set may be too "metamathematical" for the taste of an analyst. It would be interesting to get some alternative description of this class. The notion of absolute Δ_2^1 is certainly not original with the present authors, but we are unable to find a suitable reference in the literature.

We now come to the main result of this section.

Theorem. *If for all $x \in N^N$ there is a countable standard model M of ZF such that $x \in M$, then every absolute Δ_2^1 set is absolutely measurable.*

Proof. Let μ be an atomless, finite, positive Borel-measure (note that this represents no essential restriction) and M a countable standard model of ZF in which μ can be defined (— note that a measure is determined by its values on the base elements and thus may be represented by a sequence of reals). We call x *random* over M if x belongs to no Borel-set of μ-measure 0 which is codeable over M (— note that similar to the coding of the σ-algebra $\sigma(\Pi_1^1)$ introduced above, one can introduce a coding for the Borel-sets; a Borel-set is then codeable over M if the code for the set belongs to M).

We now observe, since M is countable, that the set of non-random elements has μ-measure 0, and that no element of M is random, since $\{x\}$ is codable over M whenever $x \in M$.

We shall make use of the forcing technique: Let B_1 and B_2 be Borel-sets codeable over M and define $B_1 \simeq B_2$ if $\mu(B_1 \Delta B_2) = 0$. $\simeq$ is an equivalence relation and let $\mathfrak{B}$ be the set of equivalence classes [B]. Set $\boldsymbol{P}$

$= \mathfrak{B}\setminus\{[\emptyset]\}$. Define an ordering $\leqslant$ by $[B_1] \leqslant [B_2]$ if $\mu(B_1 \setminus B_2) = 0$. It is easily verified that $\langle \boldsymbol{P}, \leqslant \rangle$ is a set of conditions.

Lemma. *Let x be random over M. Then the set*

$$G_x = \{[B];\ x \in B,\ B \text{ is codeable over } M,\ \mu(B) > 0\}$$

i $\boldsymbol{P}$-generic over M.

We content ourselves by verifying the density condition in the definition of a $\boldsymbol{P}$-generic set. So let $\Delta \in M$ be a dense subset of $\boldsymbol{P}$, we first verify that in M

$$(*) \qquad \mu\Big(\bigcup_{[B]\in\Delta} B\Big) = \mu(N^N)\,.$$

For the proof of $(*)$ assume that in M we have $\mu_*\big(\bigcup_{[B]\in\Delta} B\big) = r$ and that $\mu(N^N) > r$ (here μ_* is the inner measure associated with μ). Let C be a Borel-set in M such that $C \subseteq \bigcup_{[B]\in\Delta} B$ and such that $\mu(C) = r$. Consider $D = N^N \setminus C$. By density $[D]$ must have an extension $[E] \in \Delta$. Then $\mu(C \cup E) > r$, but $C \cup E \subseteq \bigcup_{[B]\in\Delta} B$, — a contradiction.

From $(*)$ it now follows that if $x \notin \bigcup_{[B]\in\Delta} B$, then x would belong to some Borel-set C which is codeable over M and which has μ-measure 0. But this contradicts the fact that x is random over M. Hence $x \in \bigcup_{[B]\in\Delta} B$, which shows that $\Delta \cap G_x \neq \emptyset$. This ends the proof of the lemma.

Remark. Since $\{x\} = \bigcap_{[B]\in G_x} B$, we see that $x \in M[G_x]$, for every x which is random over M.

Let now Φ be a formula which is absolute with respect to all forcing extensions of M. Define

$$\Psi(G) = \forall y \Big(y \in \bigcap_{[B]\in G} B \to \Phi(y)\Big)\,.$$

We then see that $\Psi(G_x) \leftrightarrow \Phi(x)$, for all x. Define

$$E = \bigcup \{B;\ [B] \Vdash \Psi(G)\}\,,$$

and let x be random over M. Using the completeness theorem for forcing we obtain

$$\begin{aligned} x \in E \quad &\text{iff} \quad \exists B\big([B] \Vdash \Psi(G) \wedge x \in B\big) \\ &\text{iff} \quad \exists [B] \in G_x\big([B] \Vdash \Psi(G)\big) \\ &\text{iff} \quad M[G_x] \models \Psi(G_x) \\ &\text{iff} \quad M[G_x] \models \Phi(x)\,. \end{aligned}$$

(In this part we beg the expert to overlook some looseness with respect to the languages involved.)

The proof is now finished: Let A be absolute Δ_2^1 in the parameter y and let M be a countable standard model containing y. Let Φ be the defining formula for A, we then see that for x random over M, $x \in A$ iff $x \in E$, where E is the set defined above. Since the non-random elements over M has measure 0, it follows that $A \Delta E$ is a subset of a set with measure 0. And since E obviously is a Borel-set, we conclude that A is μ-measurable.

Remarks. 1. The argument above is an analysis of the appropriate part of Solovay [8]. His purpose in that paper was to obtain a consistency result, but our result is quite easy to read off from his proof. We have been informed that Solovay some time ago also formulated the above theorem, but this was never published.

2. Adding the Shoenfield absoluteness theorem to the above argument and assuming that there exists a measurable cardinal gives the result that Σ_2^1 sets are absolutely measurable. This is due to Solovay (unpublished).

3. Restricting ourselves to provable Δ_2^1 sets we do not need the assumption about inner models, hence the result is a pure ZF result. The reason is that since the proof only uses a finite part of the axioms, we can use a Skolem-Löwenheim argument to obtain an "inner model".

4. Our assumption about inner models is stronger than ZF + con(ZF). Is this assumption a reasonable addition to ZF (i.e. can it be accepted as a true statement)? At least one of the authors are inclined to believe so.

2. Analytic measure spaces. Let E be an analytic space. There are two ways of defining the projective hierarchy on E:

(i) Starting with the Borel-sets in E^k, $k = 1, 2, \ldots$, we generate the projective hierarchy in the usual way.

(ii) Let π be a continuous and surjective map $\pi\colon N^N \to E^k$, we let $A \subseteq E^k$ belong to the class Π_n iff $\pi^{-1}(A) \in \boldsymbol{\Pi}_n^1$.

As we shall later see the two possibilities are equivalent for Polish spaces, but in general (ii) defines a larger class than (i). Since we will use condition (ii) in lifting results from $\boldsymbol{R}$ to arbitrary analytic measure spaces, we make the following definition.

Definition. Let E be an analytic space, $\pi\colon N^N \to E$ a continuous and surjective mapping, $n \geqslant 1$, and $A \subseteq E$:

A belongs to class $\Pi_n(\Sigma_n, \Delta_n)$ iff $\pi^{-1}(A) \in \boldsymbol{\Pi}_n^1(\Sigma_n^1, \Delta_n^1)$.

Lemma. a. *Every Borel-set in E is of class Δ_1.*

b. *Π_n is closed under countable intersection and unions, and Δ_n is a σ-algebra.*

c. *Let $A \subseteq E^{k+1}$ be of class Π_n and let B be the projection of A on E^k. Then B is of class Σ_{n+1}.*

d. *Let π_1, π_2: $N^N \to E$ be two Borel-continuous maps (π: $F \to E$ is called Borel continuous if Y Borel in E implies $\pi^{-1}(Y)$ Borel in F), and let $A \subseteq \mathrm{range}(\pi_1) \cap \mathrm{range}(\pi_2)$. Then*

$$\pi_1^{-1}(A) \in \Pi_n^1(\Sigma_n^1) \quad \textit{iff} \quad \pi_2^{-1}(A) \in \Pi_n^1(\Sigma_n^1)\,.$$

We omit the proofs. (For the proof of d note that the set $\{\langle x, y\rangle;\ \pi_1(x) = \pi_2(y)\}$ is Borel.) The lemma shows that method (ii) included method (i), and it shows that the definition is independent of the particular mapping π: $N^N \to E$.

DEFINITION. A set $A \subseteq E$ is called *absolute Δ_2* if $\pi^{-1}(A)$ is absolute Δ_2^1. (Part d of the lemma above holds equally well for absolute Δ_2^1 set, hence the definition of absolute Δ_2 is independent of the map π.)

We now come to the main result of this section. This was proved by D. Normann in [6]. Let $\Gamma(E)$ denote any of the classes Π_n, Σ_n, Δ_n, absolute Δ_2 in the analytic space E.

THEOREM. *The following three conditions are equivalent:*

(i) *Every set in $\Gamma(N^N)$ is absolutely measurable.*

(ii) *Let $\langle P, \mu\rangle$ be a Polish measure space: Every $\Gamma(P)$ set is μ-measurable.*

(iii) *Let $\langle E, \mu\rangle$ be an analytic measure space: Every $\Gamma(E)$ set is μ-measurable.*

Proof. It suffices to prove (ii) → (iii) and (i) → (ii).

(ii) → (iii). Recall that every compact subset of an analytic space is Polish. Further recall that the measures involved are regular and σ-finite, i.e. there exists a increasing sequence of compacts $\langle K_n\rangle_{n\in N}$ such that

$$\mu(E \setminus \bigcup_{n\in N} K_n) = 0\,, \quad \mu(K_n) < \infty\,, \quad \text{all } n\,.$$

Let $A \in \Gamma(E)$, it suffices to show that $A \cap K_n$ is μ-measurable for all n. If π_1: $N^N \to E$ is surjective and continuous, it follows that $\pi^{-1}(K_n)$ is closed in N^N, hence $\pi_1^{-1}(A \cap K_n)$ is of class $\Gamma(N^N)$. Since K_n is Polish, there is a continuous surjection π_2: $N^N \to K_n$. As in d of the lemma it follows that $\pi_2^{-1}(A \cap K_n)$ is of class $\Gamma(N^N)$, which means that $A \cap K_n$ is of class $\Gamma(K_n)$ in K_n. By (ii), this means that $A \cap K_n$ is μ-measurable.

(i) → (ii). A Polish space is homeomorphic to a G_δ set in I^N. From this it follows that if $\langle P, \mu\rangle$ is a Polish measure space there is a Borel-set $Q \subseteq N^N$ and a Borel isomorphism π: $Q \to P$. Let $A \in \Gamma(P)$; in order to show that A is μ-measurable, we define a measure μ' on N^N by

$$\mu'(Y) = \mu\pi(Y \cap Q)\,,$$

whenever the latter is defined. It is a matter of routine to verify that μ' is a complete and σ-finite Borel-measure on N^N. Since $\pi^{-1}(A)$ is of class $\Gamma(N^N)$, it follows by (i) that $\mu'(\pi^{-1}(A))$ is defined, which means that $\mu(A) = \mu\pi(\pi^{-1}(A) \cap Q)$ is defined, i.e. A is μ-measurable.

Remarks. 1. The Borel isomorphism π between P and a Borel-set $Q \subseteq N^N$ is precisely what is needed to verify that the two methods mentioned in the beginning of this section leads to the same hierarchies over P.

2. We have a strong negative result. All uncountable analytic spaces E includes a Cantorlike subspace [2, p. 118]; hence E includes a Borel-set homeomorphic to N^N. Then, if there is a set in $\Gamma(N^N)$ which is not absolutely measurable, there will be a set in $\Gamma(E)$ which is not absolutely measurable. On the other hand: If there is a non μ-measurable set $A \in \Gamma(E)$ we must, by our theorem have a set $B \in \Gamma(N^N)$ such that B is not absolutely measurable. Thus, given Γ as above, the following two statements are equivalent:

a. There is an uncountable analytic space E in which some $\Gamma(E)$-set is not absolutely measurable.

b. In all uncountable analytic spaces E, some $\Gamma(E)$-set is not absolutely measurable.

3. In Normann [6] several other results are generalized from N^N to arbitrary Polish and analytic spaces. E.g. one may show that every analytic space E is the continuous injective image of some Π_1^1 set in N^N, which suffices to show the following result: Let $\Pi_1^1 \subseteq \Gamma(N^N)$ and assume that every uncountable $\Gamma(N^N)$ set includes a perfect subset. Then the same is true for sets of class $\Gamma(E)$, where E is an arbitrary analytic space.

3. On the complexity of absolutely measurable sets. Our results so far go in one direction: every "nice" set is absolutely measurable. Is there a converse, i.e. are absolutely measurable sets necessarily nice? We recall the following classical result which suffices to answer the question in the negative in most cases.

LEMMA. *There exists a set A of reals of cardinality ω_1 such that*

(i) *$B \subseteq A \Rightarrow B$ is absolutely measurable,*

(ii) *$B \subseteq A \Rightarrow B$ is not perfect.*

Remark. In [6] the following amusing forcing-theoretic proof was given. Let A consist of one code for each countable ordinal. We show that $\mu(A) = 0$ for all atomless Borel-measures μ. Let M be a countable standard model for ZF such that μ is definable over M. The proof will be done if we can show that every $x \in A$ such that the ordinal coded by x does not belong to M, is non-random over M. Suppose not, then $x \in M[G_x]$, hence the ordinal coded by x belongs to $M[G_x]$. But M and $M[G_x]$ have the same ordinals. (Note, the proof needs only a finite part of ZF, so the assumption about inner models can be eliminated.)

Some consequences of the lemma are:

1. If $2^{\omega_1} > 2^{\omega_0}$, then there are absolutely measurable sets which are not in the projective hierarchy.

2. Let A be the set of the lemma. We can prove in the theory $ZF + \forall a \subset \omega(\omega_1^{L[a]} < \omega_1)$ that A is not Σ_2^1.

3. Let A be the set of the lemma. We can prove in the theory ZF+PD that A is not projective.

To prove 1 use a cardinality argument. To prove 2 and 3 notice that in $ZF + \forall a \subset \omega(\omega_1^{L[a]} < \omega_1)$ every uncountable Σ_2^1 set contains a perfect subset, Solovay [7], and in ZP+PD every uncountable projective set contains a perfect set, see e.g. [5].

Remark. The lemma does not answer the question about the complexity of absolutely measurable sets in every case. It has been proved consistent by Martin and Solovay [4] that every Σ_2^1 set is absolutely measurable and that every set of cardinality ω_1 is Π_1^1.

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Solution to a problem of Gandy's

by

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Abstract. Consider the hierarchy of ramified analytical sets A_β, where A_0 = finite sets of integers (for simplicity, finite *reals*), $A_{\beta+1}$ = reals definable by analytical predicates with constants from A_β and quantifiers restricted to A_β, $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$, if λ is a limit. One of the authors and Gandy independently confirmed a conjecture of Cohen by proving the existence of a smallest β-model of analysis. Moreover, they identified it to be A_{β_0}, where β_0 is the least place where the hierarchy A_β stops, i.e., the least β such that $A_\beta = A_{\beta+1}$. We prove here that for all $\beta < \beta_0$, $A_{\beta+1}$ = reals definable by analytical predicates *without constants* with quantifiers restricted to A_β. We also show that there is a constant-free predicate which uniformly well-orders the A_β (when its quantifiers are restricted to A_β), and a constant-free predicate which is satisfied by the arithmetically complete sets of order less than β.

Notation. We define the A_β as follows:

(i) $A_\emptyset = \{X \subset N\colon X \text{ is finite}\}$.

(ii) $A_{\beta+1} = \{X \subseteq N\colon X$ is 2-N.T. definable over A_β, using constants to name sets in A_β.

(iii) $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$.

Let $\beta_0 = (\mu\beta)(A_\beta = A_{\beta+1})$. The ramified analytic hierarchy (RAH) is defined to be A_{β_0}.

If $X \in A_{\beta+1} - A_\beta$ we say X is *of order* β. If X has the property, that any Y of order β (a fortiori, any $Y \in A_{\beta+1}$) is arithmetical in X, we say X is *complete of order* β. We shall use the notation "$Y \leqslant_a X$" to express "Y is arithmetical in X". We shall reserve the notation E_β to denote particular complete sets of order β.

Our notation will be otherwise that of [3]. We will assume throughout the results of [1] and [2], and especially the results on equivalences between the RAH and other hierarchies.

Gandy (in lectures in 1967) asked the following question: If we drop the mention of constants from clause (ii) above, do we still have a characterization of the same sets? In other words, if $X \in A_{\beta+1}$, does X have a constant-free definition over A_β? We shall answer Gandy's question in the affirmative. Our theorem is the following: