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Applications of the Baire-category method to the problem of independent sets

by

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Abstract. A set $F \subset X$ is said independent in $R \subset X^n$, if for every system $x_1, \ldots, x_n$ of different points of F the point $\langle x_1, \ldots, x_n \rangle$ never belongs to R. The main result states that, if X is a complete space and R is closed and nowhere dense, then the set $\boldsymbol{J}(R)$ of all compact subsets F of X independent in R is a dense G_δ in the space $\boldsymbol{C}(X)$ of all compact subsets of X. Using Baire category theorem this statement is extended to the case where R is an F_σ-set of the first category and also to the case of an infinite sequence $R_1, R_2, \ldots$, where $R_k \subset X^{n(k)}$.

The same method allows also to show the existence of Cantor sets in X (supposed dense-in-itself) independent in R (or more generally, in $R_1, R_2, \ldots$). Similar results were obtained in [10] and [11].

Applications to indecomposable continua (and others) are considered.

§ 1. Introduction.

DEFINITION. Let X be a space and R an n-ary relation in X, i.e. $R \subset X^n$. A set $F \subset X$ is said to be *independent in* R, written $F \in \boldsymbol{J}(R)$, if for every point $x = \langle x_1, \ldots, x_n \rangle \in F^n$ with distinct coordinates (i.e. $x_i \neq x_j$ for $i \neq j$), we have $x \in X^n - R$.

In particular, if R is a binary relation ($n = 2$), F is independent, if no two of its elements are in the relation R.

In many cases, it is important to know whether or not there exists an uncountable compact set $F \subset X$ independent in a given relation R.

The Main Theorem of this paper will give a possibility of proving the existence of an F independent in R (under suitable assumptions on X and R) with the use of the Baire category method; thus — avoiding individual constructions of F (ackward — in many cases).

Let us note two useful (and obvious) formulas

(1) $$\text{if} \quad R_1 \subset R_2, \quad \text{then} \quad \boldsymbol{J}(R_2) \subset \boldsymbol{J}(R_1),$$

(2) $$\boldsymbol{J}(\bigcup_k R_k) = \bigcap_k \boldsymbol{J}(R_k).$$

§ 2. Terminology and notations. X denotes in this paper an arbitrary *complete* (metric) space.

By a *Cantor set* we mean any space homeomorphic to the well known Cantor discontinuum.

$\boldsymbol{C}(X)$ denotes the space of all compact non-empty subsets of X with the exponential (Vietoris) topology; that means that the sets

(3) $$\{F\colon F\subset G\}$$

and

(4) $$\{F\colon F\cap G\neq\emptyset\},$$

where $F\in\boldsymbol{C}(X)$ and G is open, form an open subbase of $\boldsymbol{C}(X)$.

(5) *The space $\boldsymbol{C}(X)$ is topologically complete.*

More precisely, the space $\boldsymbol{C}(X)$, with the Hausdorff-distance topology (which agrees under our assumptions on X, with the exponential topology) is complete.

The fact that the exponential topology of $\boldsymbol{C}(X)$ for X complete, agrees with the Hausdorff-distance topology of $\boldsymbol{C}(X)$, was essentially shown in [3], vol. II, p. 47. (See also [7], 3.3 and 3.6.) It follows (by a known theorem of H. Hahn, see [3], p. 124) that $\boldsymbol{C}(X)$ is topologically complete.

A set $A\subset X$ is called *nowhere dense*, if its closure contains no interior points. A countable union of nowhere dense sets is called a set of the *first category*. A complement of a set of the first category is called *residual*.

By a classical theorem of Baire, *a residual set in a complete space is dense in this space.*

Since a countable union of sets of the first category is a set of the first category, it follows that a countable intersection of residual sets (in a complete space) is residual.

We shall say that *almost every* element of a space has a given property, if the set of elements which have this property is residual.

§ 3. A Lemma on topological spaces.

Let X be an arbitrary topological space and let G be an open dense set in the space X^n ($n\geqslant 1$). Let $H_1,\ldots,H_m$, where $m\geqslant n$, be open non-empty sets in X. Then there exists a system of open non-empty sets $U_1,\ldots,U_m$ such that

(6) $$U_i\subset H_i\quad \textit{for}\quad 1\leqslant i\leqslant m$$

and

(7) $$U_{i_1}\times\ldots\times U_{i_n}\subset G\quad \textit{whenever}\quad i_1<\ldots<i_n.$$

Proof (by induction relatively to $m-n$).

α. Case $m-n=0$. Put $Q=H_1\times\ldots\times H_n$. Since G is dense in X^n and Q is open, we have $Q\cap G\neq\emptyset$, and since $Q\cap G$ is an open non-empty set in X^n, there exists a system $U_1,\ldots,U_n$ of open non-empty subsets of X such that

$$U_1\times\ldots\times U_n\subset(Q\cap G),$$

i.e. $U_1\times\ldots\times U_n\subset H_1\times\ldots\times H_n$ and $U_1\times\ldots\times U_n\subset G$.

Hence the conditions (6) and (7) are satisfied (in condition (7) we put, of course, $i_j=j$).

β. Case $m+1$. Assume that the lemma for m ($\geqslant n$) is true.

Let $(\sigma_1,\sigma_2,\ldots,\sigma_t)$ be the set of all n-elements systems σ_j of the form $(i_1,\ldots,i_n)$ where $i_1<\ldots<i_n\leqslant m+1$.

We shall define in X open non-empty sets

$$H_i^j,\quad 0\leqslant j\leqslant t \text{ and } 1\leqslant i\leqslant m+1$$

so that

(8) $$H_i^0=H_i,$$

$(9)_j$ $$H_i^j\subset H_i^{j-1}\quad\text{for}\quad j\geqslant 1,$$

$(10)_j$ $$H_{i_1}^j\times\ldots\times H_{i_n}^j\subset G\quad\text{for}\quad j\geqslant 1 \text{ and } \sigma_j=(i_1,\ldots,i_n).$$

We proceed by induction relatively to $j\leqslant t$. Let $0\leqslant j\leqslant t-1$ and let $\sigma_{j+1}=(i_1,\ldots,i_n)$. As has been shown (see case α), replacing the system $H_1,\ldots,H_n$ by $H_{i_1}^j,\ldots,H_{i_n}^j$, there exists a system of open non-empty sets $H_{i_1}^{j+1},\ldots,H_{i_n}^{j+1}$ satisfying conditions $(9)_{j+1}$ and $(10)_{j+1}$.

Thus the sets H_i^{j+1} have been defined for $i\in\sigma_j$. For $i\notin\sigma_j$, we put $H_i^{j+1}=H_i^j$.

Finally, we define the system $U_1,\ldots,U_{m+1}$ by setting

(11) $$U_i=H_i^t\quad\text{for}\quad 1\leqslant i\leqslant m+1.$$

Thus defined, $U_1,\ldots,U_{m+1}$ is the required system. For, by (8), $(9)_j$ and (11), the inclusion (6) is fulfilled. Next, let $i_1<\ldots<i_n$ be given and put $\sigma_j=(i_1,\ldots,i_n)$. Since $H_i^t\subset H_i^j$ (by $(9)_j$), inclusion $(10)_j$ implies (7).

This completes the proof.

§ 4. Auxiliary properties of $\boldsymbol{C}(X)$.

Proposition 1. *The set of all finite sets is dense in $\boldsymbol{C}(X)$.*

This is an easy consequence (for each T_1-topological space) of the fact (mentioned in [7], 2.4.1), that the set of all finite sets is dense in the space (denoted 2^X) of all closed non-empty subsets of X.

Proposition 2. *If X contains no isolated points, then the set of all Cantor subsets of X is a G_δ-set dense in $\boldsymbol{C}(X)$.*

Proof. It is easily seen ([5], vol. II, p. 108) that for F compact, the necessary and sufficient condition to be a Cantor set, is the existence, for each m, of a finite system $\Gamma = (G_1, \ldots, G_{2k})$ of disjoint open sets such that

$$F \subset G_1 \cup \ldots \cup G_{2k}, \quad F \cap G_i \neq \emptyset \quad \text{and} \quad \delta(G_{2j-1} \cup G_{2j}) < 1/m .$$

Now let us denote $T_{\Gamma,m}$ the set of F satisfying the above conditions. Obviously (comp. § 2 (1)) $T_{\Gamma,m}$ is open in $C(X)$, and so is $T_m = \bigcup_{\Gamma} T_{\Gamma,m}$. Since $T = \bigcap_m T_m$, it follows that T is a G_δ-set in $C(X)$.

It remains to show that T is dense in $C(X)$.

By virtue of Proposition 1, it suffices to show that each neighbourhood of a finite set (in $C(X)$) contains a Cantor set, and this reduces to the case where this set is composed of a single point p. Now let

$$Q = \{F\colon (F \subset G_0)(F \cap G_1 \neq \emptyset) \ldots (F \cap G_n \neq \emptyset)\}$$

be an open set in $C(X)$ containing $\{p\}$, i.e. $p \in G$, where $G = G_0 \cap G_1 \cap \ldots \ldots \cap G_n$.

Since G is open in X, G contains no isolated points. Since every dense-in-itself complete space contains a Cantor set (see e.g. [5], vol. II, p. 444), there is a Cantor set $F \subset G$. Obviously $F \in Q$.

Proposition 3. *The mapping F^n is continuous.*

More precisely: *if we put $\Phi_n(F) = F^n$, then the mapping $\Phi_n\colon C(X) \to C(X^n)$ is continuous.*

We shall deduce this proposition from the following more general statement.

Proposition 4. *Let X and Y be topological spaces and let $\Psi(K, L) = K \times L$. Then the mapping*

$$\Psi\colon C(X) \times C(Y) \to C(X \times Y)$$

is continuous.

Proof. We have to show that the sets

$$A = \{\langle K, L\rangle\colon (K \times L) \subset G\} \quad \text{and} \quad B = \{\langle K, L\rangle\colon (K \times L) \cap G \neq \emptyset\}$$

are open in $C(X) \times C(Y)$ whenever G is open $C(X \times Y)$.

First, let us assume that $\langle K_0, L_0\rangle \in A$, i.e. that $(K_0 \times L_0) \subset G$. Then by a Theorem of Wallace (see [4], p. 142), there exist open sets U in X and V in Y such that

$$K_0 \subset U, \quad L_0 \subset V \quad \text{and} \quad U \times V \subset G .$$

Hence $(K \subset U)(L \subset V) \Rightarrow (K \times L \subset G)$, and

$$\langle K_0, L_0\rangle \in \{K\colon K \subset U\} \times \{L\colon L \subset V\} \subset A .$$

Since the sets $\{K\colon K \subset U\}$ and $\{L\colon L \subset V\}$ are open (comp. (1)), we have thus defined an open neighbourhood of $\langle K_0, L_0\rangle$ contained in A. Therefore A is open.

Next, suppose that $\langle K_0, L_0\rangle \in B$, i.e. that $(K_0 \times L_0) \cap G \neq \emptyset$. Let $x_0 \in K_0$, $y_0 \in L_0$ and $\langle x_0, y_0\rangle \in G$. Then there are open sets U in X and V in Y such that

$$x_0 \in U, \quad y_0 \in V \quad \text{and} \quad U \times V \subset G .$$

Therefore

$$\langle K_0, L_0\rangle \in \{K\colon K \cap U \neq \emptyset\} \times \{L\colon L \cap V \neq \emptyset\} \subset B .$$

Since the sets $\{K\colon K \cap U \neq \emptyset\}$ and $\{L\colon L \cap V \neq \emptyset\}$ are open, it follows — like before — that B is open.

This completes the proof of Proposition 4.

We deduce by induction that, assuming

$$\Psi_n(K_1, K_2, \ldots, K_n) = K_1 \times K_2 \times \ldots \times K_n ,$$

the mapping $\Psi_n\colon C(X_1) \times C(X_2) \times \ldots \times C(X_n) \to C(X_1 \times X_2 \times \ldots \times X_n)$ is continuous.

In particular, the mapping $[C(X)]^n \to C(X^n)$ is continuous and consequently so is $\Phi_n\colon C(X) \to C(X^n)$, because the mapping $C(X) \to [C(X)]^n$ is continuous (since for every space Z, the mapping of Z onto the diagonal of Z^n is continuous).

Remark. The Proposition 4 was proved for the case where X and Y are compact by Engelking (in a more general form, see [2], p. 725, Cor. 2). It is also possible to deduce Proposition 4 from the Theorem of Engelking.

Proposition 5. *If A is a G_δ-set in X, then the set $\{F\colon F \subset A\}$ is a G_δ-set in $C(X)$.*

Let $A = G_1 \cap G_2 \cap \ldots$ where all G_n are open. Then $F \subset A$ iff $F \subset G_n$ for all $n = 1, 2, \ldots$ Hence

$$\{F\colon F \subset A\} = \bigcap_n \{F\colon F \subset G_n\} ,$$

which completes the proof.

§ 5. The Main Theorem. *Let X be a complete space and let $R \subset X^n$ be nowhere dense. Then almost every $F \in C(X)$ is independent in R.*

More precisely, *if R is closed and nowhere dense, then the set $J(R)$ is a G_δ dense in $C(X)$.*

Proof. Since the first part of the Theorem follows from the second (by virtue of (1)), we may assume that R is closed.

1. *$J(R)$ is G_δ.* (Here the assumption of R being nowhere dense can be omitted.)

Denote by Δ_n the set of all points $\langle x_1, \ldots, x_n\rangle$ of X^n with, at least, two identical coordinates. Then, we have by definition of $J(R)$ (see § 2)

$$J(R) = \{F \in C(X)\colon F^n \subset G \cup \Delta_n\} \tag{12}$$

where $G = X^n - R$.

Denote, like in § 4, $\Phi_n(F) = F^n$, and put

$$Q = \{Z \in C(X^n):\ Z \subset G \cup \Delta_n\}.$$

Therefore $J(R) = \Phi_n^{-1}(Q)$.

Since G is open and Δ_n closed, so $G \cup \Delta_n$ is a G_δ-set in X^n, and hence (by § 4,5), Q is G_δ in $C(X^n)$, and it follows by the continuity of Φ_n (see Proposition 3 of § 4) that $J(R)$ is a G_δ-set in $C(X)$.

2. *$J(R)$ is dense in $C(X)$*. We have to show that, for every $A \in C(X)$ and every open set H in $C(X)$ such that $A \in H$, there exists $F \in (H \cap J(R))$, i.e. by (12)

$$F \in H \tag{13}$$

and

$$F^n \subset G \cup \Delta_n. \tag{14}$$

Since the family of finite sets is dense in $C(X)$, we may assume that A is finite: $A = (a_1, \ldots, a_m)$. Since $A \in H$, there are open sets $H_1, \ldots, H_m$ such that $a_i \in H_i$ and

$$\{(x_1 \in H_1) \ldots (x_m \in H_m)\} \Rightarrow \{(x_1, \ldots, x_m) \in H\}. \tag{15}$$

Let $U_1, \ldots, U_m$ be open non-empty subsets of X satisfying conditions (6) and (7). Let $b_i \in U_i$ for $i = 1, \ldots, n$, and let $F = (b_1, \ldots, b_m)$.

Formula (13) follows immediately from (6) and (15).

Finally, by (7), we have $\langle b_{i_1}, \ldots, b_{i_n}\rangle \in G$ whenever $i_1 < \ldots < i_n$; this implies (14).

§ 6. Corollaries and remarks.

Corollary 1. *The first part of the Theorem remains true if we assume R to be of the first category.*

In its second part we may assume that R is an F_σ-set of the first category.

This follows at once from (2) by puting $R = \bigcup_k R_k$, where R_k is closed nowhere dense.

Corollary 2. *Let $R_1, R_2, \ldots$ be a sequence of sets $R_k \subset X^{n(k)}$ of the first category. Then almost every set $F \in C(X)$ is independent in each set R_k, $k = 1, 2, \ldots$* (see [9] and [10]).

Because the family of these sets F is $\bigcap_k J(R_k)$.

Corollary 3. *If the complete space X has no isolated points, then — under the assumption of the Theorem — almost every Cantor subset of X is independent in R.*

Similar remarks apply to Corollaries 1 and 2.

This follows by virtue of Proposition 2 of § 4.

Remarks. Let C denote the Cantor discontinuum and X a complete space. As usually, X^C denotes the space of continuous mappings $f\colon C \to X$, with the topology of uniform convergence.

There is a natural mapping Γ of X^C onto $C(X)$. Namely

$$\Gamma(f) = f(C).$$

By a theorem of Michael, the mapping Γ is continuous and open (see [8]).

Now, for $R \subset X^n$, denote

$$I(R) = \{f \in X^C:\ f(C) \in J(R)\}.$$

Obviously $I(R) = \Gamma^{-1}[J(R)]$, and using Michael's Theorem one deduces from our Main Theorem the following Corollary (*).

Corollary 4. *If $R \subset X^n$ is closed and nowhere dense, then $I(R)$ is a G_δ-set dense in X^C.*

Because, an inverse image of a G_δ-set under a continuous mapping is G_δ, and the inverse of a dense set under an open mapping is dense (see e.g. [5], vol. I, p. 117 (7)).

One sees easily that, similarly, the Corollaries 1–3 can also be formulated in terms of the space X^C instead of $C(X)$.

§ 7. Applications.

1. *Let X be the space of reals and let xRy mean that the difference $x-y$ is rational.*

Here R is an F_σ-set of the first category. For, let $r_1, r_2, \ldots$ be the sequence of all rationals. Then $R = R_1 \cup R_2 \cup \ldots$, where xR_iy means that $x - y = r_i$. Obviously R_i is a closed nowhere dense set, and hence R a set of the first category.

According to the Corollary 3 of § 6, there exists a Cantor set F such that, if $x, y \in F$ and $x \neq y$, the difference $x-y$ is irrational (in fact, almost all compact sets have this property).

Call a set V of reals a Vitali (non-measurable) set if it contains exactly one point from each member of the quotient-space X/R.

According to the Corollary 3 of § 6, *there is a Vitali set containing a Cantor set.*

2. *Let X be a metric indecomposable continuum (i.e. X cannot be represented as the union of two proper subcontinua). Let xRy mean that there is a proper subcontinuum of X containing x and y, i.e.*

$$(xRy) \equiv \exists F \in C(X):\ \{(F \text{ is a continuum})\ (F \neq X)\ (x \in F)\ (y \in F)\}.$$

(*) Which is, in fact, the "Main Theorem" of Mycielski's papers [10] and [11]. This implication was pointed out to the author by Prof. J. Mycielski.

The inverse implication can be deduced — as noticed by Mr. R. Pol — from the Hausdorff theorem on the invariance of the topological completeness under open mappings (Fund. Math. 23, p. 279).

Thus, *R is the projection of the set*

$$\{\langle F, x, y\rangle: (F \text{ is a continuum})\ (F \neq X)\ (x \in F)\ (y \in F)\}$$

parallel to the $C(X)$-axis on the product space $X \times X$.

Since the spaces $C(X)$ and the space of all subcontinua of X are compact, the projected set is an F_σ-set (in $C(X) \times X \times X$) and so is R (in $X \times X$) (comp. [5], vol. II, p. 14).

Moreover, R is a boundary set, since in every neighbourhood of two given points x_0 and y_0 of X, there are two points x and y which lie in different composants of X (i.e. that x *non*-$R\,y$).

It follows from Corollary 1 of § 6 that *there exists a Cantor set $F \subset X$ such that no two of its points belong to the same composant of X* (Theorem of Mazurkiewicz [6], see also [1]). In fact, almost every compact subset of X is a Cantor set with the above property.

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Model-completeness for sheaves of structures

by

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Abstract. By combining some sheaf-theoretic ideas of Comer, with ideas from the Lipshitz-Saracino proof that the theory of commutative rings has a model-companion, we prove a general theorem enabling one to extend metamathematical results on fields to the corresponding results for certain regular rings. As an application, the model-completeness of real-closed fields yields a model-completeness result for a natural class of lattice-ordered rings.

0. Introduction. In a recent paper [15], Lipshitz and Saracino found the model-companion of the theory of commutative rings without nil-potent elements. In the present paper we will give an abstract version of their proof, apparently suitable for finding the model-companion for various other theories of rings. In particular, we will find the model-companion for the theory of commutative f-rings with identity and no non-zero nilpotent elements.

The representation of rings by sections of sheaves is now well-established [4, 5, 11, 13]. In [3], Comer generalized the Feferman-Vaught results [8] to cover certain structures of sections of sheaves over Boolean spaces, and thereby proved decidability results for various classes of rings. It should be noted that for constant sheaves Comer's results are essentially contained in some publications of the Wrocław group [23, 24, 25, 26] in 1968–69.

Our main result gives a sufficient condition that, for a sheaf of models of a model-complete theory, the theory of the structure of sections should be model-complete.

Before treating the main result, we give a rapid discussion, in Section 1, of model-completeness properties of reduced products.

We are very grateful to Professors Comer, Lipshitz and Saracino for making preprints of [3] and [15] available to us. It will be seen that this paper is based on a synthesis of their ideas.

1. Model-completeness and reduced products.

1.1. It is well-known that theorems of Feferman-Vaught type can be used to give an algebraic structure to the space of complete theories