$= \omega^{\delta+a}$ where r is the final segment of s corresponding to t, we see that either $\Sigma(s) = \omega^{\delta+a}$ or $\Sigma(s) = (\omega^{\delta+a})2$.

Thus under the assumption made above, we have proved that $S(s)$ is finite; and we can now obtain the full result in the usual manner.

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Properties of the gimel function and a classification of singular cardinals

by

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Abstract. The paper gives a list of properties of the function $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\varkappa}$.

1. The continuum problem and computation of cardinal exponentiation from the function $\gimel$. The subject of our investigation is the cardinal function $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\varkappa}$. The gimel function is instrumental in cardinal arithmetic; Bukovský [1] proved that both the continuum function $2^\varkappa$ and the exponential function $\varkappa^\lambda$ are computable from the gimel function.

The book of Vopěnka and Hájek [7] gives inductive definitions of $2^\varkappa$ and $\varkappa^\lambda$ in terms of $\gimel$ and lists a few obvious properties of the function $\gimel$. In the present article we give a list of seven properties of the gimel function. The author believes that these properties describe the function $\gimel$ completely, in the sense that no other laws about $\gimel$ can be proved in set theory alone (without the assumption of large cardinals). This conjecture is based on the expectations (shared by others) that the *singular cardinal problem* (discussed later) will be solved in the generality analogous to Easton's result [2].

The situation is different if the existence of large cardinals is assumed. A recent result of Solovay [5] indicates that the presence of large cardinals has a strong influence on the behaviour of the gimel function at singular cardinals. These questions are discussed in the last section.

Throughout the paper, we use Greek letters $\varkappa, \lambda, \ldots$ to denote infinite cardinals (alephs) which are identified with initial ordinals. Ordinals are generally denoted by the letters $a, \beta, \ldots$ The *cofinality* of a limit ordinal a, denoted $\mathrm{cf}\,a$, is the least ordinal cofinal with a in the natural ordering of ordinals; $\mathrm{cf}\,a$ is always a regular cardinal. The cardinal $\varkappa^\lambda$ is the cardinality of the set ${}^\lambda\varkappa$ of all functions from λ to $\varkappa$; if $\lambda \leqslant \varkappa$ then also $\varkappa^\lambda = |\mathcal{S}_\varkappa(\lambda)|$ where $\mathcal{S}_\varkappa(\lambda)$ is the set of all subsets X of λ such that $|X| \leqslant \varkappa$.

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(Of course, $\varkappa^{\varkappa} = 2^{\varkappa} = |\mathcal{P}(\varkappa)|$, where $\mathcal{P}(\varkappa)$ is the power set of $\varkappa$). For each $\varkappa$, $\varkappa^{+}$ is the least cardinal greater than $\varkappa$.

In [1], Bukovský proves that both $2^{\varkappa}$ and $\varkappa^{\lambda}$ are computable from the function $\varkappa^{\mathrm{cf}\varkappa}$. The inductive definition of $2^{\varkappa}$ is as in [1]; the inductive definition of $\varkappa^{\lambda}$ given below is somewhat simpler than that given by Bukovský. Both are based on the following lemma:

1.1. LEMMA (Bukovský [1]).

(a) *If $\varkappa$ is a singular cardinal, then* $2^{\varkappa} = (\lim_{\xi\to\varkappa} 2^{\xi})^{\mathrm{cf}\varkappa}$.

(b) *If $\varkappa$ is a singular cardinal and* $\mathrm{cf}\varkappa \leqslant \lambda < \varkappa$ *then* $\varkappa^{\lambda} = (\lim_{\xi\to\varkappa} \xi^{\lambda})^{\mathrm{cf}\varkappa}$.

The proof of the lemma uses the fact that every function from λ to $\varkappa$ (every subset of $\varkappa$) is a limit of a $(\mathrm{cf}\varkappa)$-sequence of *bounded* functions (of *bounded* subsets). For details, cf. [1] or [7].

1.1. COMPUTATION OF THE CONTINUUM FUNCTION (Bukovský). *By induction on $\varkappa$*:

(i) *If $\varkappa$ is a regular cardinal then* $2^{\varkappa} = \gimel(\varkappa)$.

(ii) *If $\varkappa$ is a singular cardinal and if there is $\xi_0 < \varkappa$ such that $2^{\xi} = 2^{\xi_0}$ for all $\xi \geqslant \xi_0$, $\xi < \varkappa$, then $2^{\varkappa} = 2^{\xi_0}$.*

(iii) *If $\varkappa$ is a singular cardinal and for each $\xi < \varkappa$ there is $\eta > \xi$, $\eta < \varkappa$ such that $2^{\xi} < 2^{\eta}$ then* $2^{\varkappa} = \gimel(\lim_{\xi\to\varkappa} 2^{\xi})$.

For the proof, see [1] or [7].

1.3. COMPUTATION OF THE EXPONENTIAL FUNCTION. *For a fixed λ, by induction on $\varkappa$*:

(i) *If $\varkappa \leqslant \lambda$ then $\varkappa^{\lambda} = 2^{\lambda}$.*

(ii) *If there is a $\xi < \lambda$ such that $\xi^{\lambda} \geqslant \varkappa$ then $\varkappa^{\lambda} = \xi^{\lambda}$.*

(iii) *If $\xi^{\lambda} < \varkappa$ for all $\xi < \varkappa$ and $\varkappa$ is either a regular cardinal or a singular cardinal with $\mathrm{cf}\varkappa > \lambda$ then $\varkappa^{\lambda} = \varkappa$.*

(iv) *If $\xi^{\lambda} < \varkappa$ for all $\xi < \varkappa$ and $\mathrm{cf}\varkappa \leqslant \lambda < \varkappa$ then $\varkappa^{\lambda} = \gimel(\varkappa)$.*

Proof. (i) Obvious.

(ii) On the one hand, $\xi^{\lambda} \leqslant \varkappa^{\lambda}$. On the other hand, $\varkappa^{\lambda} \leqslant (\xi^{\lambda})^{\lambda} \leqslant \xi^{\lambda}$.

(iii) If $\varkappa$ is a successor cardinal, $\varkappa = \eta^{+}$, then $\varkappa^{\lambda} = \eta^{+}\cdot\eta^{\lambda}$ by Hausdorff formula. If $\varkappa$ is a limit cardinal with $\mathrm{cf}\varkappa > \lambda$ then $\varkappa^{\lambda} = \lim_{\xi\to\varkappa}\xi^{\lambda}$ by Tarski formula.

(iv) Follows from Bukovský's lemma.

One may ask whether the gimel function is definable from the continuum function. Recent results of Prikry and Silver show that it is not so, at least if one assumes the existence of large cardinals. Namely, consider a transitive model M of ZFC (Zermelo-Fraenkel set theory + Axiom of Choice) + GCH (generalized continuum hypothesis) + "There is a 2-extendable cardinal $\varkappa$". Silver in his paper [4] (not yet published) constructs an extension M_1 of M, in which all cardinals and cofinalities are preserved, $\varkappa$ remains measurable, $2^{\aleph_0} = \aleph_1$, $2^{\varkappa} = \varkappa^{++}$, and GCH holds above $\varkappa$. Using the method of Prikry [3], one can get an extension M_2 of M_1, in which all cardinals are preserved, so are all cofinalities except $\varkappa$, $\mathrm{cf}\varkappa = \omega$ and the continuum function behaves as in M_1. In particular, $2^{\varkappa} = \varkappa^{++}$ in M_2 and since $2^{\lambda} < \varkappa$ for each $\lambda < \varkappa$, it follows that $\varkappa^{\aleph_0} = \varkappa^{++}$ in M_2. Now let M_3 be an extension of M_2 obtained by adding $\varkappa^{++}$ subsets of $\aleph_1$. On the other hand, let N_2 be an extension of M obtained by a direct application of Prikry's method and let N_3 be an extension of N_2 by adding $\varkappa^{++}$ subsets of $\aleph_1$. The above constructions lead to two models, M_3 and N_3 that have the same cardinals and cofinalities, the same continuum function ($2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \varkappa^{++}$ and GCH above $\varkappa$), but $\varkappa^{\aleph_0}$ is $\varkappa^{+}$ in N_3 and $\varkappa^{++}$ in M_3.

2. Properties of the gimel function. The behaviour of the function $\varkappa^{\mathrm{cf}\varkappa}$ on regular cardinals is well known. If $\varkappa$ is regular then $\varkappa^{\mathrm{cf}\varkappa} = \varkappa^{\varkappa} = 2^{\varkappa}$ and the continuum function has the following properties:

(C1) $2^{\varkappa} > \varkappa$.

(C2) If $\varkappa \leqslant \lambda$ then $2^{\varkappa} \leqslant 2^{\lambda}$.

(C3) $\mathrm{cf}(2^{\varkappa}) > \varkappa$.

Moreover, a theorem of Easton [2] states that given any function $C(\varkappa)$ with properties (C1), (C2), (C3), one can construct a model of ZFC in which $2^{\varkappa} = C(\varkappa)$ for all *regular* cardinals.

There is however no construction known that would establish a result analogous to Easton's result, but for *all* cardinals. This open problem is known as the *singular cardinal problem.*

It is obvious that the conditions (C1)-(C3) do not sufficiently describe the continuum function on singular cardinals. For it follows from Bukovský's Theorem 1.2 (ii) that the continuum function has to satisfy at least one additional condition:

(C4) If $\varkappa$ is singular and if there is $\xi_0 < \varkappa$ such that $2^{\xi} = 2^{\xi_0}$ for all $\xi \geqslant \xi_0$, $\xi < \varkappa$, then $2^{\varkappa} = 2^{\xi_0}$.

The question we are interested in, is to describe the function $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\varkappa}$. It turns out that there are four categories of singular cardinals and the gimel function behaves differently on cardinals in each category.

DEFINITION. Let $\varkappa$ be a singular cardinal. We say that

(i) $\varkappa$ is *free* (*) if $\gimel(\xi) < \varkappa$ for all $\xi < \varkappa$.

(ii) $\varkappa$ is *properly bound* if $\varkappa \leqslant 2^{\mathrm{cf}\varkappa} = \gimel(\mathrm{cf}\varkappa)$.

(iii) $\varkappa$ is *improperly bound* if $2^{\mathrm{cf}\varkappa} < \varkappa$ and there exists $\xi < \varkappa$ with $\mathrm{cf}\xi \leqslant \mathrm{cf}\varkappa$ such that $\varkappa \leqslant \gimel(\xi)$.

(*) Or, $\varkappa$ is a *strongly limit* singular cardinal.

(iv) $\varkappa$ is *captive* if it is neither free nor bound (properly or otherwise), i.e. if $\varkappa \leqslant \gimel(\eta)$ for some $\eta < \varkappa$ of greater cofinality but $\varkappa > \gimel(\xi)$ for all $\xi < \varkappa$ such that $\mathrm{cf}\,\xi \leqslant \mathrm{cf}\,\varkappa$.

Obviously, it depends on the behavior of the gimel function on smaller cardinals, to which of the four categories a given singular cardinal belongs. If GCH holds then all singular cardinals are free. By standard Cohen technique, one can obtain examples of either properly bound or captive cardinals. To get an example of an improperly bound cardinal, however, one has to solve the singular cardinal problem first.

2.1. THEOREM. *The function* $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\,\varkappa}$ *has the following properties:*

(G1) $\gimel(\varkappa) > \varkappa$.

(G2) $\mathrm{cf}\,\gimel(\varkappa) > \mathrm{cf}\,\varkappa$.

(G3) *If $\varkappa$ is a regular cardinal then* $\gimel(\varkappa) \geqslant \gimel(\lambda)$ *for all* $\lambda < \varkappa$.

(G4) *If $\varkappa$ is a free singular cardinal then* $\mathrm{cf}\,\gimel(\varkappa) > \varkappa$.

(G5) *If $\varkappa$ is properly bound then* $\gimel(\varkappa) = \gimel(\mathrm{cf}\,\varkappa)$.

(G6) *If $\varkappa$ is improperly bound, let λ be the least cardinal such that* $\mathrm{cf}\,\lambda \leqslant \mathrm{cf}\,\varkappa$ *and* $\gimel(\lambda) \geqslant \varkappa$. *Then* $\gimel(\varkappa) = \gimel(\lambda)$.

(G7) *If $\varkappa$ is captive, let λ be a cardinal of least cofinality such that* $\gimel(\lambda) \geqslant \varkappa$. *Then* $\gimel(\varkappa) \leqslant \gimel(\lambda)$ *and* $\mathrm{cf}\,\gimel(\varkappa) \geqslant \mathrm{cf}\,\lambda$.

Proof. It follows from König's theorem that $\mathrm{cf}(\varkappa^{\mathrm{cf}\,\varkappa}) > \mathrm{cf}(\mathrm{cf}\,\varkappa) = \mathrm{cf}\,\varkappa$. Consequently, $\varkappa^{\mathrm{cf}\,\varkappa} > \varkappa$ because they have different cofinalities. This proves (1) and (2).

If $\varkappa$ is a regular cardinal then $\gimel(\varkappa) = 2^{\varkappa}$ and if $\lambda < \varkappa$ then $\gimel(\lambda) \leqslant 2^{\lambda} \leqslant 2^{\varkappa}$. If $\varkappa$ is a free singular cardinal, then by (1.2.iii), $\gimel(\varkappa) = 2^{\varkappa}$, and $\mathrm{cf}\,2^{\varkappa} > \varkappa$ by König's theorem. If $\varkappa$ is properly bound then $2^{\mathrm{cf}\,\varkappa} \leqslant \varkappa^{\mathrm{cf}\,\varkappa} \leqslant (2^{\mathrm{cf}\,\varkappa})^{\mathrm{cf}\,\varkappa} = 2^{\mathrm{cf}\,\varkappa}$ and we have (G3), (G4) and (G5).

If $\varkappa$ is improperly bound, let λ be the least cardinal such that $\lambda^{\mathrm{cf}\,\lambda} \geqslant \varkappa$. We have

$$\varkappa^{\mathrm{cf}\,\varkappa} \leqslant (\lambda^{\mathrm{cf}\,\lambda})^{\mathrm{cf}\,\varkappa} = \lambda^{\mathrm{cf}\,\varkappa} \leqslant \varkappa^{\mathrm{cf}\,\varkappa}$$

and therefore $\varkappa^{\mathrm{cf}\,\varkappa} = \lambda^{\mathrm{cf}\,\varkappa}$. Now we use the inductive computation of the exponential function (Theorem 1.3). One can see that for all η and ξ, ξ^{η} equals either 2^{η} or ξ or $\gimel(\zeta)$ for some ζ. In our case, since $\varkappa$ is bound improperly, we have $\lambda^{\mathrm{cf}\,\varkappa} \geqslant \varkappa > 2^{\mathrm{cf}\,\varkappa}$ and therefore $\lambda^{\mathrm{cf}\,\varkappa} = \gimel(\mu)$ for some $\mu \leqslant \lambda$. However, λ is the least cardinal such that $\varkappa \leqslant \gimel(\lambda)$. Therefore,

$$\gimel(\varkappa) = \varkappa^{\mathrm{cf}\,\varkappa} = \lambda^{\mathrm{cf}\,\varkappa} = \gimel(\lambda)$$

which proves (G6).

Finally, if $\varkappa$ is a captive singular cardinal, let λ be a cardinal of least cofinality such that $\gimel(\lambda) \geqslant \varkappa$. Since $\mathrm{cf}\,\lambda > \mathrm{cf}\,\varkappa$, it is easy to see that

$$\gimel(\varkappa) = \varkappa^{\mathrm{cf}\,\varkappa} \leqslant (\lambda^{\mathrm{cf}\,\lambda})^{\mathrm{cf}\,\varkappa} = \lambda^{\mathrm{cf}\,\lambda} = \gimel(\lambda)$$

proving the first part of (G7). To show that $\mathrm{cf}(\varkappa^{\mathrm{cf}\,\varkappa}) \geqslant \mathrm{cf}\,\lambda$, it suffices to show that for each regular μ such that $\mathrm{cf}\,\varkappa < \mu < \mathrm{cf}\,\lambda$, we have $\mathrm{cf}(\varkappa^{\mathrm{cf}\,\varkappa}) > \mu$. To do that, we prove that $\varkappa^{\mu} = \varkappa^{\mathrm{cf}\,\varkappa}$ for each regular μ such that $\mathrm{cf}\,\varkappa < \mu < \mathrm{cf}\,\lambda$. Then by König's theorem, $\mathrm{cf}(\varkappa^{\mathrm{cf}\,\varkappa}) = \mathrm{cf}(\varkappa^{\mu}) > \mu$.

Let μ be as above. Since $\mu > \mathrm{cf}\,\varkappa$, clause (iii) of Theorem 1.3 does not apply and hence $\varkappa^{\mu}$ is equal to either 2^{μ} or $\gimel(\eta)$ for some η such that $\mu < \eta \leqslant \varkappa$. The first alternative is impossible since μ is regular, therefore $2^{\mu} = \gimel(\mu)$, which would imply $\gimel(\mu) \geqslant \varkappa$ and that would contradict the minimality of $\mathrm{cf}\,\lambda$ because $\mu < \mathrm{cf}\,\lambda$ by the assumption. In the second alternative, η must necessarily satisfy clause (iv) of Theorem 1.3 which means that $\mathrm{cf}\,\eta \leqslant \mu$. Again, it is impossible that $\eta < \varkappa$ since then $\gimel(\eta) \geqslant \varkappa$ and $\mathrm{cf}\,\eta < \mathrm{cf}\,\lambda$, a contradiction. Hence $\varkappa^{\mu} = \gimel(\varkappa)$ and so $\mathrm{cf}(\gimel(\varkappa)) > \mu$.

Remarks. It seems to us unlikely that other conditions on the gimel function than those above can be found using simple cardinal arithmetic. The natural question is then whether (G1)-(G7) are the only rules provable in ZFC. In other words, the problem is, given a cardinal function $G(\varkappa)$ which satisfies (G1-(G7), to construct a model of ZFC such that $\varkappa^{\mathrm{cf}\,\varkappa} = G(\varkappa)$ for all $\varkappa$. This is of course what Easton has done for regular cardinals in [2]. In his model, $\gimel(\varkappa) = \varkappa^{+}$ for each free or captive singular cardinal (and there are no improperly bound cardinals). Therefore the problem seems to be to make $\gimel(\varkappa) > \varkappa^{+}$ for free or captive singular cardinals. A typical special case is to make $2^{\aleph_\omega} \geqslant \aleph_{\omega+2}$ together with $2^{\aleph_n} < \aleph_\omega$ for all n. The results of Prikry and Silver mentioned earlier give a partial solution of the singular cardinal problem but only under the assumption of large cardinals.

Let us also remark that if (G1)-(G7) are the only rules for the gimel function then (C1)-(C4) are the only rules for the continuum function. For, given a function C satisfying (C1)-(C4), one can define a function G which satisfies (G1)-(G7) and such that C is the function computed from G by Theorem 1.2. (G is defined inductively, taking the only possible value if $\varkappa$ is not captive. If $\varkappa$ is captive and a limit of the function C, $\varkappa = \lim_{\xi \to \eta} C(\xi)$ for some singular η, then $G(\varkappa) = C(\eta)$; otherwise, $G(\varkappa) = \varkappa^{+}$.)

3. Effect of large cardinals on the behavior of the gimel function. It has been known for a long time that existence of large cardinals influences the behavior of the continuum function. The first important result to this effect was discovered by Dana Scott:

If $\varkappa$ is a measurable cardinal and $2^{\xi} = \xi^{+}$ *for all* $\xi < \varkappa$ *then* $2^{\varkappa} = \varkappa^{+}$.

Here we shall briefly discuss the impact of large cardinals on the behavior of the gimel function on singular cardinals.

Recently [5] (not yet published), Solovay discovered the following remarkable theorem:

If $\varkappa$ is a strongly compact cardinal and $\lambda > \varkappa$ is a free or captive singular cardinal then $\lambda^{\mathrm{cf}\lambda} = \lambda^+$.

Consequently, the gimel function for cardinals $>\varkappa$ is computable from the continuum function (note also that no singular cardinals above $\varkappa$ are improperly bound).

Even under the assumption of only measurable cardinals, we have to consider additional rules for the gimmel function on singular cardinals. For the rest of the paper, let $\varkappa$ denote a measurable cardinal.

We start with an analog of Scott's theorem:

3.1. PROPOSITION. *Let $\lambda > \varkappa$ be a cardinal of cofinality $\varkappa$ and assume that $2^\xi = \xi^+$ for all $\xi < \lambda$. Then $2^\lambda = \lambda^+$.*

Proof. Let M be an ultrapower by a normal ultrafilter and let $i: V \to M$ be the corresponding elementary embedding. Let $\lambda = [f]$, i.e. let f be a function representing λ in M. Since $\mathrm{cf}\lambda = \varkappa$, $f(a) < \lambda$ and $\mathrm{cf}(f(a)) = a$ a.e. Since $f(a)^a = a^+$ for all $a < \varkappa$, we have $M \models (\lambda^\varkappa = \lambda^+)$. Since M contains all $\varkappa$-sequences of ordinals, it follows that $\lambda^\varkappa = \lambda^+$. Hence $2^\lambda = \lambda^\varkappa = \lambda^+$.

The proof gives us actually a somewhat stronger result:

If $\lambda > \varkappa$, $\mathrm{cf}\lambda = \varkappa$ and if $\gimel(\xi) = \xi^+$ for all singular $\xi < \lambda$ of cofinality $< \varkappa$ then $\gimel(\lambda) = \lambda^+$.

Even if we do not have G.C.H. below a λ of cofinality $\varkappa$, it is often possible to get a bound on $\gimel(\lambda)$. In [6], Vopěnka announced the following theorem:

If $\aleph_\xi > \varkappa$, $\mathrm{cf}(\aleph_\xi) = \varkappa$ and if $2^{\aleph_\beta} < \aleph_\xi$ for all $\beta < \xi$ then $2^{\aleph_\xi} < \aleph_\delta$, where $\delta = (|\xi|^\varkappa)^+$.

Hence, if $\xi < \aleph_\xi$, then $2^{\aleph_\xi} < \aleph_{\aleph_\xi}$. (If $\aleph_\xi = \xi$, the theorem does not give any information.)

We will present a generalization of this result. Let $\lambda > \varkappa$ be a singular cardinal of cofinality $\varkappa$. First we observe that λ does not have to be a free singular cardinal to get an estimate for $\gimel(\lambda)$; if λ is captive then we can get the same estimate. Since for bound singular cardinals, the value $\gimel(\lambda)$ is determined anyway, it would be nice to have an upper bound for $\gimel(\lambda)$ for all free or captive cardinals of cofinality $\varkappa$. This, however, we were unable to do, although an estimate exists for a large number of "describable" such cardinals.

Let us say that an ordinal a is *attainable* if there is an ordinal operation $\mathcal{F}$ with the property

(∗) if $\mathfrak{M}, \mathfrak{N}$ are models of ZF and $\mathfrak{M} \subseteq \mathfrak{N}$ then $\mathcal{F}^{\mathfrak{M}}(\gamma) \leqslant \mathcal{F}^{\mathfrak{N}}(\gamma)$ for all γ,

such that $a = \mathcal{F}(\beta)$ for some $\beta < a$.

3.2. PROPOSITION. *Let $\lambda > \varkappa$ be a cardinal of cofinality $\varkappa$, either free or captive. If λ is attainable by $\mathcal{F}$, then*

$$\lambda^\varkappa < \sup\{\mathcal{F}(\gamma)\colon\ \gamma < \lambda\}.$$

Proof. Let M and i have the same meaning as in 3.1. Since λ is either free or captive, we have $\xi^\varkappa < \lambda$ for all $\xi < \lambda$. Therefore the following is true in M: if $\xi < i\lambda$ and $\eta < i\varkappa$ then $\xi^\eta < i\lambda$. Since $\mathrm{cf}\lambda = \varkappa$, we have $\lambda < i\lambda$ and hence $M \models \lambda^\varkappa < i\lambda$. Moreover, $\lambda^\varkappa \leqslant (\lambda^\varkappa)_M$ (because $({}^\varkappa\lambda)_M = {}^\varkappa\lambda$), and so $\lambda^\varkappa < i\lambda$.

Next note that if $\xi < \lambda$ then $|i\xi| \leqslant \xi^\varkappa < \lambda$ and hence $i\xi < \lambda$. If $\lambda = \mathcal{F}(a)$ for some $a < \lambda$, then $i\lambda = \mathcal{F}^M(ia)$ and we have

$$\lambda^\varkappa < i\lambda = \mathcal{F}^M(ia) \leqslant \mathcal{F}(ia) \leqslant \sup\{\mathcal{F}(\gamma)\colon\ \gamma < \lambda\}.$$

It remains to say a few words about attainable ordinals. For instance the function $\mathcal{F}(a) = \aleph_a$ satisfies (∗) and hence we can get a bound for $\gimel(\lambda)$ if λ is not a fixed point of the aleph function. More generally, if $\mathcal{F}$ is an increasing enumeration of a Π_1 class of cardinals then $\mathcal{F}$ satisfies (∗).

Now we shortly describe one way to attain more and more ordinals. Let $\mathcal{F}$ be an increasing continuous ordinal function satisfying (∗). Let $\mathcal{F}^\beta$ be defined as follows, for any ordinal β: Let C be the class of all values of $\mathcal{F}$ (a closed unbounded class), and let $\mathcal{F}^\beta$ be the increasing enumeration of C^β where $C^{\beta+1}$ is the class of all fixed points of C^β and $C^\gamma = \bigcap_{\delta<\gamma} C^\delta$ if γ is a limit ordinal. Finally, let $C^\infty = \{a\colon\ a \in C^a\}$ and let $\mathcal{F}^\infty$ be the corresponding function. Then $\mathcal{F}^\infty$ satisfies (∗) and enables us to attain more ordinals than $\mathcal{F}$.

Finally, there is an estimate of $\gimel(\lambda)$ in some cases of singular cardinals of cofinality $<\varkappa$.

3.3. PROPOSITION. *Let $\lambda > \varkappa$ be a cardinal of cofinality $\lambda_0 < \varkappa$ such that $\lambda < 2^\varkappa$. If $\gimel(\xi) = \xi^+$ for all $\xi < \varkappa$ of cofinality λ_0 then $\gimel(\lambda) = \lambda^+$.*

(The theorem remains true if $\varkappa < \lambda < 2^\varkappa$ is replaced by $\mu < \lambda < \mu^\varkappa$ where μ is a free or captive cardinal of cofinality $\varkappa$, and if $\gimel(\xi) = \xi^+$ for all $\xi < \eta$ of cofinality λ_0.)

Proof. M and i have the same meaning as before. We have $i\varkappa > 2^\varkappa$ and therefore $i\varkappa > \lambda$. It is true in M that for each $\xi < i\varkappa$ of cofinality λ_0, $\xi^{\lambda_0} = \xi^+$. Therefore $M \models (\lambda^{\lambda_0} = \lambda^+)$ and $\lambda^{\lambda_0} \leqslant (\lambda^{\lambda_0})_M = (\lambda^+)_M \leqslant \lambda^+$.

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Applications of the Baire-category method to the problem of independent sets

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Abstract. A set $F \subset X$ is said independent in $R \subset X^n$, if for every system $x_1, \ldots, x_n$ of different points of F the point $\langle x_1, \ldots, x_n\rangle$ never belongs to R. The main result states that, if X is a complete space and R is closed and nowhere dense, then the set $\boldsymbol{J}(R)$ of all compact subsets F of X independent in R is a dense G_δ in the space $\boldsymbol{C}(X)$ of all compact subsets of X. Using Baire category theorem this statement is extended to the case where R is an F_σ-set of the first category and also to the case of an infinite sequence $R_1, R_2, \ldots$, where $R_k \subset X^{n(k)}$.

The same method allows also to show the existence of Cantor sets in X (supposed dense-in-itself) independent in R (or more generally, in $R_1, R_2, \ldots$). Similar results were obtained in [10] and [11].

Applications to indecomposable continua (and others) are considered.

§ 1. Introduction.

DEFINITION. Let X be a space and R an n-ary relation in X, i.e. $R \subset X^n$. A set $F \subset X$ is said to be *independent in* R, written $F \in \boldsymbol{J}(R)$, if for every point $x = \langle x_1, \ldots, x_n\rangle \in F^n$ with distinct coordinates (i.e. $x_i \neq x_j$ for $i \neq j$), we have $x \in X^n - R$.

In particular, if R is a binary relation ($n = 2$), F is independent, if no two of its elements are in the relation R.

In many cases, it is important to know whether or not there exists an uncountable compact set $F \subset X$ independent in a given relation R.

The Main Theorem of this paper will give a possibility of proving the existence of an F independent in R (under suitable assumptions on X and R) with the use of the Baire category method; thus — avoiding individual constructions of F (ackward — in many cases).

Let us note two useful (and obvious) formulas

(1) $$\text{if} \quad R_1 \subset R_2, \quad \text{then} \quad \boldsymbol{J}(R_2) \subset \boldsymbol{J}(R_1),$$

(2) $$\boldsymbol{J}(\bigcup_k R_k) = \bigcap_k \boldsymbol{J}(R_k).$$