Thus, R *is the projection of the set*

$$\{\langle F, x, y\rangle : (F \text{ is a continuum})\ (F \neq X)\ (x \in F)\ (y \in F)\}$$

parallel to the $C(X)$*-axis on the product space* $X \times X$.

Since the spaces $C(X)$ and the space of all subcontinua of X are compact, the projected set is an F_σ-set (in $C(X)\times X\times X$) and so is R (in $X\times X$) (comp. [5], vol. II, p. 14).

Moreover, R is a boundary set, since in every neighbourhood of two given points x_0 and y_0 of X, there are two points x and y which lie in different composants of X (i.e. that x *non*-$R\,y$).

It follows from Corollary 1 of § 6 that *there exists a Cantor set* $F \subset X$ *such that no two of its points belong to the same composant of* X (Theorem of Mazurkiewicz [6], see also [1]). In fact, almost every compact subset of X is a Cantor set with the above property.

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Model-completeness for sheaves of structures

by

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Abstract. By combining some sheaf-theoretic ideas of Comer, with ideas from the Lipshitz-Saracino proof that the theory of commutative rings has a model-companion, we prove a general theorem enabling one to extend metamathematical results on fields to the corresponding results for certain regular rings. As an application, the model-completeness of real-closed fields yields a model-completeness result for a natural class of lattice-ordered rings.

0. Introduction. In a recent paper [15], Lipshitz and Saracino found the model-companion of the theory of commutative rings without nilpotent elements. In the present paper we will give an abstract version of their proof, apparently suitable for finding the model-companion for various other theories of rings. In particular, we will find the model-companion for the theory of commutative f-rings with identity and no non-zero nilpotent elements.

The representation of rings by sections of sheaves is now well-established [4, 5, 11, 13]. In [3], Comer generalized the Feferman-Vaught results [8] to cover certain structures of sections of sheaves over Boolean spaces, and thereby proved decidability results for various classes of rings. It should be noted that for constant sheaves Comer's results are essentially contained in some publications of the Wrocław group [23, 24, 25, 26] in 1968–69.

Our main result gives a sufficient condition that, for a sheaf of models of a model-complete theory, the theory of the structure of sections should be model-complete.

Before treating the main result, we give a rapid discussion, in Section 1, of model-completeness properties of reduced products.

We are very grateful to Professors Comer, Lipshitz and Saracino for making preprints of [3] and [15] available to us. It will be seen that this paper is based on a synthesis of their ideas.

1. Model-completeness and reduced products.

1.1. It is well-known that theorems of Feferman-Vaught type can be used to give an algebraic structure to the space of complete theories in a given logic L. For example [8, 24], if $T_1 = \mathrm{Th}(\mathcal{A}_1)$ and $T_2 = \mathrm{Th}(\mathcal{A}_2)$ then we can unambiguosly define $T_1 \times T_2$ as $\mathrm{Th}(\mathcal{A}_1 \times \mathcal{A}_2)$. More generally, if $\mathcal{A}_i$ $(i \in I)$ is a family of L-structures and $T_i = \mathrm{Th}(\mathcal{A}_i)$, and if $\mathfrak{D}$ is a filter on I, then we can unambiguosly define $\prod_i T_i/\mathfrak{D}$ as $\mathrm{Th}(\prod_i \mathcal{A}_i/\mathfrak{D})$.

1.2. A number of useful preservation theorems are known. For example, the properties of being decidable, ω_0-categorical, and totally transcendental are preserved under the product operation [8, 10, 16, 24].

However, the property of being model-complete is not preserved under products.

EXAMPLE. Let 2 be the two element Boolean algebra and let B be an atomless Boolean algebra. Then $\mathrm{Th}(2)$ and $\mathrm{Th}(B)$ are model-complete, but $\mathrm{Th}(2 \times B)$ is not model-complete. This follows from Theorem 1 below.

THEOREM 1. *Let M be an infinite Boolean algebra with at least one atom. Then* $\mathrm{Th}(M)$ *is not model-complete.*

Proof. Assume we have proved

(1) If M_1 and M_2 are infinite Boolean algebras, then M_1 and M_2 satisfy the same universal sentences.

Suppose M_1 is an infinite Boolean algebra, with $\mathrm{Th}(M_1)$ model-complete. By (1), $\mathrm{Th}(M_1)$ is a model-companion of the theory of Boolean algebras. But model-companions are unique [20] and the theory of atomless Boolean algebras in the model companion. The theorem follows.

To prove (1), we first use the facts that every Boolean algebra is embeddable in an atomless Boolean algebra, and that the theory of atomless Boolean algebras is complete. Thus to prove (1) we need only show that for any infinite M_1 $\mathrm{Th}(M_1)$ has a model with an atomless subalgebra. This is easy by the compactness theorem.

1.3. From Theorem 1, it follows that the map $T \mapsto T^I/\mathfrak{D}$ cannot preserve model-completeness unless $2^I/\mathfrak{D}$ is either finite or atomless.

If $2^I/\mathfrak{D}$ is finite then $2^I/\mathfrak{D} \simeq 2^n$ for some $n \in \omega$, and so by [7, 23] $T^I/\mathfrak{D} = T^n$.

Suppose $2^I/\mathfrak{D}$ is atomless. Let F be the Fréchet filter on ω. Then $2^\omega/F \equiv 2^I/\mathfrak{D}$, since both are atomless Boolean algebras. Then, by [7, 23], $T^I/\mathfrak{D} = T^\omega/F$.

This raises the problem:

PROBLEM. a) Do the maps $T \mapsto T^n$ $(n \in \omega)$ preserve model-completeness?

b) Does the map $T \to T^\omega/F$ preserve model-completeness?

We conjecture that the answer to both a and b is negative in general, but we have been unable to construct counterexamples.

The main result of this paper implies that the answer to b is positive, if we restrict attention to theories T in a certain class.

1.4. We shall explain briefly the connection between reduced powers and the papers of Comer and Lipshitz-Saracino.

We refer to [14, 23, 24, 25, 26] for the basic information on limit powers $M^J|\mathcal{F}$, where $\mathcal{F}$ is a filter on J^2. Let C be Cantor space and let M^C be the structure of continuous functions from C to M (where M has the discrete topology). (See [23, 24].) Then M^C is in fact a limit power $M^J|\mathcal{F}$, where J and $\mathcal{F}$ are independent of M [23]. Moreover, there is an index set I and a filter $\mathfrak{D}$ on I such that for all M $M^C = M^J|\mathcal{F} \equiv M^I/\mathfrak{D}$.

By taking M as the Boolean algebra $\mathfrak{B}$, and using [7], we see that we can take I as ω and $\mathfrak{D}$ as the Fréchet filter F.

Now, M^C is a structure of sections over a constant sheaf on the Boolean space C, so we make contact with Comer's [3]. On the other hand, the paper of Lipshitz-Saracino is essentially concerned with rings of sections of sheaves whose stalks are algebraically closed fields. Indeed in their proof of their Theorem 1, Lipshitz and Saracino explicitly consider products of rings M^C where M is an algebraically closed field.

2. Sheaves of L-structures.

2.1. Let L be a fixed first-order logic. We will be considering sheaves whose stalks are L-structures.

It will be convenient to make the usual abuse of notation that uses the same symbol for a relation-symbol (or operation-symbol) and its interpretation in a particular model.

DEFINITION. $\mathcal{S}$ is a *sheaf of L-structures over X* if $\mathcal{S}$ is a quadruple $\langle S, \pi, X, \mu\rangle$ where

i) S and X are topological spaces;

ii) π is a continuous onto map from S to X;

iii) each point in S has an open neighbourhood which is mapped homeomorphically onto an open set in X under π;

iv) μ is a map with domain X;

v) for each $x \in X$, $\mu(x)$ is an L-structure with underlying set $\pi^{-1}(x)$;

vi) for each $(n+1)$-ary operation-symbol τ of L, the map $(s_0, \ldots, s_n) \to \tau(s_0, \ldots, s_n)$ from $\bigcup_{x \in X} \mu(x)^{n+1}$ to S is continuous, where the domain is given the topology induced as a subset of S^{n+1};

vii) for each individual constant a of L, the map $X \to S$, that assigns to x the denotation of a in $\mu(x)$, is continuous;

viii) for each $(n+1)$-ary relation-symbol R of L, the map $(s_0, \ldots, s_n) \to \chi_R(s_0, \ldots, s_n)$ from $\bigcup_{x \in X} \mu(x)^{n+1}$ to $\mathbb{Q}$ is continuous, where $\mathbb{Q}$ has the discrete topology, the domain has the topology of (vi), and χ_R is the characteristic function of R.

EXAMPLE. Let M be an L-structure, and X a topological space. Give M the discrete topology. Let $S = X \times M$, and let π be the projection map $S \to X$. Let $\mu(x) = M$ for each x. Then $\langle S, X, \pi, \mu \rangle$ is a sheaf, the constant M-sheaf over X.

Remark. Comer [3] appears to exclude relation-symbols from L. We have not seen [6], but presumably the definition of sheaf used there is close to that used above.

2.2. Sections. Let $\langle S, X, \pi, \mu \rangle$ be a sheaf of L-structures. Let U be a subset of X, with the induced topology. A section over U is a continuous σ: $U \to S$ such that $\pi \circ \sigma$ is the identity on U. Let $\Gamma(U, S)$ be the set of sections over U.

We note that $\Gamma(U, S) \subseteq \prod_{x \in U} \mu(x)$, so $\Gamma(U, S)$ inherits the relational structure of the product. On the other hand, conditions (vi) and (vii) readily imply that $\Gamma(U, S)$ is closed under the operations of the product. Therefore $\Gamma(U, S)$ is a substructure of the L-structure $\prod_{x \in U} \mu(x)$. By the usual abuse of notation, let $\Gamma(U, S)$ be the above L-structure.

The most basic point about sections is

LEMMA 1. *Let $f, g \in \Gamma(U, S)$. Then $\{x \in U: f(x) = g(x)\}$ is open.*

Proof. The proof is identical to that in [21, page 25].

COROLLARY. *If S is Hausdorff, $\{x \in U: f(x) = g(x)\}$ is clopen.*

Proof. Since f and g are continuous maps into S, $\{x \in U: f(x) = g(x)\}$ is closed.

DEFINITION. $\langle S, X, \pi, \mu \rangle$ is T_2 if S is Hausdorff.

LEMMA 2. *Suppose X is Boolean and for all $f, g \in \Gamma(X, S)$ $\{x \in X: f(x) = g(x)\}$ is clopen. Then S is Hausdorff.*

Proof. Let $s, t \in S$, $s \neq t$. If $\pi(s) \neq \pi(t)$, s and t have disjoint open neighbourhoods by condition (iii) and the fact that X is Hausdorff. Suppose next $\pi(s) = \pi(t) = x_0$. Then by [17, pages 13–14] there are $f, g \in \Gamma(X, S)$ such that $f(x_0) = s$, $g(x_0) = t$. Let $W = \{x \in X: f(x) \neq g(x)\}$. Then W is open. But sections are open maps [17, 21], so $f(W)$ and $g(W)$ are disjoint open neighbourhoods of s, t respectively. Thus S is Hausdorff.

Remark. It is not difficult to construct examples where X is Boolean and S is not Hausdorff. Indeed, one can take L as pure logic with $=$, and $\langle S, X, \pi, \mu \rangle$ a sheaf of infinite sets. This contradicts the remark of Comer [3], following his introduction of condition (C). For, by Lemma 2, if Comer's remark were correct, any sheaf of models of a model-complete theory over a Boolean space would have S Hausdorff. But the theory of infinite sets is model-complete.

This observation does not affect any of Comer's applications, for the

sheafs there are T_2. In the next subsection we will give a natural condition which implies Comer's condition (C).

2.3. Henceforward it will be convenient to write "S_x" for $\mu(x)$.

LEMMA 3. *Let $\Phi(v_0, \ldots, v_n)$ be a quantifier-free L-formula, and $f_0, \ldots, f_n \in \Gamma(X, S)$.*

a) *If Φ is positive, $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ is open.*

b) *If S is Hausdorff, $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ is clopen.*

Proof. We shall just prove b. A trivial modification gives a.

It suffices to prove the result for atomic Φ. So, assume S is Hausdorff and Φ is atomic. Φ is of one of two possible types:

α) $\tau_1(v_0, \ldots, v_n) = \tau_2(v_0, \ldots, v_n)$ where τ_1 and τ_2 are terms;

β) $R(\tau_0, \ldots, \tau_m)$ where $\tau_0, \ldots, \tau_m$ are terms and R is an $(m+1)$-ary relation-symbol.

For α, the result follows from vi and vii and the corollary to Lemma 1.

For β, we use vi, vii and viii. Let χ_R be the characteristic function of R. Then χ_R: $\bigcup_x S_x^{m+1} \to \mathbf{2}$ is continuous, by vii.

By vi and vii, the map $X \to \bigcup_x S_x^{m+1}$ given by $x \to \big(\tau_0(f_0(x), \ldots, f_n(x)), \ldots, \tau_m(f_0(x), \ldots, f_n(x))\big)$ is continuous.

The result follows by composing these two maps.

COROLLARY.

a) *Suppose X is Boolean, $\Phi(v_0, \ldots, v_n)$ is a positive existential L-formula, and $f_0, \ldots, f_n \in \Gamma(X, S)$. Then $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ is open.*

b) *Suppose S is Hausdorff, $\Phi(v_0, \ldots, v_n)$ is an existential L-formula and $f_0, \ldots, f_n \in \Gamma(X, S)$. Then $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ is open.*

Proof. We just prove a. b is similar.

By a basic extension theorem for sections over Boolean spaces [17, pages 13–14], we see that, for an L-formula $\Psi(w_0, \ldots, w_m, v_0, \ldots, v_n)$,

$$\{x \in X: S_x \models (\exists w_0) \ldots (\exists w_m) \Psi(w_0, \ldots, w_m, f_0(x), \ldots, f_n(x))\}$$
$$= \bigcup_{g_0, \ldots, g_m \in \Gamma(X, S)} \{x \in X: S_x \models \Psi(g_0(x), \ldots, g_m(x), f_0(x), \ldots, f_n(x))\}.$$

With Ψ positive and quantifier-free, the result now follows by the theorem.

Remark. Even for S Hausdorff, X Boolean, and Φ existential, $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ need not be closed. In ring theory, a counterexample is provided by Arens-Kaplansky [1, page 477], with Φ as the sentence $(\exists w_0)[w_0^2 \neq w_0]$.

Comer's paper [3] deals with sheaves over Boolean spaces, satisfying (C): For every L-formula $\Phi(v_0, \ldots, v_n)$, and every $f_0, \ldots, f_n \in \Gamma(X, S)$, $\{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}$ is clopen.

As remarked before, we dispute Comer's claim that (C) holds if X is

Boolean and $\mathrm{Th}(\{S_x: x \in X\})$ is model-complete. To make an appropriate modification, we introduce

DEFINITION. A theory T is *positively model-complete* if T is model-complete and, relative to T, every existential formula is equivalent to a positive existential formula.

Then we have:

LEMMA 4. *If X is Boolean and $\mathrm{Th}(\{S_x: x \in X\})$ is positively model-complete, then* (C) *holds.*

Proof. This is immediate from Corollary a to Lemma 3.

3. The main proof.

3.1. For this section, let $\langle S, X, \pi, \mu\rangle$ be a sheaf of L-structures. We define the *stalk theory* of this sheaf as $\mathrm{Th}(\{S_x: x \in X\})$. We define the *section theory* as $\mathrm{Th}(\Gamma(X, S))$.

Our objective is a theorem of the form:

If the stalk theory is model-complete, and..., then the section theory is model-complete.

3.2. Consider the following conditions:

(A) X is Boolean and has no isolated points;

(B) The stalk theory is positively model-complete;

(C) L includes the language for ring theory, with the usual $+, \cdot, 0, 1$;

(D) The stalk theory includes the axioms for non-trivial rings with 1, and the axiom that 0 and 1 are the only idempotents.

Our main effort goes into proving

THEOREM 2. *If the stalk theory is complete, and* (A), (B), (C), (D) *hold, then the section theory is model-complete.*

Remark. The assumption of completeness of the stalk theory is a nuisance, but we do not see how to remove it in the general case. Because of this assumption, Theorem 2 does not cover the Lipshitz-Saracino result. However, we also prove

THEOREM 3. *Suppose L is the language of ring theory. If* (A), (B) *and* (D) *hold, then the section theory is model-complete.*

The proofs of the two theorems differ at only one point, so we will prove Theorem 2 and indicate the modification needed to get Theorem 3. We will divide the proof into sections, so as to show the part played by each assumption.

NOTATION. Let $\Phi(v_0, \ldots, v_n)$ be an L-formula. Let $\vec{f} = \langle f_0, \ldots, f_n\rangle$ where each $f_i \in \Gamma(X, S)$. We define

$$K_\Phi(\vec{f}) = \{x \in X: S_x \models \Phi(f_0(x), \ldots, f_n(x))\}.$$

3.3. Proof of Theorem 2. Assume that the stalk theory is complete and (A), (B), (C), (D) hold. To prove that the section theory is model-complete, it suffices to prove that if Ψ is a primitive formula then $\neg\Psi$ is equivalent to an existential formula relative to the section theory.

Thus, let Ψ be

$$(\exists w_0)\ldots(\exists w_h)\bigwedge_{j<m}\Phi_j(w_0, \ldots, w_h, v_0, \ldots, v_n)\wedge\bigwedge_{l<k}\neg\theta_l(w_0, \ldots, w_h, v_0, \ldots, v_n)]$$

where each Φ_j and θ_l is atomic.

Let $\vec{w}, \vec{v}$ be the sequences $w_0, \ldots, w_h$ and $v_0, \ldots, v_n$ respectively. We use the customary notations such as $\Psi(\vec{v})$, $\Phi_j(\vec{w}, \vec{v})$.

For $l < k$, define Ψ_l as $(\exists\vec{w})[\bigwedge_{j<m}\Phi_j(\vec{w}, \vec{v})\wedge\neg\theta_l(\vec{w}, \vec{v})]$.

Part 1. We claim that, for $f_0, \ldots, f_n \in \Gamma(X, S)$,

$$\Gamma(X, S) \models \Psi(f_0, \ldots, f_n)$$

if and only if the sets $K_{\Psi_l}(\vec{f})$ $(l < m)$ are non-empty and cover X.

Necessity is clear.

Sufficiency. Let $\Delta_l = K_{\Psi_l}(\vec{f})$, for $l < m$, and suppose the sets Δ_l cover X and are each non-empty.

Since the stalk theory is positively model-complete and X is Boolean, each Δ_l is clopen. We want to obtain non-empty clopen sets $C_0, \ldots, C_t$ such that

a) $\bigcup_r C_r = X$, $C_r \cap C_s = \emptyset$ if $r \neq s$;

b) $(\forall r)(\exists l)[C_r \subseteq \Delta_l]$;

c) $(\forall l)(\exists r)[C_r \subseteq \Delta_l]$.

To get these, let B be the Boolean algebra of clopen subsets of X, and let $C_0, \ldots, C_t$ be the atoms of the (necessarily finite) subalgebra generated by $\Delta_0, \ldots, \Delta_k$.

Now we use the assumption that X has no isolated points. This means that B is atomless. It follows easily that for each $r \leqslant t$ there are non-empty clopen C_{rl} $(l < m)$ such that $C_r = \bigcup_l C_{rl}$, and $C_{rl_1} \cap C_{rl_2} = \emptyset$ if $l_1 \neq l_2$.

It follows that we can define a map σ from $\{C_{rl}: r \leqslant t, l < m\}$ to $\{\Delta_l: l < m\}$ such that

d) $\sigma(C_{rl}) = \Delta_l$ if $C_r \subseteq \Delta_l$;

e) $C_{rl} \subseteq \sigma(C_{rl})$.

Now, since the C_{rl} form a clopen partition of X, a standard argument [17] gives that

$$\Gamma(X, S) \cong \prod_{r,l}\Gamma(C_{rl}, S).$$

We now prove that

$$C_{rl} \subseteq \Delta_p \Rightarrow \Gamma(C_{rl}, S) \models \Psi_p(f_0|C_{rl}, \dots, f_n|C_{rl}) .$$

From a–e and the above isomorphism it follows easily that $\Gamma(X, S) \models \Psi(f_0, \dots, f_n)$.

This will complete the proof of sufficiency.

So, suppose

$$C_{rl} \subseteq \Delta_p = \{x \in X \colon S_x \models \Psi_p(f_0(x), \dots, f_n(x))\} .$$

At each point y in C_{rl} we select $\xi_0(y), \dots, \xi_h(y)$ in S_y such that

$$S_y \models \bigwedge_{j<m} \Phi_j(\xi_0(y), \dots, \xi_h(y), f_0(y), \dots, f_n(y)) \wedge \wedge \neg \theta_p(\xi_0(y), \dots, \xi_h(y), f_0(y), \dots, f_n(y)) .$$

By [17, pages 13–14] there exist $g_{0y}, \dots, g_{hy}$ in $\Gamma(C_{rl}, S)$ such that $g_{\gamma y}(y) = \xi_\gamma(y)$ for $\gamma \leqslant h$.

Thus by Lemma 2, the sets

$$\{x \in C_{rl} \colon S_x \models \bigwedge_{j<m} \Phi_j(g_{0y}(x), \dots, g_{hy}(x), f_0(x), \dots, f_n(x)) \wedge \wedge \neg \theta_p(g_{0y}(x), \dots, g_{hy}(x), f_0(x), \dots, f_n(x))\}$$

for $y \in C_{rl}$, form a clopen cover of C_{rs}. As before, this cover can be refined to a finite clopen partition of C_{rs}. From the latter we can patently obtain $g_0, \dots, g_h$ in $\Gamma(C_{rl}, S)$ such that for all x in C_{rl}

$$S_x \models \bigwedge_{j<m} \Phi_j(g_0(x), \dots, g_h(x), f_0(x), \dots, f_n(x)) \wedge \wedge \neg \theta_p(g_0(x), \dots, g_h(x), f_0(x), \dots, f_n(x)) .$$

Thus

$$\Gamma(C_{rl}, S) \models \Psi_p(f_0|C_{rl}, \dots, f_n|C_{rl}) .$$

This concludes Part 1. Note that we have used only (A) and the model-completeness of the stalk theory.

Part 2. By Part 1,

$$\Gamma(X, S) \models \neg \Psi(f_0, \dots, f_n)$$

if and only if either some $\Delta_l = \emptyset$ or $X \neq \bigcup_l \Delta_l$.

Let $\Psi^-(v_0, \dots, v_n)$ be $\bigwedge_{l<m} \neg \Psi_l(v_0, \dots, v_n)$.

Thus $\Gamma(X, S) \models \neg \Psi(f_0, \dots, f_n)$ if and only if either

α) for some l $K_{\Psi_l}(\vec{f}) = \emptyset$, or

β) $K_{\Psi}(\vec{f}) \neq \emptyset$.

By (B), it follows that there exists an integer q and positive primitive formulas $\Sigma_{ls}(v_0, \dots, v_n)$, for $l < k$ and $s < q$, such that relative to the stalk theory $\neg \Psi_l$ is equivalent to $\bigvee_{s<q} \Sigma_{ls}$.

Thus α is equivalent to

α′) for some l, $X = \bigcup_{s<q} K_{\Sigma_{ls}}(\vec{f})$.

Essentially the same argument gives positive primitive $\Omega_u(v_0, \dots, v_n)$, for $u < U$, such that β is equivalent to

β′) for some $u < U$, $K_{\Omega_u}(\vec{f}) \neq \emptyset$.

Now, using (B) and the Corollary a Lemma 3 once more, we get α″ and β″ below, equivalent respectively to α′ and β′.

α″) There are clopen sets D_{ls} ($l < m$, $s < q$) such that for each l, s $D_{ls} \subseteq K_{\Sigma_{ls}}(\vec{f})$, and, for some l, $X = \bigcup_{s<q} D_{ls}$.

β″) There are clopen sets E_u ($u < U$) such that for each u $E_u \subseteq K_{\Omega_u}(\vec{f})$, and, for some u, $E_u \neq \emptyset$.

This concludes Part 2. Note that (C) and (D) are still unused.

Part 3. We now know that

$$\Gamma(X, S) \models \neg \Psi(f_0, \dots, f_n)$$

if and only if either α″ or β″ holds. It is important to note that the positive primitive formulas used in α″ and β″ are independent of $\vec{f}$, though of course dependent on Ψ.

Suppose Y is clopen, and $\Sigma(v_0, \dots, v_n)$ is positive primitive. Then clearly

$$Y \subseteq K_\Sigma(\vec{f}) \Leftrightarrow \Gamma(Y, S) \models \Sigma(f_0|Y, \dots, f_n|Y) .$$

The reason for having (C) and (D) in our conditions is to enable us to express in the theory of $\Gamma(X, S)$ such statements as

$$(\exists Y)[Y \text{ clopen} \wedge \Gamma(Y, S) \models \Sigma(f_0|Y, \dots, f_n|Y)] .$$

Of course, the key idea is to code up clopen sets by idempotents.

Assume (C) and (D).

For $x \in X$, let 0_x and 1_x be respectively the zero and unit elements of S_x. Let C be a clopen subset of X. Define χ_C by

$$\chi_C(x) = 1_x \quad \text{if} \quad x \in C ,$$
$$\chi_C(x) = 0_x \quad \text{if} \quad x \notin C .$$

Then obviously $\chi_C \in \Gamma(X, S)$, and χ_C is an idempotent.

Conversely, let e be an idempotent in $\Gamma(X, S)$. Then by (D),

$$e(x) = 0_x \text{ or } 1_x , \quad \text{for all } x \in X .$$

Let $C = \{x \in X \colon e(x) = 1_x\}$. Then clearly C is clopen and $\chi_C = e$.

Thus we have a bijection $C \mapsto \chi_C$ from clopen subsets of X to idempotents of $\Gamma(X, S)$, under which $\emptyset$ goes to 0, X to 1.

From this viewpoint we get α‴ and β‴ below, respectively equivalent to α″ and β″.

α''') There are idempotents d_{ls} ($l < m$, $s < q$) such that for each l, s d_{ls} vanishes outside $K_{\Sigma_{ls}}(\vec{f})$, and, for some l, $1 = \prod_{s<q}(1-d_{ls})$.

β''') There are idempotents e_u ($u < U$) such that for each u e_u vanishes outside $K_{\Omega_u}(\vec{f})$, and, for some u, $e_u \neq 0$.

It is now clear that the theorem will be proved as soon as we have established (∗) below.

(∗) For each positive primitive $\Sigma(v_0, \ldots, v_n)$ there is an existential $\Sigma^{\#}(v_0, \ldots, v_n, v_{n+1})$ such that for all $g_0, \ldots, g_{n+1} \in \Gamma(X, S)$ $\Gamma(X, S) \models \Sigma^{\#}(g_0, \ldots, g_n, g_{n+1}) \Leftrightarrow g_{n+1}$ is an idempotent and

$$\{x \in X\colon\ g_{n+1}(x) = 1_x\} \subseteq K_\Sigma(g_0, \ldots, g_n)\,.$$

Obviously, once we have (∗) we know that α''' and β''' are equivalent to existential L-formulas relative to the section theory, and so $\neg\Psi$ is equivalent to an existential L-formula.

Part 4. Proof of (∗). Let $\Sigma(v_0, \ldots, v_n)$ be positive primitive. Then $\Sigma(v_0, \ldots, v_n)$ is of the form $(\exists\vec{w})M(\vec{w}, \vec{v})$, where M is a conjunction of atomic formulas.

Now assume the stalk theory is complete.

Let $\Sigma^{\#}(v_0, \ldots, v_n, v_{n+1})$ be

$$v_{n+1} \neq 0 \to \Big[v_{n+1}^2 = v_{n+1} \wedge (\exists v_{n+2}) \ldots (\exists v_{2n+2})\Big[\bigwedge_{i \leq n} (v_{n+2+i}v_{n+1} = v_i v_{n+1}) \wedge \wedge \Sigma(v_{n+2}, \ldots, v_{2n+2})\Big]\Big]\,.$$

We claim that $\Sigma^{\#}$ has the properties required for (∗). $\Sigma^{\#}$ is certainly existential. It is obvious that $\Gamma(X, S) \models \Sigma^{\#}(g_0, \ldots, g_n, g_{n+1}) \Rightarrow g_{n+1}$ is an idempotent and

$$\{x \in X\colon\ g_{n+1}(x) = 1_x\} \subseteq K_\Sigma(g_0, \ldots, g_n)\,.$$

Suppose conversely that g_{n+1} is an idempotent and g_{n+1} vanishes outside $K_\Sigma(g_0, \ldots, g_n)$. If $g_{n+1} = 0$, then clearly

$$\Gamma(X, S) \models \Sigma^{\#}(g_0, \ldots, g_n, g_{n+1})\,.$$

Suppose $g_{n+1} \neq 0$. Let $C = \{x \in X\colon\ g_{n+1}(x) = 1_x\}$. Then $C \neq \emptyset$. Select $x \in X$. Then $S_x \models \Sigma\big(g_0(x), \ldots, g_n(x)\big)$. Therefore

$$S_x \models (\exists v_0) \ldots (\exists v_n)\,\Sigma(v_0, \ldots, v_n)\,.$$

Since the stalk theory is complete,

$$S_y \models (\exists v_0) \ldots (\exists v_n)\,\Sigma(v_0, \ldots, v_n)\,, \quad \text{for all } y \in X\,.$$

C is clopen. By a now familiar argument, since

$$S_y \models (\exists\vec{w})(\exists v_0) \ldots (\exists v_n) M(\vec{w}, v_0, \ldots, v_n) \quad \text{for each } y \text{ in } X\,,$$

we get $g_0', \ldots, g_n' \in \Gamma(X \setminus C, S)$ such that

$$\Gamma(X \setminus C, S) \models \Sigma(g_0', \ldots, g_n')\,.$$

For $i \leq n$, let g_{n+2+i} be the unique element of $\Gamma(X, S)$ such that $g_{n+2+i}|C = g_i$ and $g_{n+2+i}|(X \setminus C) = g_i'$. Then

$$g_{n+2+i} \cdot g_{n+1} = g_i \cdot g_{n+1} \quad \text{and} \quad \Gamma(X, S) \models \Sigma(g_{n+2}, \ldots, g_{2n+2})\,.$$

Thus $\Gamma(X, S) \models \Sigma^{\#}(g_0, \ldots, g_n, g_{n+1})$. This concludes the proof.

3.3. Proof of Theorem 3. We have only to alter Part 4 of the preceding proof, replacing the assumption of completeness of the stalk theory by the new assumption that L is the language of ring theory. (Note that (C) follows from this assumption).

We have to prove (∗). With our new assumption, $\Sigma(\vec{v})$ is of the form

$$(\exists\vec{w}) \bigwedge_{j<m} \big(p_j(\vec{w}, \vec{v}) = q_j(\vec{w}, \vec{v})\big)$$

where each p_j, q_j is a polynomial.

Now it is trivial that we can take $\Sigma^{\#}(v_0, \ldots, v_n, v_{n+1})$ as

$$v_{n+1}^2 = v_{n+1} \wedge (\exists\vec{w}) \bigwedge_{j<m} \big(p_j(\vec{w}, \vec{v}) \cdot v_{n+1} = q_j(\vec{w}, \vec{v}) \cdot v_{n+1}\big)\,.$$

This proves Theorem 3.

Remark. One can generalize Theorem 3. Drop the assumption that L is the language of ring theory. Replace it by (C), and (E) below.

(E) For each atomic $\Phi(v_0, \ldots, v_n)$ in which 1 does not occur, the sentence $\Phi(0, \ldots, 0)$ is in the stalk theory.

We use the proof of Theorem 2, this time using (E) to get (∗). We omit the details.

4. The main theorems.

4.1. From the point of view of applications, Theorems 2 and 3 are not too useful. The problem is that both theorems say that a certain *complete* theory is model-complete, and leave us the problem of identifying the complete theory by means of intelligible axioms.

However, there is a uniformity in the proofs of Theorems 2 and 3, which leads to more useful theorems.

Let us look back at the proof of Theorem 2. What we in fact proved was that, provided the hypotheses are satisfied, for any universal $U(\vec{v})$ there is an existential $E(\vec{v})$ such that $U(\vec{v})$ is equivalent to $E(\vec{v})$ relative to $\mathrm{Th}\big(\Gamma(X, S)\big)$, and E depends only on U and the stalk theory. (Such a uniformity can be seen in a more general setting in Comer's Theorem 1.1 [3]).

We immediately deduce:

THEOREM 4. *Suppose L includes the language for ring theory. Let T be* *a complete positively model-complete L-theory. Suppose T contains the axioms for the theory of non-trivial rings and the axiom that* 0 *and* 1 *are the only idempotents.*

Let $\mathfrak{C}$ *be the class of all* $\Gamma(X, S)$, *where*

i) S *is a sheaf of L-structures over* X;

ii) X *is Boolean with no isolated points*;

iii) *for each* $x \in X$, $S_x \models T$.

Then $\mathrm{Th}(\mathfrak{C})$ *is model-complete.*

Remarks 1. Because of the existence of constant sheaves, $\mathfrak{C}$ is never empty. If C is Cantor space, and $M \models T$, $M^C \in \mathfrak{C}$.

3. From the completeness of T, and ii, it follows that $\mathrm{Th}(\mathfrak{C})$ is complete. This is easily deduced from Comer's Theorem.

4.2. Analogous considerations of uniformity easily give an improvement of Theorem 3.

THEOREM 5. *Let L be the language of ring theory. Let T be a positively model-complete L-theory. Suppose T contains the axioms for the theory of non-trivial rings and the axiom that* 0 *and* 1 *are the only idempotents.*

Let $\mathfrak{C}$ *be the class of all* $\Gamma(X, S)$ *where*

i) S *is a sheaf of L-structures over* X;

ii) X *is Boolean with no isolated points*;

iii) *for each* $x \in X$, $S_x \models T$.

Then $\mathrm{Th}(\mathfrak{C})$ *is model-complete.*

COROLLARY. *If T includes the axioms of field theory, and is model-complete, then* $\mathrm{Th}(\mathfrak{C})$ *is model-complete.*

Proof. We just use the remark of [22] that in field theory every existential formula is equivalent to a positive existential formula.

Remark. We can also use the uniformity to improve the remark following Theorem 3. We omit the details.

5. The Lipshitz-Saracino Theorem. We now indicate how the above ideas can be applied to theories of commutative rings with 1, without non-zero nilpotent elements.

Suppose R is a non-trivial commutative ring with 1, without non-zero nilpotent elements. Then [12] R is semi-simple and is embeddable as a subring of $\prod_M R/M$, where M ranges over all maximal ideals of R. Each R/M is a field. To illustrate the general idea to be used below, select for each M a field F_M with $R/M \subseteq F_M$. Select also for each M a copy C_M of Cantor space, and let $F_M^{C_M}$ be the usual structure of continuous functions. We have the embedding $R \to \prod_M F_M^{C_M} = R_1$, say.

Now each $F_M^{C_M}$ is regular and so R_1 is regular. The maximal ideal space of R_1 is naturally homeomorphic to $\prod_M C_M$ and so is Boolean without isolated points. By Dauns-Hofmann [5], R_1 is isomorphic to a ring $\Gamma(X, S)$ where $X = \prod_M C_M$ and each S_x is isomorphic to some F_M. This enables us to apply Theorem 5.

Application 1. Let T_0 be the theory of commutative rings with 1, without non-zero nilpotent elements. We will indicate the proof of the Lipshitz-Saracino theorem [15] that T_0 has a model-companion T_1.

Let T be the theory of algebraically closed fields. Let $\mathfrak{C}$ be the class defined in Theorem 5. Since T is model-complete [19,22], $\mathrm{Th}(\mathfrak{C})$ is model-complete. Now, by taking each F_M as algebraically closed above, we see that each model of T_0 is embeddable in a model of $\mathrm{Th}(\mathfrak{C})$. On the other hand, it is clear that each model of $\mathrm{Th}(\mathfrak{C})$ is a model of T_0. Thus T_0 and $\mathrm{Th}(\mathfrak{C})$ are mutually model-consistent, so T_0 has a model-companion T_1, namely $\mathrm{Th}(\mathfrak{C})$.

Lipshitz and Saracino exhibit axioms for T_1. This can be done naturally in our approach, too. Firstly, it is clear that T_1 includes the axioms for non-trivial commutative regular rings with 1. Secondly, since the base spaces have no isolated points, T_1 includes the axiom that there are no minimal idempotents. Finally, since T includes the axioms saying that each monic polynomial has a root, the usual lifting argument for Boolean spaces implies that T_1 also includes these axioms.

Conversely, if M satisfies these axioms just found, we apply the Dauns-Hofmann approach to show that M is in $\mathfrak{C}$, so $M \models T_1$. For M must be, by regularity, of the form $\Gamma(X, S)$ with X Boolean and S a sheaf of fields. Since there are no minimal idempotents, X has no isolated points. Finally, the third group of axioms imply that the stalks are algebraically closed. Thus M is in $\mathfrak{C}$.

6. Application 2. Lattice-ordered rings.

6.1. This paper was inspired by the question: What happens if we replace algebraically closed fields by real closed fields in [15]?

It turns out that we can obtain an analogue of Application 1. This applies to a certain natural class of lattice-ordered rings, namely the class of non-trivial commutative f-rings [2] with 1 and with no non-zero nilpotent elements.

For the basic facts about lattice-ordered rings, one should consult [2, 9, 13].

Let L be a logic containing the usual logic for ring theory with 1, and in addition two 2-ary operation-symbols $\wedge$ and $\vee$.

An l-ring with 1, or lattice-ordered ring, is an L-structure which is a ring with 1, and a lattice under $\wedge$ and $\vee$, such that if $\leqslant$ is the lattice partial ordering then

(1) $$x \geqslant y \Rightarrow a+x \geqslant a+y,$$

(2) $$x \geqslant 0 \text{ and } y \geqslant 0 \Rightarrow xy \geqslant 0 .$$

An example is the ring of real-valued continuous functions on $[0,1]$, with $\vee$ and $\wedge$ as the usual vector-lattice operations [2].

It is well-known [2] that in an l-ring $\vee$ is definable from the other operations. However, we won't utilize this fact.

DEFINITION. An *f-ring* (function ring) is an l-ring in which $a\wedge b = 0$ and $c \geqslant 0 \Rightarrow ca\wedge b = ac\wedge b = 0$.

In [2] it is shown that an l-ring with no non-zero nilpotent elements is an f-ring if and only if it satisfies the neat condition

$$a\wedge b = 0 \Rightarrow ab = 0 .$$

Let T_0 be the theory of commutative f-rings with 1, with no non-zero nilpotent elements. We are going to outline a proof that T_0 has a model-companion T_1.

Let T be the theory of l-rings with 1 which are linearly ordered by $\leqslant$ and are real closed fields. The usual proofs of model-completeness for real closed fields [19,22] show that T is a model-complete L-theory. Indeed, T is obviously complete and positively model-complete. Thus, we prepare to apply Theorem 4.

Suppose R is a commutative f-ring with 1, with no non-zero nilpotent elements. Then by a theorem of Pierce [18] R is embeddable in a product of linearly ordered domains. Thus R is embeddable in a product of ordered fields, and so in a product of real closed ordered fields. By the same idea as in Section 5 we get an embedding $R \to \prod_M R_M^{C_M}$ where each R_M is real closed and each C_M is a Cantor space. In this case we can go further. Because of the joint-embedding property for real closed fields, we get an embedding $R \to S^C$ where S is real closed and C is a Cantor space.

Let $\mathcal{C}$ be the class defined in Theorem 4. Then $\mathrm{Th}(\mathcal{C})$ is complete and model-complete, since T is complete and positively model-complete. Now $S^C \in \mathcal{C}$. On the other hand, $\mathcal{C} \models T_0$. Let $T_1 = \mathrm{Th}(\mathcal{C})$. We have proved that T_1 and T_0 are mutually model-consistent, and T_1 is model-complete. Thus T_0 has a model-companion, namely T_1.

6.2. Axioms for T_1. From the completeness of T_1, it is clear that $T_1 = \mathrm{Th}(\boldsymbol{R}^C)$, where $\boldsymbol{R}^C$ is the l-ring of real-valued continuous functions on Cantor space C.

Firstly it is clear that $\boldsymbol{R}^C$ is a regular commutative ring with 1, with no minimal idempotents. Secondly, $\boldsymbol{R}^C$ is an f-ring. Thirdly, the usual lifting argument for Boolean spaces, applied to a basic property of $\boldsymbol{R}$, implies that if g is a monic polynomial in one variable over $\boldsymbol{R}^C$, of odd degree, then g has a root in $\boldsymbol{R}^C$.

Finally, another lifting argument gives the following property of $\boldsymbol{R}^C$: If $x\wedge 0 = 0$ then there exists a y such that $y^2 = x$.

Let T_1^- be the axioms for the theory of L-structures M such that
1) M is a commutative non-trivial regular ring with 1;
2) M has no minimal idempotents;
3) M is an f-ring;
4) all monic polynomials of odd degree over M have roots in M;
5) if $x\wedge 0 = 0$, then there exists y such that $y^2 = x$.

THEOREM 6. *T_1^- is a set of axioms for T_1.*

Proof. Clearly $T_1^- \subseteq T_1$.

Conversely, suppose $M \models T_1^-$. We want to apply a result of Keimel [13].

Suppose $a \in M$ and $0 \leqslant x \leqslant \lambda a$, where $\lambda, x \in M$. By regularity, $\lambda a = \mu e$ where e is an idempotent in the ideal generated by a. $1-e = (1-e)^2$, so $1-e \geqslant 0$, since M is an f-ring. We have

$$x = xe + x(1-e)\,, \quad \text{so} \quad 0 \leqslant xe + x(1-e) \leqslant \mu e\,,$$

so, multiplying by $1-e$,

$$0 \leqslant x(1-e) \leqslant 0\,, \quad \text{so} \quad x(1-e) = 0\,, \quad \text{so} \quad x = xe\,.$$

This proves that in M the principal l-ideal generated by a is precisely the principal ring ideal generated by a. This implies that M is a *quasi-regular f-ring* in the sense of Keimel [13], and we will be able to apply Keimel's Theorem 7.4 once we check some facts about irreducible l-ideals in M.

First note that, by regularity, if I is an l-ideal and $x^2 \in I$ then $x \in I$. This implies that in any homomorphic image of M there are no non-zero elements t with $t^2 = 0$.

Suppose I is an irreducible l-ideal in M. Then M/I is a totally ordered l-ring. By the preceding paragraph, M/I has no non-zero elements t with $t^2 = 0$. It follows by a very trivial argument that M/I is an integral domain. Thus I is a prime ring-ideal. Since M is regular, I is a maximal ring-ideal, so M/I is a field. Thus M/I is an ordered field.

It follows from the above that Keimel's space Spec M of irreducible l-ideals is Boolean in this case. Also, since M has no minimal idempotents, Spec M has no isolated points. We apply Keimel's Third Special Representation Theorem [13]. By this, and the above, M is isomorphic to $\Gamma(\mathrm{Spec}\, M, S)$ for a sheaf S of ordered fields.

We have only to show that the stalks S_x are real closed fields. From Axiom (4) it is clear that S_x has no algebraic extensions of odd degree.

Suppose $a \in S_x$, and $a \geqslant 0$. Then $a\wedge 0 = 0$. Since Spec(M) is Boolean, there exists $f \in \Gamma(\mathrm{Spec}\, M, S)$ such that $f(x) = a$. Consider the sections

$f\wedge 0$ and 0. These agree at x, and so agree in a clopen neighbourhood of x. It follows that there is an idempotent $e\neq 0$ such that $ef\wedge 0=0$, and $e(x)=1$. By Axiom (5) for M, and the isomorphism $M\cong\Gamma(\mathrm{Spec}\,M, S)$, there exists a section g such that $g^2=ef$. Then $g^2e=fe$. Thus $g^2(x)=f(x)=a$, so a is a square. We have proved that non-negative elements of S_x are squares.

It follows that each stalk is real closed. We conclude that $M\in\mathfrak{C}$, so $M\models T_1$. This proves the theorem.

COROLLARY. *T_1 is decidable.*

Proof. T_1 is complete and recursively axiomatizable.

To summarize:

The theory of commutative f-rings, with 1 and no non-zero nilpotent elements, has a decidable model-companion, which is complete. The model-companion is the theory of the l-ring $\boldsymbol{R}^C$ of continuous real-valued functions on Cantor space.

Remark. The theory of commutative f-rings with 1 and no non-zero nilpotents has no model-completion, since it does not have the amalgamation property. This can in fact be seen from the example used by Lipshitz-Saracino in their paper for commutative rings with no non-zero nilpotent elements.

In contrast, we do not know if the theory of commutative regular f-rings with 1 has the amalgamation property.

7. Concluding remarks. We hope the above method will be useful elsewhere. It appears to provide a powerful way of extending meta mathematical work on fields to regular rings. There may be some interesting algebra involved in doing the analogue of Section 6 for p-adic fields.

We would like to see the above method extended to non-commutative biregular rings. This seems plausible in view of [5]. We have had some success with sheaf-theoretic ideas in connection with ω_0-categorical theories of rings [27].

Added in proof.

a) Comer now has an improved version of [13].

b) Professor Andrew B. Carson of Seattle has independently obtained the Lipshitz-Saracino result, in a paper to be published in the Journal of Algebra.

c) Dr. Volker Weisspfenning of Yale has shown that the theory of commutative regular f-rings with 1 has the amalgamation property.

d) Dr Dan Saracino of Yale pointed out an example of a model-complete T such that T^2 is not model-complete. T is the theory of an equivalence relation with a constant a. Each equivalence class has exactly two elements, except the class of a which is a singleton. There are infinitely many classes. It is easily seen that T has the required properties.

e) Problem b of 1.3 has a negative answer. Take T as the theory of the abelina group $\boldsymbol{Z}_2\times\boldsymbol{Z}_4$.

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