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A problem on series of ordinals

by

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Abstract. Since ordinal addition is not commutative, the sum of a series of ordinals will in general depend upon the order of the terms. Thus the question arises, how many different sums can we obtain from a given sequence simply by permuting the terms? This seems to be a most difficult question to answer in detail, and in this paper we concentrate on formulating conditions under which the number of different sums is finite. One of the principal techniques for obtaining results of this kind seems to be the temporary "elimination" of terms that seem likely to cause trouble: we prove a general result validating this method.

It is of course well-known that the sum of a series of ordinals is in general dependent upon the order of the terms. Thus the following problem presents itself: given a certain series of ordinals, how many different sums can we obtain by rearranging the terms? In order to make this precise, we introduce the following terminology and notation.

DEFINITION 1. Let $s = (s_\xi)_{\xi<\alpha}$ be an α-sequence of ordinals, where α is a given ordinal, and let $t = (t_\xi)_{\xi<\alpha}$ be another α-sequence of ordinals. We say that t is an *arrangement of* s if there exists a permutation $\mathfrak{p}$ of α such that $t_\xi = s_{\mathfrak{p}(\xi)}$ for $\xi < \alpha$. Clearly $\mathfrak{p}$, if it exists, is unique, and in this case we shall denote t by "$\mathfrak{p}[s]$". For any α-sequence $s = (s_\xi)$ of ordinals, we denote by "$\Sigma(s)$" the sum of the associated series: $\Sigma(s) = \sum_{\xi<\alpha} s_\xi$. Finally, given an α-sequence s, we define the ordinal set $S(s)$ by $S(s) = \{\Sigma(\mathfrak{p}[s]);\ \mathfrak{p}$ is a permutation of $\alpha\}$.

Couched in this terminology, the preceding problem becomes: "Given an α-sequence s of ordinals, what is the cardinality of $S(s)$?".

Now Sierpiński in [1] showed that if s is any ω-sequence, then $S(s)$ is finite. It can be shown, in a relatively straightforward manner, that this result of Sierpiński's characterizes ω amongst the transfinite ordinals: for any $\alpha > \omega$, there exists an α-sequence s for which $S(s)$ is infinite. This result still holds even if we demand that the s-terms be positive and pairwise distinct; indeed, it would be somewhat surprising if this were not the case. It can also be shown that if α is countable then for any α-sequence s, $S(s)$ is countable: the proof of this last result is not quite so straightforward.

Can we formulate conditions on the α-sequence s that will enable us to say whether $S(s)$ is finite or not? In the general case, this seems to be a rather complex question, and this paper is devoted to an examination of some of the simpler cases that arise. We adopt the following conventions. The small letters "r", "s", "t", ..., possibly with superscripts, will denote sequences of positive ordinals, a general term of which will be denoted by the same symbol with an ordinal subscript. Ordinals in general will be denoted by small Greek letters, possibly with subscripts, and finite ordinals will be denoted by "i", "j", "k", "m", "n". The cardinality of a set or ordinal will be denoted by "| |". Finally, german letters "$\mathfrak{p}$", "$\mathfrak{q}$", "$\mathfrak{r}$", will denote permutations of ordinals.

We note that we are concerned only with sequences of positive ordinals: if the "interested reader" so desires, he can generalize our results by permitting the appearance of zeroes: nothing startling emerges.

The following ordinal-valued function appears with monotonous regularity throughout this paper.

DEFINITION 2. For any ordinal $\alpha \neq 0$, we define $\lg(\alpha)$ to be the unique β such that $\omega^\beta \leqslant \alpha < \omega^{\beta+1}$, and we call $\lg(\alpha)$ the *primary exponent of* α.

We start with a simple result, a special case of which will be required later.

THEOREM 1. *Let s be an α-sequence, where α is transfinite and limit, and where $\lg(s_\xi) = \gamma$, $\xi < \alpha$. Then $\Sigma(s) = \omega^\gamma \alpha$, and so $|S(s)| = 1$.*

Proof. This is an easy induction on α. The result is obvious for $\alpha = \omega$. Suppose the result true for $\alpha_0 = \omega\zeta$, and let $\alpha = \omega(\zeta+1)$. Then we have $\Sigma(s) = \omega^\gamma \alpha_0 + \omega^\gamma \omega = \omega^\gamma \alpha$. Now suppose that $\alpha = \lim \beta_\psi$, where each β_ψ is a limit ordinal for which the result holds. Then $\Sigma(s) = \lim \omega^\gamma \beta_\psi = \omega^\gamma \alpha$. Since both γ and α are independent of the arrangement of s, the equality $|S(s)| = 1$ follows immediately.

COROLLARY. *Let α be a transfinite limit ordinal, and let s be an α-sequence of positive integers. Then $\Sigma(s) = \alpha$.*

Proof. In this case we have $\gamma = 0$.

A common method of proving $S(s)$ finite for some given s is the following one. We delete from s a finite number of terms that seem likely to "make trouble"; let the resulting sequence be s°. It may now be possible to prove by some direct method (usually by "slogging") that $S(s^\circ)$ is finite. If we had a theorem saying that the insertion of a finite number of terms in s° only altered $|S(s^\circ)|$ by a finite amount, we could then go back and conclude that $S(s)$ is finite.

This is the motivation for Theorem 2, the proof of which requires a lemma.

LEMMA 1. *Let $\alpha \neq 0$ be any ordinal with smallest positive remainder ϱ, and suppose that $\varrho < \alpha$. Then for any ordinals β, γ such that $\beta + \varrho = \alpha = \gamma + \varrho$, we have $\lg(\beta) = \lg(\gamma)$.*

Proof. Suppose that $\lg(\beta) < \lg(\gamma)$; then $\beta < \gamma$ and $\gamma - \beta = \gamma$. However, if $\beta + \varrho = \gamma + \varrho$ and $\beta < \gamma$, then $(\gamma - \beta) + \varrho = \varrho$. Thus $\varrho = \gamma + \varrho = \alpha$, a contradiction. Hence $\lg(\beta) \geqslant \lg(\gamma)$, and by symmetry, $\lg(\gamma) \geqslant \lg(\beta)$. Thus $\lg(\beta) = \lg(\gamma)$.

THEOREM 2. *Let s be an α-sequence, and let β be a given ordinal. For any $\xi < \alpha$, define the sequence s^ξ as follows.*

(1) *For $\zeta < \xi$ or $\xi + \omega \leqslant \zeta < \alpha$, put $s^\xi_\zeta = s_\zeta$;*

(2) *For $\zeta = \xi$, put $s^\xi_\zeta = \beta$;*

(3) *For $\zeta = \xi + n + 1$, put $s^\xi_\zeta = s_{\xi+n}$.*

Then s^ξ is either an α-sequence or an $(\alpha+1)$-sequence, and the set $T(s) = \{\Sigma(s^\xi);\ \xi < \alpha\}$ is finite.

Proof. Since the first assertion is obvious, we turn immediately to the second. Suppose this assertion to be false, and choose s so that $T(s)$ is infinite and $\Sigma(s) = \gamma$ is minimal. Let ϱ be the smallest positive remainder of γ; from the assumption that $T(s)$ is infinite, we shall show that $\varrho < \gamma$. Suppose that $\varrho = \gamma$. Now if $\lg(\beta) < \lg(\varrho)$, then for any final segment r of s^ξ, we have $\beta + \Sigma(r) = \beta + \varrho = \varrho$, from which it follows that $\Sigma(s^\xi) = \Sigma(s)$. Since $\xi < \alpha$ is arbitrary, this contradicts the assumption that $T(s)$ is infinite; hence we must have $\lg(\beta) \geqslant \lg(\varrho)$. However, if t is any proper initial segment of s, it follows easily from $\varrho = \gamma$ that $\lg(\Sigma(t)) < \lg(\varrho)$: thus if $\lg(\beta) \geqslant \lg(\varrho)$, we have $\Sigma(s^\xi) = \beta + \tau$, where τ is some remainder of γ. But the number of remainders of γ is finite, and so once again we have contradicted our assumption that $T(s)$ is infinite. Thus we must have $\varrho < \gamma$. Now, however, we can apply Lemma 1 and assert the existence of an ordinal δ with $\delta + \varrho = \gamma$ and $\lg(\delta)$ maximal. It is easily seen that there is no loss of generality in assuming that there is an initial segment t of s with $\Sigma(t) = \delta$. We now consider two cases.

(1) $\lg(\beta) \geqslant \lg(\delta)$. Choose any $\xi < \alpha$ such that the initial segment $t^\xi(\beta)$ of s^ξ determined by β contains t. Then we have either $\Sigma(s^\xi) = \beta + \tau$ in the case $\lg(\beta) > \lg(\delta)$, or else $\Sigma(s^\xi) = \omega^{\lg(\delta)} n + \beta + \tau$ in the case $\lg(\beta) = \lg(\delta)$, where $\Sigma(t^\xi(\beta)) = \omega^{\lg(\delta)} n + \sigma$ with $\lg(\sigma) < \lg(\delta)$: in each case τ is a remainder of γ that depends only upon ξ. However, since γ has only a finite number of remainders, it follows that the set $\{\Sigma(s^\xi);\ t^\xi(\beta)$ contains $t\}$ is finite. As $T(s)$ is assumed infinite, it must be the case that $T(t)$ is infinite. However, $\Sigma(t) = \delta < \gamma$, contradicting the minimality of γ.

(2) $\lg(\beta) < \lg(\delta)$. Now if $\lg(\varrho) = \lg(\delta)$, then for any remainder τ of γ, we have $\beta + \tau = \tau$, whence $\Sigma(s^\xi) = \Sigma(s)$ for each $\xi < \alpha$. Thus the as-

sumption that $T(s)$ is infinite forces us to conclude that $\lg(\varrho) < \lg(\delta)$. Let $r^{\xi}(\beta)$ be the final segment of s^{ξ} determined by β. It follows that there is a smallest ξ such that $\lg\big(\Sigma(r^{\xi}(\beta))\big) < \lg(\delta)$. Now if

$\zeta < \xi$ then, as we have just seen in the proof of $\lg(\varrho) < \lg(\delta)$, we must have $\Sigma(s^{\zeta}) = \Sigma(s)$. Thus $T\big(r^{\xi}(\beta)\big)$ must be infinite. But $\lg\big(\Sigma(r^{\xi}(\beta))\big) < \lg(\delta) \leqslant \lg(\gamma)$, and so $\Sigma\big(r^{\xi}(\beta)\big) < \gamma$, again contradicting the minimality of γ. This proves Theorem 2.

Reverting to our main problem, we consider a case in which $S(s)$ is infinite.

THEOREM 3. *Let s be an α-sequence where α is a successor ordinal. If the set $T = \{s_\xi\colon \xi < \alpha\}$ of distinct s-terms is infinite, then $S(s)$ is infinite.*

Proof. Clearly α is transfinite. For each ordinal δ, let $E(\delta)$ be the set $\{s_\xi \in T;\ \lg(s_\xi) = \delta\}$. Since T is infinite, either $E(\delta) \neq \emptyset$ for an infinite number of δ, or $E(\delta)$ is infinite for at least one δ. Consider the first case, and let $\delta_0, \delta_1, \ldots$, be those δ, arranged by magnitude, for which $E(\delta) \neq \emptyset$. Let s be initially arranged so that the following conditions are satisfied, where β is defined by $\alpha = \beta+1$.

(1) s_β is minimal in magnitude.

(2) For each ξ with $\xi+\omega < \alpha$, if $s_\xi \in E(\delta)$ and δ is not maximal amongst $\delta_0, \delta_1, \ldots$, then for some $\zeta > \xi$ and some $\gamma > \delta$, we have $s_\zeta \in E(\gamma)$.

Obviously these conditions are possible. Let s° be the β-sequence obtained from s by omitting s_β, and let μ be the largest limit ordinal $< \alpha$. Now let X be the ordinal set $\{\xi < \mu;\ \lg(s_\xi)$ is not maximal$\}$; from the assumption that $E(\delta) \neq \emptyset$ for an infinite number of δ, we deduce that X is infinite. For each $\xi \in X$, let $\mathfrak{p}_\xi$ be the transposition $\beta \leftrightarrow \xi$, considered as a permutation of α, and let $\mathfrak{p}_\xi[s]^\circ$ be the β-sequence obtained from $\mathfrak{p}_\xi[s]$ by omitting $\mathfrak{p}_\xi[s]_\beta$. Then from (1) and (2) it is easy to see that $\Sigma(\mathfrak{p}_\xi[s]^\circ) = \Sigma(s^\circ)$ for each $\xi \in X$, and that for each such ξ, $\Sigma(s^\circ) > s_\xi$. Thus as $\Sigma(\mathfrak{p}_\xi[s]) = \Sigma(\mathfrak{p}_\xi[s]^\circ) + s_\xi$, it follows from the fact that $E(\delta) \neq \emptyset$ for an infinite number of δ, that the set $\{\Sigma(\mathfrak{p}_\xi[s]);\ \xi \in X\}$ is an infinite subset of $S(s)$. Thus in this case, $S(s)$ is infinite.

Suppose now that $E(\delta)$ is infinite for at least one δ, and let Y be the set $\{\xi < \alpha;\ s_\xi \in E(\delta)$ and for some $\zeta > \xi$, $s_\zeta \in E(\delta)\}$. For $\xi \in Y$, consider the permutation $\mathfrak{p}_\xi$ defined as above: then it is easily seen that $\Sigma(\mathfrak{p}_\xi[s]^\circ) = \Sigma(s^\circ) > s_\xi$, and so, since the s_ξ for $\xi \in Y$ are pairwise distinct, it follows that the set $\{\Sigma(\mathfrak{p}_\xi[s]);\ \xi \in Y\}$ is an infinite subset of $S(s)$. Hence in this case also $S(s)$ is infinite. (We are assuming, in this second case, that the initial arrangement of s satisfies condition (1) above.) This proves our theorem.

We note in passing that by Theorem 1, the hypothesis in Theorem 3

that α be a successor ordinal is an essential one. Conditions analogous to those given by Theorem 3 in the case where α is a limit ordinal appear not to be known.

Our next result shows that in the case where the set T (defined as in Theorem 3) is finite, then we need not worry about whether α is successor or limit.

THEOREM 4. *Let s be an α-sequence such that the set $T = \{s_\xi;\ \xi < \alpha\}$ is finite. Let β be the largest limit ordinal $\leqslant \alpha$, and for any α-sequence t, let t° be the initial segment of t of type β. Then $S(s)$ is finite if and only if $S(\mathfrak{p}[s]^\circ)$ is finite for every permutation $\mathfrak{p}$ of α.*

Proof. The result is trivial if $\beta = \alpha$; thus we suppose that $\alpha = \beta+n$, $n \neq 0$, and we put $m = |T|$. For each arrangement t of s, we therefore have $\Sigma(t) = \Sigma(t^\circ)+\sigma$, where σ is an ordinal selected from an ordinal set of at most m^n elements. Thus if $S(t^\circ)$ is finite for each arrangement t of s, then $S(s)$ must be finite.

Now suppose that $S(s)$ is finite, but that $S(t^\circ)$ is infinite for some (henceforth fixed) arrangement t of s. This means that there exist an infinite number of permutations $\mathfrak{p}_i$ of β such that $\Sigma(\mathfrak{p}_i[t^\circ]) \neq \Sigma(\mathfrak{p}_j[t^\circ])$ for $i \neq j$. However, $\Sigma(\mathfrak{p}[t^\circ])+\sigma = \Sigma(\mathfrak{q}[s])$ for some permutation $\mathfrak{q}$ of α, where as before there are at most m^n choices for σ. Since $S(s)$ is finite, it follows that for one such σ, there are an infinite number of permutations $\mathfrak{p}$ amongst the $\mathfrak{p}_i$ for which $\Sigma(\mathfrak{p}[t^\circ])+\sigma = \sigma$; as we are only concerned with having an infinite number of these permutations, we can assume that every $\mathfrak{p}_i$ satisfies this equality.

Thus for each $\mathfrak{p}_i$ we must have $\lg\big(\Sigma(\mathfrak{p}_i[t^\circ])\big) < \lg(\sigma)$. But since $\sigma = s_{\xi_0} + \ldots + s_{\xi_{m-1}}$ for some s-terms s_{ξ_k}, it follows that for each s_{ξ_k} and each t°-term t_ζ, the inequality $\lg(s_{\xi_k}) > \lg(t_\zeta)$ holds. Let t_i, $i < m$, be the first m terms of t, and consider the permutation $\mathfrak{r}$ of α induced by the interchanges $t_i \longleftrightarrow s_{\xi_i}$, $i < m$. Since clearly $\lg\big(\Sigma(t^\circ)\big) \geqslant \lg(t_i)$, $i < m$, a moment's reflection shows that $S(\mathfrak{r}[t]^\circ)$ must be infinite, from which we immediately deduce, via the above inequalities $\lg\big(\Sigma(t^\circ)\big) \geqslant \lg(t_i) < \lg(s_{\xi_k})$, that $S(\mathfrak{r}[t])$ is infinite. Since of course $S(\mathfrak{r}[t]) = S(s)$, we have reached a contradiction and so have proved the theorem.

Instead of demanding that the set of distinct s-terms be finite, we can consider the more general case in which the set of distinct primary exponents is finite. Under this condition, we have the following result.

THEOREM 5. *Let s be an α-sequence, α transfinite and limit, and suppose that the set of primary exponents of the s-terms is finite; let these exponents be $\delta_0, \delta_1, \ldots, \delta_k$. For each $i \leqslant k$, define the set $D(i)$ by $D(i) = \{\xi < \alpha;\ \lg(s_\xi) = \delta_i\}$. Let ϱ be the smallest positive remainder of α, and suppose that for each $i \leqslant k$, if $\delta_i\omega \geqslant \lg(\varrho)$, then $D(i)$ is finite. Then $S(s)$ is finite.*

Proof. The idea of the proof is as follows. We let s have any initial arrangement, delete all those s_ξ for which $\lg(s_\xi)\omega \geqslant \lg(\varrho)$ — by assumption, there are only a finite number of these — and show that for the resulting sequence s°, we have $S(s^\circ)$ finite. We then apply Theorem 2 a finite number of times and conclude that $S(s)$ is finite.

Thus we may as well assume that $\delta_i\omega < \lg(\varrho)$ for each $i \leqslant k$; under this assumption we shall in fact show that for any arrangement t of s, $\Sigma(t) = \alpha$. Therefore we let s have an arbitrary but fixed arrangement. Let δ be the maximum of the δ_i; then $\delta\omega < \lg(\varrho)$. First of all we consider a particular case, namely the case where $\lg(\varrho)$ is finite. Thus $\delta = 0$, i.e. s is an α-sequence of positive integers, and so we have $\Sigma(s) = \alpha$ by the Corollary to Theorem 1.

It now suffices to consider the case of $\lg(\varrho)$ being infinite; in this case, since $\delta\omega < \lg(\varrho)$, we have $\delta+1+\lg(\varrho) = \lg(\varrho)$. Now ϱ, being the smallest positive remainder of α, is a prime component, and so $\varrho = \omega^{\lg(\varrho)}$. Thus $\alpha = \omega^{\beta_0}+ \ldots +\omega^{\beta_n}$, where $\beta_i \geqslant \lg(\varrho)$ for each $i \leqslant n$. Thus the following inequalities and equalities hold:

$$\Sigma(s) \leqslant \omega^{\delta+1}\alpha = \omega^{\delta+1+\beta_0}+ \ldots +\omega^{\delta+1+\beta_n} = \omega^{\beta_0}+ \ldots +\omega^{\beta_n} = \alpha\,.$$

However, each s-term being positive, we obviously have $\Sigma(s) \geqslant \alpha$. Thus in each case we have $\Sigma(s) = \alpha$, and so, by a finite number of applications. Of Theorem 2, we obtain the full version of Theorem 5.

Corollary. *Let s be an α-sequence, α being arbitrary, with smallest positive remainder ϱ. Suppose that the set of distinct s-terms is finite, and that the set $\{\xi < \alpha;\ \lg(s_\xi)\omega \geqslant \lg(\varrho)\}$ is also finite. Then $S(s)$ is finite.*

Proof. If α is finite, the result is obvious. If α is transfinite, use Theorem 5 and, possibly, Theorem 4.

For the remainder of this paper we consider α-sequences s in which the set of distinct primary exponents of the s-terms is finite, and where α is initial, $\alpha > \omega$. As in Theorem 5, we let $\delta_0, \ldots, \delta_k$ be the primary exponents, and define the sets $D(i)$ by $D(i) = \{\xi < \alpha;\ \lg(s_\xi) = \delta_i\}$. For at least one i we must have $|D(i)| = |\alpha|$; we let δ be the maximum of those δ_i for which $|D(i)| = |\alpha|$.

Theorem 6. *Suppose that for some i with $|\omega| \leqslant |D(i)| < |\alpha|$, there exists an ordinal β such that $|\beta| = |D(i)|$ and $\delta_i+\lg(\beta) \geqslant \delta+\alpha$. Then $S(s)$ is infinite.*

Proof. Let γ be any fixed δ_i satisfying the above hypothesis, with the ordinal β being defined as above. Let n be any fixed positive integer, and consider any α-sequence s^n satisfying the following conditions.

(1) Let D be the union of all those $D(i)$ for which $\delta_i \neq \gamma$, and $|D(i)| < |\alpha|$, and let μ be the initial ordinal for which $|\mu| = |D|$; then $\mu < \alpha$. Let the initial segment t^n of s^n, of type μ, contain precisely those s-terms s_ξ (in an arbitrary but fixed order) for which $\xi \in D$. Since α is initial, the final segment r^n of s^n corresponding to t^n has type α.

(2) Let the initial segment u^n of r^n, of type βn, contain precisely those s-terms s_ξ for which $\lg(s_\xi) = \gamma$: this is possible because $|\beta n| = |D(i)|$ with i defined by $\delta_i = \gamma$. Let these terms have an arbitrary but fixed order. Since α is initial, the final segment v^n of r^n corresponding to u^n has type α.

(3) Let v^n contain, in an arbitrary but fixed order, those s-terms s_ξ such that for some i, $\lg(s_\xi) = \delta_i$ and $|D(i)| = |\alpha|$.

It is easily seen that s^n is an arrangement of s, Now $\Sigma(s^n) = \Sigma(t^n)+ +\Sigma(u^n)+\Sigma(v^n)$, and $\Sigma(t^n)$, $\Sigma(v^n)$ are clearly independent of n: we let $\Sigma(t^n) = \sigma$, and a simple calculation shows that $\Sigma(v^n) = \omega^{\delta+\alpha}$. Since we can always choose β to be a limit ordinal, another easy calculation shows that $\Sigma(u^n) = \omega^{\gamma}\beta n$. Thus $\Sigma(s^n) = \sigma+\omega^{\gamma}\beta n+\omega^{\delta+\alpha}$. However, $\lg(\omega^{\delta+\alpha}) = \delta+\alpha \leqslant \gamma+\lg(\beta) = \lg(\omega^{\gamma}\beta n)$, from which it follows easily that $\Sigma(s^n) \neq \Sigma(s^m)$ for $n \neq m$. Thus $S(s)$ is infinite.

Before proceeding with the alternative cases, we give a simple lemma, which will be required in the proof of our next result.

Lemma 2. *Let $\varkappa$, σ, μ, ν be ordinals, with $\varkappa$ initial and $\sigma < \varkappa$. If $\mu+\sigma \geqslant \nu+\varkappa$, then $\mu \geqslant \nu+\varkappa$.*

Proof. If $\mu+\sigma \geqslant \nu+\varkappa$, then $\lg(\mu+\sigma) \geqslant \lg(\nu+\varkappa)$, and clearly $\lg(\mu+\sigma) = \max\{\lg(\mu), \lg(\sigma)\}$. Since $\varkappa$ is initial and $\sigma < \varkappa$ we have $\lg(\sigma) < \lg(\varkappa) \leqslant \lg(\nu+\varkappa)$, whence it follows that $\lg(\mu) \geqslant \lg(\nu+\varkappa)$. Clearly then, $\mu \geqslant \varkappa$, and so $\lg(\sigma) < \lg(\mu)$. From this it is clear that $\mu \geqslant \nu+\varkappa$.

Theorem 7. *Suppose that for each i with $|D(i)| < |\alpha|$, if there exists an ordinal β such that $|\beta| = |D(i)|$ and $\delta_i+\lg(\beta) \geqslant \delta+\alpha$, then $D(i)$ is finite. Then $S(s)$ is finite.*

Proof. We use the same method that we used in the proof of Theorem 5, namely we delete from s all terms s_ξ for which $\lg(s_\xi) = \delta_i$ and and $D(i)$ is finite. If we can show of the resulting sequence s° that $S(s^\circ)$ is finite, we can then apply Theorem 2 a finite number of times and conclude that $S(s)$ is finite. Thus we may as well assume that for no i with $|D(i)| < |\alpha|$ is there an ordinal β such that $|\beta| = |D(i)|$ and $\delta_i+\lg(\beta) \geqslant \delta+\alpha$. Let s have any arrangement, and let t be the smallest initial segment of s that contains each s_ξ for which $\lg(s_\xi) = \delta_i$ and $|D(i)| < |\alpha|$. Let β be the type of t; then $\beta < \alpha$. Now let γ be the maximum of those δ_i for which $|D(i)| < |\alpha|$; since we must have $|\gamma| = |D(i)|$ for some such i, it follows from Lemma 2 that $\gamma+\lg(\beta) < \delta+\alpha$. Now put $\sigma = \max\{\gamma, \delta\}$; then $\sigma+\lg(\beta) \leqslant \delta+\alpha$. But $\Sigma(t) \leqslant \omega^{\sigma+\lg(\beta)}$, and so, since clearly $\Sigma(r)$

$= \omega^{\delta+a}$ where r is the final segment of s corresponding to t, we see that either $\Sigma(s) = \omega^{\delta+a}$ or $\Sigma(s) = (\omega^{\delta+a})2$.

Thus under the assumption made above, we have proved that $S(s)$ is finite; and we can now obtain the full result in the usual manner.

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Properties of the gimel function and a classification of singular cardinals

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Abstract. The paper gives a list of properties of the function $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\varkappa}$.

1. The continuum problem and computation of cardinal exponentiation from the function $\gimel$. The subject of our investigation is the cardinal function $\gimel(\varkappa) = \varkappa^{\mathrm{cf}\varkappa}$. The gimel function is instrumental in cardinal arithmetic; Bukovský [1] proved that both the continuum function $2^\varkappa$ and the exponential function $\varkappa^\lambda$ are computable from the gimel function.

The book of Vopěnka and Hájek [7] gives inductive definitions of $2^\varkappa$ and $\varkappa^\lambda$ in terms of $\gimel$ and lists a few obvious properties of the function $\gimel$. In the present article we give a list of seven properties of the gimel function. The author believes that these properties describe the function $\gimel$ completely, in the sense that no other laws about $\gimel$ can be proved in set theory alone (without the assumption of large cardinals). This conjecture is based on the expectations (shared by others) that the *singular cardinal problem* (discussed later) will be solved in the generality analogous to Easton's result [2].

The situation is different if the existence of large cardinals is assumed. A recent result of Solovay [5] indicates that the presence of large cardinals has a strong influence on the behaviour of the gimel function at singular cardinals. These questions are discussed in the last section.

Throughout the paper, we use Greek letters $\varkappa, \lambda, \ldots$ to denote infinite cardinals (alephs) which are identified with initial ordinals. Ordinals are generally denoted by the letters $a, \beta, \ldots$ The *cofinality* of a limit ordinal a, denoted $\mathrm{cf}\,a$, is the least ordinal cofinal with a in the natural ordering of ordinals; $\mathrm{cf}\,a$ is always a regular cardinal. The cardinal $\varkappa^\lambda$ is the cardinality of the set ${}^\lambda\varkappa$ of all functions from λ to $\varkappa$; if $\lambda \leqslant \varkappa$ then also $\varkappa^\lambda = |\mathcal{P}_\varkappa(\lambda)|$ where $\mathcal{P}_\varkappa(\lambda)$ is the set of all subsets X of λ such that $|X| \leqslant \varkappa$.

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