Thus, \( R \) is the projection of the set
\[
\{ (F, x, y) : (F \text{ is a continuum}) \ (F \neq X) \ (x \in F) \ (y \in F) \}
\]
parallel to the \( C(X) \)-axis on the product space \( X \times X \).

Since the spaces \( C(X) \) and the space of all subcontinua of \( X \) are compact, the projected set is an \( F^* \)-set in \( (C(X) \times X) \times X \) and so is \( R \) (in \( X \times X \)) (comp. [5], vol. II, p. 14).

Moreover, \( R \) is a boundary set, since in every neighbourhood of two given points \( x_0 \) and \( y_0 \) of \( X \), there are two points \( x \) and \( y \) which lie in different components of \( X \) (i.e. that \( x \) non-\( R \) \( y \)).

It follows from Corollary 1 of § 6 that there exists a Cantor set \( F \subset X \) such that no two of its points belong to the same component of \( X \) (Theorem of Mazurkiewicz [6], see also [1]). In fact, almost every compact subset of \( X \) is a Cantor set with the above property.

References


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Model-completeness for sheaves of structures

by

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Abstract. By combining some sheaf-theoretic ideas of Comer, with ideas from the Lipshitz-Saracino proof that the theory of commutative rings has a model-companion, we prove a general theorem enabling one to extend metamathematical results on fields to the corresponding results for certain regular rings. As an application, the model-completeness of real-closed fields yields a model-completeness result for a natural class of lattice-ordered rings.

0. Introduction. In a recent paper [15], Lipshitz and Saracino found the model-companion of the theory of commutative rings without nilpotent elements. In the present paper we will give an abstract version of their proof, apparently suitable for finding the model-companion for various other theories of rings. In particular, we will find the model-companion for the theory of commutative \( f \)-rings with identity and no-zero nilpotent elements.

The representation of rings by sections of sheaves is now well-established [4, 5, 11, 13]. In [3], Comer generalized the Feferman-Vaught results [8] to cover certain structures of sections of sheaves over Boolean spaces, and thereby proved decidability results for various classes of rings. It should be noted that for constant sheaves Comer's results are essentially contained in some publications of the Wroclaw group [23, 24, 25, 26] in 1968-69.

Our main result gives a sufficient condition that, for a sheaf of models of a model-complete theory, the theory of the structure of sections should be model-complete.

Before treating the main result, we give a rapid discussion, in Section 1, of model-completeness properties of reduced products.

We are very grateful to Professors Comer, Lipshitz and Saracino for making preprints of [3] and [15] available to us. It will be seen that this paper is based on a synthesis of their ideas.

1. Model-completeness and reduced products.

1.1. It is well-known that theorems of Feferman-Vaught type can be used to give an algebraic structure to the space of complete theories
in a given logic $L$. For example [8, 24], if $T_1 = \text{Th}(A_1)$ and $T_2 = \text{Th}(A_2)$ then we can unambiguously define $T_1 \times T_2$ as $\text{Th}(A_1 \times A_2)$. More generally, if $A_i (i \in I)$ is a family of $L$-structures and $T_i = \text{Th}(A_i)$, and if $\mathcal{D}$ is a filter on $I$, then we can unambiguously define $\bigotimes_{i \in I} T_i(\mathcal{D})$ as $\text{Th}(\bigotimes_{i \in I} A_i(\mathcal{D})).$

1.2. A number of useful preservation theorems are known. For example, the properties of being decidable, $\omega$-categorical, and totally transcendental are preserved under the product operation [8, 10, 16, 24].

However, the property of being model-complete is not preserved under products.

**Example.** Let $2$ be the two element Boolean algebra and let $B$ be an atomless Boolean algebra. Then $\text{Th}(2)$ and $\text{Th}(B)$ are model-complete, but $\text{Th}(2 \times B)$ is not model-complete. This follows from Theorem 1 below.

**Theorem 1.** Let $M$ be an infinite Boolean algebra with at least one atom. Then $\text{Th}(M)$ is not model-complete.

**Proof.** Assume we have proved

(1) If $M_1$ and $M_2$ are infinite Boolean algebras, then $M_1$ and $M_2$ satisfy the same universal sentences.

Suppose $M_1$ is an infinite Boolean algebra, with $\text{Th}(M_1)$ model-complete. By (1), $\text{Th}(M_1)$ is a model-companion of the theory of Boolean algebras. But model-companions are unique [20] and the theory of atomless Boolean algebras in the model companion. The theorem follows.

To prove (1), we first use the facts that every Boolean algebra is embeddable in an atomless Boolean algebra, and that the theory of atomless Boolean algebras is complete. Thus to prove (1) we need only show that for any infinite $M_1$, $\text{Th}(M_1)$ has a model with an atomless subalgebra.

This is easy by the compactness theorem.

1.3. From Theorem 1, it follows that the map $T \mapsto T(\mathcal{D})$ cannot preserve model-completeness unless $T(\mathcal{D})$ is either finite or atomless.

If $T(\mathcal{D})$ is finite then $T(\mathcal{D}) \cong 2^2$ for some $n \in \omega$, and so by [7, 23] $T(\mathcal{D}) = T$.

Suppose $T(\mathcal{D})$ is atomless. Let $F$ be the Fréchet filter on $\omega$. Then $2^\omega/F = 2^\mathcal{D}/\mathcal{D}$, since both are atomless Boolean algebras. Then, by [7, 23], $T(\mathcal{D}) = T/F$.

This raises the problem:

**Problem.** a) Do the maps $T \mapsto T(\mathcal{D}) (n \in \omega)$ preserve model-completeness?

b) Does the map $T \mapsto T/F$ preserve model-completeness?

We conjecture that the answer to both a and b is negative in general, but we have been unable to construct counterexamples.

The main result of this paper implies that the answer to b is positive, if we restrict attention to theories $T$ in a certain class.

1.4. We shall explain briefly the connection between reduced powers and the papers of Comer and Lipshitz-Saracino.

We refer to [14, 23, 24, 25, 26] for the basic information on limit powers $M^\omega/F$, where $F$ is a filter on $\omega$. Let $C$ be Cantor space and let $M^C$ be the structure of continuous functions from $C$ to $M$ (where $M$ has the discrete topology). (See [23, 24].) Then $M^C$ is in fact a limit power $M^\omega/F$, where $F$ and $T$ are independent of $M$ [23]. Moreover, there is an index set $I$ and a filter $\mathcal{D}$ on $I$ such that for all $M, M^I = M^\omega/F$.

By taking $M$ as the Boolean algebra $B$, and using [7], we see that we can take $I$ as $\omega$ and $\mathcal{D}$ as the Fréchet filter $F$.

Now, $M^C$ is a structure of sections over a constant sheaf on the Boolean space $C$, so we make contact with Comer's [3]. On the other hand, the paper of Lipshitz-Saracino is essentially concerned with rings of sections of sheaves whose stalks are algebraically closed fields. Indeed, in their proof of their Theorem 1, Lipshitz and Saracino explicitly consider products of rings $M^C$ where $M$ is an algebraically closed field.

2. Sheaves of $L$-structures.

2.1. Let $L$ be a fixed first-order logic. We will be considering sheaves whose stalks are $L$-structures.

It will be convenient to make the usual abuse of notation that uses the same symbol for a relation-symbol (or operation-symbol) and its interpretation in a particular model.

**Definition.** $S$ is a sheaf of $L$-structures over $X$ if $S$ is a quadruple $\langle S, \pi, X, \mu \rangle$ where

i) $S$ and $X$ are topological spaces;

ii) $\pi$ is a continuous onto map from $S$ to $X$;

iii) each point in $S$ has an open neighborhood which is mapped homeomorphically onto an open set in $X$ under $\pi$;

iv) $\mu$ is a map with domain $X$;

v) for each $x \in X, \mu(x)$ is an $L$-structure with underlying set $\pi^{-1}(\{x\})$;

vi) for each $(n + 1)$-ary operation-symbol $\tau$ of $L$, the map $\langle s_0, \ldots, s_n \rangle \mapsto \tau(s_0, \ldots, s_n)$ from $\prod_{\pi^{-1}(x)} \mu(x)\pi^{-1}(x)$ to $S$ is continuous, where the domain is given the topology induced on $S$ as a subset of $S^{n+1}$;

vii) for each individual constant $a$ of $L$, the map $X \to \pi^{-1}(x)$, that assigns to $x$ the denotation of $a$ in $\mu(x)$, is continuous;

viii) for each $(n + 1)$-ary relation-symbol $R$ of $L$, the map $\langle s_0, \ldots, s_n \rangle \mapsto \pi^{-1}(x)$ from $\prod_{\pi^{-1}(x)} \mu(x)\pi^{-1}(x)$ to $\pi^{-1}(x)$ is continuous, where $\pi^{-1}(x)$ has the discrete topology, the domain has the topology of (vii), and $\pi^{-1}(x)$ is the characteristic function of $R$. 


EXAMPLE. Let $M$ be an $L$-structure, and $X$ a topological space. Give $M$ the discrete topology. Let $S = X \times M$, and let $\pi$ be the projection map $S \to X$. Let $\mu(x) = M$ for each $x$. Then $\langle S, X, \pi, \mu \rangle$ is a sheaf, the constant $M$-sheaf over $X$.

Remark. Comer [3] appears to exclude relation-symbols from $L$. We have not seen [6], but presumably the definition of sheaf used there is close to that used above.

2.2. Sections. Let $\langle S, X, \pi, \mu \rangle$ be a sheaf of $L$-structures. Let $U$ be a subset of $X$, with the induced topology. A section over $U$ is a continuous function $\sigma: U \to S$ such that $\pi \circ \sigma = \text{id}$ on $U$. Let $\Gamma(U, S)$ be the set of sections over $U$.

We note that $\Gamma(U, S) \subseteq \bigsqcup_{x \in U} \mu(x)$, so $\Gamma(U, S)$ inherits the relational structure of the product. On the other hand, conditions (vi) and (vii) readily imply that $\Gamma(U, S)$ is closed under the operations of the product. Therefore $\Gamma(U, S)$ is a substructure of the $L$-structure $\bigsqcup_{x \in U} \mu(x)$. By the usual abuse of notation, let $\Gamma(U, S)$ be the arrow $U$-structure.

The most basic point about sections is

LEMMA 1. Let $f, g \in \Gamma(U, S)$. Then $\{x \in U: f(x) = g(x)\}$ is open.

Proof. The proof is identical to that in [21, page 25].

COROLLARY. If $S$ is Hausdorff, then $\{x \in U: f(x) = g(x)\}$ is clopen.

Proof. Since $f$ and $g$ are continuous maps into $S$, $\{x \in U: f(x) = g(x)\}$ is clopen.

DEFINITION. $\langle S, X, \pi, \mu \rangle$ is a $T_4$ if $S$ is Hausdorff.

LEMMA 2. Suppose $X$ is Boolean and for all $f, g \in \Gamma(X, S)$, $\{x \in X: f(x) = g(x)\}$ is clopen. Then $S$ is Hausdorff.

Proof. Let $s, t \in S$, $s \neq t$. If $\pi(s) \neq \pi(t)$, $s$ and $t$ have disjoint open neighbourhoods by condition (ii) and the fact that $X$ is Hausdorff. Suppose next that $\pi(s) = \pi(t) = a_0$. Then by [17, pages 13-14] there are $f, g \in \Gamma(X, S)$ such that $f(a_0) = s$ and $g(a_0) = t$. Let $W = \{x \in X: f(x) \neq g(x)\}$. Then $W$ is open. But sections are open maps [17, 21], so $f(W)$ and $g(W)$ are disjoint open neighbourhoods of $s$ and $t$, respectively. Thus $S$ is Hausdorff.

Remark. It is not difficult to construct examples where $X$ is Boolean and $S$ is not Hausdorff. Indeed, one can take $L$ as pure logic with $\to$, and $\langle S, X, \pi, \mu \rangle$ a sheaf of infinite sets. This contradicts the remark of Comer [3], following his introduction of condition (O). For, by Lemma 2, if Comer’s remark were correct, any sheaf of models of a model-complete theory over a Boolean space would have $S$ Hausdorff. But the theory of infinite sets is model-complete.

This observation does not affect any of Comer’s applications, for the theory of infinite sets is model-complete.

sheaves there are $T_4$. In the next subsection we will give a natural condition which implies Comer’s condition (O).

2.3. Henceforward it will be convenient to write “$S_\mu$” for $\mu(x)$.

LEMMA 3. Let $\Phi(\nu_0, \ldots, \nu_n)$ be a quantifier-free $L$-formula, and $f_0, \ldots, f_n \in \Gamma(X, S)$. Then $\Phi(\nu_0, \ldots, \nu_n)$ is open.

a) If $\Phi$ is positive, $\{x \in X: S_\mu \models \Phi(f_0(x), \ldots, f_n(x))\}$ is open.

b) If $S$ is Hausdorff, $\{x \in X: S_\mu \models \Phi(f_0(x), \ldots, f_n(x))\}$ is clopen.

Proof. We shall just prove b. A trivial modification gives a.

It suffices to prove the result for atomic $\Phi$. So, assume $S$ is Hausdorff and $\Phi$ is atomic. $\Phi$ is of one of two possible types:

a) $\tau_1(\nu_0, \ldots, \nu_m) = \tau_2(\nu_0, \ldots, \nu_n)$ where $\tau_1$ and $\tau_2$ are terms;

b) $\tau_1(\nu_0, \ldots, \nu_n) = \tau_2(\nu_0, \ldots, \nu_n)$ where $\tau_1$ and $\tau_2$ are terms and $\tau$ is an $(m+1)$-ary relation-symmetric.

For a, the result follows from vii and viii and the corollary to Lemma 1. For b, we use vi, vii and viii. Let $\chi_\tau$ be the characteristic function of $R$. Then $\chi_\tau \cup \bigcup_{\nu \in \mathbb{Q}} S_{\nu \rightarrow \tau}^\nu$ is continuous, by vii.

By vii and viii, the map $\mathcal{X} \to \bigcup_{\nu \in \mathbb{Q}} S_{\nu \rightarrow \tau}^\nu$ given by $\mathcal{X} \to \{f_0(x), \ldots, f_n(x)\}$ is continuous.

The result follows by composing these two maps.

COROLLARY. Suppose $X$ is Boolean, $\Phi(\nu_0, \ldots, \nu_n)$ is a positive existential $L$-formula, and $f_0, \ldots, f_n \in \Gamma(X, S)$. Then $\{x \in X: S_\mu \models \Phi(f_0(x), \ldots, f_n(x))\}$ is open.

Proof. We just prove a. $b$ is similar.

By a basic extension theorem for sections over Boolean spaces [17, pages 13-14], we see that, for an $L$-formula $\Psi(\nu_0, \ldots, \nu_n, \nu, \ldots, \nu)$,

$$\{x \in X: S_\mu \models \langle \{\nu_0\}, \ldots, \{\nu_n\}, \Psi(f_0(x), \ldots, f_n(x))\} = \bigcup_{\nu_0, \ldots, \nu_n \in \mathbb{R}(X, X)} \{x \in X: S_\mu \models \Psi(\nu_0(x), \ldots, \nu_n(x), f_0(x), \ldots, f_n(x))\}.$$

With $\Psi$ positive and quantifier-free, the result now follows by the theorem.

Remark. Even for $S$ Hausdorff, $X$ Boolean, and $\Phi$ existential, $\{x \in X: S_\mu \models \Phi(f_0(x), \ldots, f_n(x))\}$ need not be closed. In ring theory, a counterexample is provided by Arens-Kaplansky [1, page 477], with $\Phi$ as the sentence $\langle \{\nu_0\}, \nu_0 \neq \nu_1 \rangle$.

Comer’s paper [3] deals with sheaves over Boolean spaces, satisfying (O): For every $L$-formula $\Phi(\nu_0, \ldots, \nu_n)$, and every $f_0, \ldots, f_n \in \Gamma(X, S)$, $\{x \in X: S_\mu \models \Phi(f_0(x), \ldots, f_n(x))\}$ is closed.

As remarked before, we dispute Comer’s claim that (O) holds if $X$ is
3.3. Proof of Theorem 3. Assume that the stalk theory is complete and (A), (B), (C), (D) hold. To prove that the section theory is model-complete, it suffices to prove that if $\mathcal{V}$ is a primitive formula then $\forall \mathcal{V}$ is equivalent to an existential formula relative to the section theory.

Thus, let $\mathcal{V}$ be

$$(\forall \mathcal{V}_1 \ldots \forall \mathcal{V}_n) \bigwedge_{i=1}^{n} \Phi_i(\mathcal{w}_1, \ldots, \mathcal{w}_k, \mathcal{v}_1, \ldots, \mathcal{v}_n) \wedge \bigwedge_{i=1}^{n} \neg \theta_i(\mathcal{w}_1, \ldots, \mathcal{w}_k, \mathcal{v}_1, \ldots, \mathcal{v}_n, \mathcal{v})$$

where each $\Phi_i$ and $\theta_i$ is atomic.

Let $\mathcal{w}_i, \mathcal{v}$ be the sequences $w_1, \ldots, w_k$ and $v_1, \ldots, v_n$ respectively. We use the customary notations such as $\mathcal{V}(\mathcal{v})$, $\Phi_i(\mathcal{w}, \mathcal{v})$.

For $i < h$, define $\mathcal{V}_i$ as $(\forall \mathcal{V}_i)[\bigwedge_{j=1}^{i} \Phi_j(\mathcal{w}, \mathcal{v}) \wedge \neg \theta_i(\mathcal{w}, \mathcal{v})]$.

Part 1. We claim that, for $f_1, \ldots, f_n \in \Gamma(X, S)$,

$$\Gamma(X, S) = \langle f_1, \ldots, f_n \rangle$$

if and only if the sets $K_{\mathcal{V}_i}(f)$ ($1 < m$) are non-empty and cover $X$.

Necessity is clear.

Sufficiency. Let $A_i = K_{\mathcal{V}_i}(f_i)$ for $1 < m$, and suppose the sets $A_i$ cover $X$ and are each non-empty.

Since the stalk theory is positively model-complete and $X$ is Boolean, each $A_i$ is clopen. We want to obtain non-empty clopen sets $C_0, \ldots, C_t$ such that

a) $\bigcup C_r = X, C_r \cap C_s = \emptyset$ if $r \neq s$;

b) $(\forall r)(\forall s)(C_r \subseteq A_r)$;

c) $(\forall i)(\forall r)(C_r \subseteq A_i)$.

To get these, let $B$ be the Boolean algebra of clopen subsets of $X$, and let $C_0, \ldots, C_t$ be the atoms of the (necessarily finite) subalgebra generated by $A_0, \ldots, A_t$.

Now we use the assumption that $X$ has no isolated points. This means that $B$ is atomless. It follows easily that for each $r < t$ there are non-empty clopen $C_{rt}$ ($1 < m$) such that $C_r = \bigcup_{1}^{t} C_{rt}$ and $C_{rt} \cap C_{st} = \emptyset$ if $s \neq t$.

It follows that we can define a map $\sigma$ from $\{C_t : r < t, 1 < m\}$ such that

da) $\sigma(C_{st}) = A_t$ if $C_{st} \subseteq A_t$;

d) $C_{nt} \subseteq \sigma(C_{nt})$.

Now, since the $C_{rt}$ form a clopen partition of $X$, a standard argument [17] gives that

$$\Gamma(X, S) \equiv \bigwedge_{1}^{t} \Gamma(C_{rt}, S)$$

We now prove that...
\[ G_{t_i} \subseteq \mathcal{A} \Rightarrow \Gamma(G_{t_i}, S) \models \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right). \]

From a–e and the above isomorphism it follows easily that \( \Gamma(X, S) \models \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right). \)

This will complete the proof of sufficiency.

So, suppose
\[ G_{t_i} \subseteq \mathcal{A} \Rightarrow \exists \alpha \in \mathcal{F}: S_{\alpha} = \{ \alpha \in \mathcal{F} \mid \exists \beta \forall \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \}. \]

At each point \( y \in G_{t_i} \) we select \( \xi_{0}(y), \ldots, \xi_{k}(y) \) in \( S_{\alpha} \) such that
\[ S_{\alpha} = \{ \xi_{0}(y), \ldots, \xi_{k}(y), f_{0}(y), \ldots, f_{k}(y) \} \wedge \left( \forall \beta \left( \xi_{0}(y), \ldots, \xi_{k}(y), f_{0}(y), \ldots, f_{k}(y) \right) \right). \]

By [17], pages 13–14 there exist \( g_{0}(y), \ldots, g_{k}(y) \) in \( \Gamma(G_{t_i}, S) \) such that
\[ g_{0}(y) = \xi_{0}(y) \quad \text{for} \quad y \in t. \]

Thus by Lemma 2, the sets
\[ \{ x \in G_{t_i} : S_{x} = \bigwedge_{i \in \mathcal{I}} \Phi_{x}(g_{0}(x), \ldots, g_{k}(x), f_{0}(x), \ldots, f_{k}(x)) \wedge \left( \forall \beta \left( g_{0}(x), \ldots, g_{k}(x), f_{0}(x), \ldots, f_{k}(x) \right) \right) \} \]

for \( y \in G_{t_i} \), form a clopen cover of \( G_{t_i} \). As before, this can be refined to a finite clopen partition of \( G_{t_i} \). From the latter we can partially obtain \( g_{0}, \ldots, g_{k} \) in \( \Gamma(G_{t_i}, S) \) such that for all \( x \in G_{t_i} \)
\[ S_{x} = \bigwedge_{i \in \mathcal{I}} \Phi_{x}(g_{0}(x), \ldots, g_{k}(x), f_{0}(x), \ldots, f_{k}(x)) \wedge \left( \forall \beta \left( g_{0}(x), \ldots, g_{k}(x), f_{0}(x), \ldots, f_{k}(x) \right) \right). \]

Thus
\[ \Gamma(G_{t_i}, S) \models \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right). \]

This concludes Part 1. Note that we have used only (A) and the model-completeness of the stalk theory.

Part 2. By Part 1,
\[ \Gamma(X, S) \models \neg \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \]

if and only if either some \( \alpha_{i} = \emptyset \) or \( X \neq \bigcup_{i} A_{i}. \)

Let \( \alpha_{i}(s), \ldots, s_{k} \) be
\[ \bigwedge_{i \in \mathcal{I}} \neg \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \]

Thus \( \Gamma(X, S) \models \neg \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \) if and only if either
\[ \alpha \text{ for some } t \in c_{2} \] or
\[ \beta \text{ for some } c_{2} \in c_{2} \text{ if } \beta \in c_{2} \text{ and } \Gamma_{t} \neq \emptyset. \]

By (B), it follows that there exists an integer \( q \) and positive primitive formulas \( \Sigma_{a} \in c_{2} \), for \( 1 \leq k \) and \( s \leq q \), such that relative to the stalk theory \( \Gamma(X, S) \models \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right). \)

Thus \( \alpha \) is equivalent to \( \alpha' \) for some \( t \in c_{2} \).

Essentially the same argument gives positive primitive \( \Theta(u, \ldots, v, u) \), for \( s < U \), such that \( \beta \) is equivalent to \( \beta' \) for some \( u < U, \beta_{c_{2}} \in c_{2} \).

Now, using (B) and the Corollary 3 once more, we get \( \alpha'' \) and \( \beta'' \) below, equivalent respectively to \( \alpha' \) and \( \beta' \).

By [17], pages 13–14 there exist \( h_{0}(y), \ldots, h_{k}(y) \) in \( \Gamma(G_{t_i}, S) \) such that
\[ h_{0}(y) = \xi_{0}(y) \quad \text{for} \quad y \in c_{2}. \]

Thus by Lemma 2, the sets
\[ \{ x \in G_{t_i} : S_{x} = \bigwedge_{i \in \mathcal{I}} \Phi_{x}(h_{0}(x), \ldots, h_{k}(x), f_{0}(x), \ldots, f_{k}(x)) \wedge \left( \forall \beta \left( h_{0}(x), \ldots, h_{k}(x), f_{0}(x), \ldots, f_{k}(x) \right) \right) \} \]

for \( y \in G_{t_i} \), form a clopen cover of \( G_{t_i} \). As before, this can be refined to a finite clopen partition of \( G_{t_i} \). From the latter we can partially obtain \( h_{0}, \ldots, h_{k} \) in \( \Gamma(G_{t_i}, S) \) such that for all \( x \in G_{t_i} \)
\[ S_{x} = \bigwedge_{i \in \mathcal{I}} \Phi_{x}(h_{0}(x), \ldots, h_{k}(x), f_{0}(x), \ldots, f_{k}(x)) \wedge \left( \forall \beta \left( h_{0}(x), \ldots, h_{k}(x), f_{0}(x), \ldots, f_{k}(x) \right) \right). \]

Thus
\[ \Gamma(G_{t_i}, S) \models \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right). \]

This concludes Part 2. Note that (C) and (D) are still unused.

Part 3. We now know that
\[ \Gamma(X, S) \models \neg \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \]

if and only if either \( \alpha'' \) or \( \beta'' \) holds. It is important to note that the positive primitive formulas used in \( \alpha'' \) and \( \beta'' \) are independent of \( \vec{f} \), though of course dependent on \( \bar{y} \).

Suppose \( Y \) is clopen, and \( \Sigma(u, \ldots, v) \) is positive primitive. Then clearly
\[ Y \subseteq \mathcal{K}(f) \Leftrightarrow \Gamma(Y, S) \models \Sigma(f_{0}, \ldots, f_{k}) \]

The reason for having (C) and (D) in our conditions is to enable us to express in the theory of \( \Gamma(X, S) \) statements as
\[ \left( \forall \bar{y} \left( \forall \alpha \exists \beta \exists \gamma \exists \delta \left( \alpha(s) \Leftrightarrow \beta(s) \wedge \gamma(s) \implies \delta(s) \right) \right) \right). \]

Of course, the key idea is to code up clopen sets by idempotents. Assume (C) and (D).

For \( \in \mathcal{F}, \) let \( 0_{a} \) and \( 1_{a} \) be respectively the zero and unit elements of \( S_{a} \). Let \( C \) be a clopen subset of \( X \). Define \( \chi_{C} \) by
\[ \chi_{C}(s) = 1_{a} \quad \text{if} \quad s \in C \],
\[ \chi_{C}(s) = 0_{a} \quad \text{if} \quad s \notin C \].

Then obviously \( \chi_{C} \in \Gamma(X, S) \), and \( \chi_{C} \) is an idempotent.

Conversely, let \( \in \mathcal{F} \) be an idempotent in \( \Gamma(X, S) \). Then by (D),
\[ \chi(s) = 0_{a} \quad \text{or} \quad 1_{a} \quad \text{for all} \quad s \in \mathcal{F}. \]

Let \( C = \{ s \in \mathcal{F} : \chi(s) = 1_{a} \} \). Then clearly \( C \) is clopen and \( \chi_{C} = \chi \).
There are idempotents \( d_{rs} \) (\( 1 < m, s < q \) such that for each \( I, s \), \( d_{rs} \) vanishes outside \( K_{rs} \)) and, for some \( i \), \( i = \sum_{s=0}^{q-1} c_s (1 - d_{rs}) \).

\( \beta'' \) There are idempotents \( e_u \) (\( u < U \)) such that for each \( u \) \( e_u \) vanishes outside \( K_{e_u} \), and, for some \( u \), \( e_u \neq 0 \).

It is now clear that the theorem will be proved as soon as we have established (*) below.

(*) For each positive primitive \( \Sigma(\emptyset_0, ..., \emptyset_n) \) there is an existential \( \Sigma^*(\emptyset_0, ..., \emptyset_n; \emptyset_{n+1}) \) such that for all \( \emptyset_0, ..., \emptyset_{n+1} \in \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_{n+1}) \) \( \equiv g_{n+1} \), it is an idempotent and

\[ \{ x \in \mathcal{X} : g_{n+1}(x) = 1 \} \subseteq K_{g_{n+1}}(\emptyset_0, ..., \emptyset_n) \] .

Obviously, once we have (*) we know that \( \alpha'' \) and \( \beta'' \) are equivalent to existential \( L \)-formulas relative to the section theory, and so \( \alpha'' \) is equivalent to an existential \( L \)-formula.

4. Proof of branch. Let \( \Sigma(\emptyset_0, ..., \emptyset_n) \) be positive primitive. Then \( \Sigma(\emptyset_0, ..., \emptyset_n) \) is of the form \( (\mathfrak{G}(\emptyset_0)) M(\emptyset_0) \), where \( M \) is a conjunction of atomic formulas.

Now assume the stalk theory is complete.

Let \( \emptyset_{n+1} \neq 0 \) be \( \emptyset_{n+1} = \emptyset_{n+1} \wedge (\mathfrak{G}(\emptyset_{n+2}) \wedge \cdots \wedge (\mathfrak{G}(\emptyset_{n+q})) \wedge \cdots \wedge \Sigma(\emptyset_{n+3}, ..., \emptyset_{n+q+1})) \).

We claim that \( \Sigma^* \) is the properties required for (*) \( \Sigma^* \) is certainly existential. It is obvious that \( \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) \) is an idempotent and

\[ \{ x \in \mathcal{X} : g_{n+1}(x) = 1 \} \subseteq K_{g_{n+1}}(\emptyset_0, ..., \emptyset_n) \] .

Suppose conversely that \( g_{n+1} \) is an idempotent and \( g_{n+1} \) vanishes outside \( K_{g_{n+1}}(\emptyset_0, ..., \emptyset_n) \). If \( g_{n+1} = 0 \), then clearly

\[ \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) \equiv \mathfrak{G}(\emptyset_0, ..., \emptyset_n) \] .

Suppose \( g_{n+1} \neq 0 \). Let \( G = \{ x \in \mathcal{X} : g_{n+1}(x) = 1 \} \). Then \( G \neq \emptyset \). Select \( x \in G \). Then \( S_x \equiv \mathfrak{G}(\emptyset_0, ..., \emptyset_n) \). Therefore

\[ S_x = (\mathfrak{G}(\emptyset_0) \wedge (\mathfrak{G}(\emptyset_0) \Sigma(\emptyset_0, ..., \emptyset_n)) \] .

Since the stalk theory is complete,

\[ S_x = (\mathfrak{G}(\emptyset_0) \wedge (\mathfrak{G}(\emptyset_0) \Sigma(\emptyset_0, ..., \emptyset_n)), \text{ for all } x \in \mathcal{X} .\]

\( \mathcal{C} \) is clopen. By a now familiar argument, since

\[ S_y = (\mathfrak{G}(\emptyset_0) \wedge (\mathfrak{G}(\emptyset_0) \Sigma(\emptyset_0, ..., \emptyset_n))) \text{ for each } y \in \mathcal{X} ,\]

we get \( \emptyset_0, ..., \emptyset_n \in \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) \) such that

\[ \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) = \Sigma(\emptyset_0, ..., \emptyset_n) .\]

For \( i < n \), let \( h_{g_{n+1}} \) be the unique element of \( \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) \) such that \( h_{g_{n+1}}(\emptyset_0, ..., \emptyset_q) = g_{n+1} \). Then

\[ h_{g_{n+1}}(\emptyset_0, ..., \emptyset_q) = h_{g_{n+1}}(\emptyset_0, ..., \emptyset_q) \] .

Thus \( \Gamma(\emptyset_0, \emptyset_1, ..., \emptyset_q) = \Sigma(\emptyset_0, ..., \emptyset_q) \). This concludes the proof.

3.3. Proof of Theorem 3. We have only to alter Part 4 of the preceding proof, replacing the assumption of completeness of the stalk theory by the new assumption that \( L \) is the language of ring theory. (Note that (1) follows from this assumption.)

We have to prove (*) with our new assumption, \( \Sigma(\emptyset) \) is of the form

\[ (\mathfrak{G}(\emptyset) \wedge (\mathfrak{G}(\emptyset) \Sigma(\emptyset, ..., \emptyset))) \] .

where each \( \emptyset, \emptyset \) is a polynomial.

Now it is trivial that we can take \( \Sigma^*(\emptyset_0, ..., \emptyset_n) \) as

\[ \emptyset_{n+1} = \emptyset_{n+1} \wedge (\mathfrak{G}(\emptyset)) \wedge (\mathfrak{G}(\emptyset) \Sigma(\emptyset_0, ..., \emptyset_n)) \] .

This proves Theorem 3.

Remark. One can generalize Theorem 3. Drop the assumption that \( L \) is the language of ring theory. Replace it by (C) and (D) below.

(C) For each atomic \( \emptyset(\emptyset_0, ..., \emptyset_n) \) in which \( 1 \) does not occur, the sentence \( \emptyset(1, ..., 0) \) is in the stalk theory.

We use the proof of Theorem 2, this time using (C) to get (*). We omit the details.

4. The main theorems.

4.1. From the point of view of applications, Theorems 2 and 3 are not too useful. The problem is that both theorems say that a certain complete theory is model-complete, and leave us the problem of identifying the complete theory by means of interpretable axioms.

However, there is a uniformity in the proofs of Theorems 2 and 3, which leads to more useful theorems.

Let us look back at the proof of Theorem 2. What we in fact proved was that, provided the hypotheses are satisfied, for any universal \( U(\emptyset) \) there is an existential \( H(\emptyset) \) such that \( U(\emptyset) \) is equivalent to \( E(\emptyset) \) relative to \( Th(\emptyset, \emptyset) \), and \( H(\emptyset) \) depends only on \( U \) and the stalk theory. (Such a uniformity can be seen in a more general setting in Comer’s Theorem 1.1 [3].)

We immediately deduce:

**Theorem 4.** Suppose \( L \) includes the language for ring theory. Let \( T \) be
a complete positively model-complete L-theory. Suppose T contains the axioms for the theory of non-trivial rings and the axioms that 0 and 1 are the only idempotents.

Let C be the class of all I(X, S), where
i) S is a sheaf of L-structures over X;
ii) X is Boolean with no isolated points;
iii) for each x ∈ X, Sx = T.

Then Th(C) is model-complete.

Remarks 1. Because of the existence of constant sheaves, C is never empty. If C is Cantor space, and M |= T, MG ∈ C.

3. From the completeness of T, and ii, it follows that Th(C) is complete. This is easily deduced from Corner’s Theorem.

4.2. Analogous considerations of uniformity easily give an improvement of Theorem 3.

Theorem 5. Let L be the language of ring theory. Let T be a positively model-complete L-theory. Suppose T contains the axioms for the theory of non-trivial rings and the axioms that 0 and 1 are the only idempotents.

Let C be the class of all I(X, S), where
i) S is a sheaf of L-structures over X;
ii) X is Boolean with no isolated points;
iii) for each x ∈ X, Sx = T.

Then Th(C) is model-complete.

Corollary. If T includes the axioms of field theory, and is model-complete, then Th(C) is model-complete.

Proof. We just use the remark of [22] that in field theory every existential formula is equivalent to a positive existential formula.

Remark. We can also use the uniformity to improve the remark following Theorem 3. We omit the details.

5. The Lipshitz-Saracino Theorem. We now indicate how the above ideas can be applied to theories of commutative rings with 1, without non-zero nilpotent elements.

Suppose R is a non-trivial commutative ring with 1, without non-zero nilpotent elements. Then [12] R is semi-simple and is embeddable as a subring of \( \bigoplus M \) where M ranges over all maximal ideals of R.

Each \( R/M \) is a field. To illustrate the general idea to be used below, select for each \( M \) a field \( F/M \) with \( E/M \subseteq F/M \). Select also for each \( M \) a copy \( G_M \) of Cantor space, and let \( F_M \) be the usual structure of continuous functions. We have the embedding \( R \to \bigoplus M F_M = E_1 \), say.

Now each \( F_M \) is regular and so \( E_1 \) is regular. The maximal ideal space of \( E_1 \) is naturally homeomorphic to \( \bigcup M \) and so is Boolean without isolated points. By Dauns-Hofmann [5], \( E_1 \) is isomorphic to a ring \( I(X, S) \) where \( X = \bigcup M \) and each \( S_x \) is isomorphic to some \( F_M \). This enables us to apply Theorem 5.

Application 1. Let \( T_b \) be the theory of commutative rings with 1, without non-zero nilpotent elements. We will indicate the proof of the Lipshitz-Saracino theorem [15] that \( T_b \) has a model-companion \( T_5 \).

Let T be the theory of algebraically closed fields. Let C be the class defined in Theorem 5. Since T is model-complete [19,22], Th(C) is model-complete. Now, by taking each \( F_M \) as algebraically closed above, we see that each model of \( T_b \) is embeddable in a model of Th(C). On the other hand, it is clear that each model of Th(C) is a model of \( T_b \). Thus \( T_b \) and Th(C) are mutually model-consistent, so \( T_b \) has a model-companion \( T_5 \), namely T(C).

Lipshitz and Saracino exhibit axioms for \( T_5 \). This can be done naturally in our approach, too. Firstly, it is clear that \( T_b \) includes the axioms for non-trivial commutative regular rings with 1. Secondly, since the base spaces have no isolated points, \( T_b \) includes the axiom that there are no minimal idempotents. Finally, since \( T_b \) includes the axioms saying that each monic polynomial has a root, the usual lifting argument for Boolean spaces implies that \( T_5 \) also includes these axioms.

Conversely, if \( M \) satisfies these axioms just found, we apply the Dauns-Hofmann approach to show that \( M \) is in C, so \( M \to T_b \). For \( M \) must be, by regularity, of the form \( I(X, S) \) with X Boolean and S a sheaf of fields. Since there are no minimal idempotents, \( X \) has no isolated points. Finally, the third group of axioms imply that the stalks are algebraically closed. Thus \( M \) is in C.


6.1. This paper was inspired by the question: What happens if we replace algebraically closed fields by real closed fields in [15]? It turns out that we can obtain an analogue of Application 1. This applies to a certain class of lattice-ordered rings, namely the class of non-trivial commutative f-rings [2] with 1 and with no non-zero nilpotent elements.

For the basic facts about lattice-ordered rings, one should consult [2, 9, 13].

Let L be a logic containing the usual logic for ring theory with 1, and in addition two 2-ary operation-symbols \( \wedge \) and \( \vee \).

An L-ring with 1, or lattice-ordered ring, is an L-structure which is a ring with 1, and a lattice under \( \wedge \) and \( \vee \), such that if \( \preceq \) is the lattice partial ordering then

\[
\forall x, y \in R \quad x \preceq y \iff x + y \wedge x \preceq x + y \vee x.
\]
Finally, another lifting argument gives the following property of $R^2$:

If $a \cdot 0 = 0$ then there exists a $y$ such that $y^2 = a$.

Let $T'_n$ be the axioms for the theory of $L$-structures $M$ such that
1) $M$ is a commutative non-trivial regular ring with 1;
2) $M$ has no minimal idempotents;
3) $M$ is an $f$-ring;
4) all monic polynomials of odd degree over $M$ have roots in $M$;
5) if $a \cdot 0 = 0$, then there exists $y$ such that $y^2 = a$.

**Theorem 6.** $T'_n$ is a set of axioms for $T_n$.

**Proof.** Clearly $T'_n \subseteq T_n$.

Conversely, suppose $M \models T'_n$. We want to apply a result of Keimel [13].

Suppose $a \in M$ and $0 \leq \sigma \leq \lambda a$, where $\lambda, \sigma \in M$. By regularity, $\lambda a = \mu e$ where $e$ is an idempotent in the ideal generated by $a$. $1 - e = (1 - e)^2$, so $1 - e \geq 0$, since $M$ is an $f$-ring. We have

$$a = ae + e(1 - e),$$

so, multiplying by $1 - e$,

$$0 \leq e(1 - e) \leq 0,$$

$$a(1 - e) = 0,$$

$$a = ae.$$
f(x) and 0. These agree at x, and so agree in a clopen neighbourhood of x. It follows that there is an idempotent \( e \neq 0 \) such that \( ef(x) = 0 \), and \( e(x) = 1 \). By Axiom (5) for \( M \), and the isomorphism \( \mathcal{M} \cong \mathcal{G}(\text{Spec } M, S) \), there exists a section \( g \) such that \( g^2 = 0 \). Then \( g^2 = g \). Thus \( g^2 = \alpha \), so \( \alpha \) is a square. We have proved that non-negative elements of \( S \) are squares.

It follows that each stalk is real closed. We conclude that \( \mathcal{M} \in \mathcal{C} \). This proves the theorem.

Corollary. \( T \) is isomorphic and recursively axiomatizable.

Proof. \( T \) is complete and recursively axiomatizable.

To summarize:

The theory of commutative \( f \)-rings, with 1 and no non-zero nilpotent elements, has a decidable model-companion, which is complete. The model-companion is the theory of the \( I \)-ring \( K \) of continuous real-valued functions on Cantor space.

Remark. The theory of commutative \( f \)-rings with 1 and no non-nilpotent has no model-completeness, since it does not have the amalgamation property. This can in fact be seen from the example used by Lipshitz-Saracino in their paper for commutative rings with no nilpotent elements.

In contrast, we do not know if the theory of commutative regular \( f \)-rings with 1 has the amalgamation property.

7. Concluding remarks. We hope the above method will be useful elsewhere. It appears to provide a powerful way of extending metamathematical work on fields to regular rings. There may be some interesting algebra involved in doing the analogue of Section 6 for \( p \)-adic fields.

We would like to see the above method extended to non-commutative regular rings. This seems plausible in view of [5]. We have had some success with sheaf-theoretic ideas in connection with \( \omega \), categorical theories of rings [27].

Added in proof. a) Conner now has an improved version of [13].

b) Professor Andrew B. (Queen of Seattle) has independently obtained the Lipshitz-Saracino result, in a paper to be published in the Journal of Algebra.

c) Dr. Volker Weispfenning of Yale has shown that the theory of commutative regular \( f \)-rings with 1 has the amalgamation property.

d) Dr. Dan Saracino of Yale pointed out an example of a model-complete \( T \) such that \( T \) is not model-complete. \( T \) is the theory of an equivalence relation with a constant \( a \). Each equivalence class has exactly two elements, except the class of \( a \) which is a singleton. There are infinitely many classes. It is easily seen that \( T \) has the required properties.

e) Problem b of 1.3 has a negative answer. Take \( T \) as the theory of the abelian group \( Z \times Z \).

References


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