Almost every function is independent

by

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Abstract. The main result of this paper is a theorem analogous to the theorem of Menger and Nöbeling which says that if X is a d-dimensional compact metric space then the set of continuous functions $f\colon X\to R^{2d+1}$ which are one-to-one is residual. This means that almost all f's have the property that $\{(f(x), f(y))\colon (x, y)\in X^2, x\neq y\}\cap D=\emptyset$, where $D=\{(u, v)\colon u, v\in Y, u=v\}$ and $Y=R^{2d+1}$. In our theorem D becomes an arbitrary n-ary relation in Y^n which is of the first category (i.e., meagre) Y is an arbitrary complete metric space but X is only the Cantor set. We give some applications of our theorem in algebra. The problem of finding a common generalization of the theorem of Menger-Nöbeling and ours is still open. A refinement and other applications of our theorem were recently obtained by Professor K. Kuratowski.

Main theorems. In this paper I prove a refinement of the main theorem of [12]. Corresponding refinements of all the applications given in [12] follow immediately (and will not be stated here).

Let $\langle M, R_n\rangle_{n<\omega}$ be a relational structure, i.e., M is a non-empty set and $R_n\subseteq M^{r(n)}$, where $1\leqslant r(n)<\omega$ for all $n<\omega$.

For every set X and every function $f\colon X\to M$ f will be called *independent in* $\langle M, R_n\rangle_{n<\omega}$ if for every $n<\omega$ and every sequence $(x_1, \ldots, x_{r(n)})$ of *distinct* elements of X the sequence $(f(x_1), \ldots, f(x_{r(n)}))$ does not belong to R_n.

Let now M be a *complete metric* space with distance function $\varrho(\cdot, \cdot)$ and C be the Cantor discontinuum $\{0, 1\}^\omega$. We denote by M^C the space of all continuous maps $f\colon C\to M$ with the distance

$$d(f_1, f_2)=\max\{\varrho(f_1(x), f_2(x))\colon x\in C\}.$$

Thus M^C is a complete metric space (*meagre* = of the first category; *residual* = complement of meagre in the appropriate space).

Our main theorem is the following (it was announced in [14]).

THEOREM 1. *If each R_n is meagre in $M^{r(n)}$ then the set of all $f\in M^C$ which are independent in $\langle M, R_n\rangle_{n<\omega}$ is residual.*

Remark. If M is dense in itself then the relation $=$ is meagre in M^2. Hence, in this case, the set of f's which are one-to-one is residual.

Proof. For every $a \in \{0,1\}^m$ we put

$$C(a) = \{x \in C\colon (x_0, \ldots, x_{m-1}) = a\}.$$

Let G be an open dense subset of M^r $(1 \leqslant r < \omega)$ and V be an open non-empty subset of M^C. Then for every $k < \omega$ there exists an $m(k) < \omega$ and open non-empty sets $V(a)$ in M for all $a \in \{0,1\}^{m(k)}$, such that

(1) $$m(k) \geqslant k,$$

(2) if $f \in M^C$ and $f[C(a)] \subseteq V(a)$ for all $a \in \{0,1\}^{m(k)}$ then $f \in V$,

(3) $V(a_1) \times \ldots \times V(a_r) \subseteq G$ for every sequence $a_1, \ldots, a_r$ of distinct elements of $\{0,1\}^{m(k)}$.

The construction of $m(k)$ and of the sets $V(a)$ is the following. First we find $m(k)$ and $W(a)$'s satisfying (1) and (2). Then since (3) constitutes finitely many, namely $r!\binom{2^{m(k)}}{r}$, additional conditions we can satisfy them one by one taking appropriate subsets of the $W(a)$'s. And so the $V(a)$'s are produced.

Let $U(k, G, V)$ be the set of all f's satisfying the assumption of (2). Clearly $U(k, G, V)$ is open in M^C and included in V. Let now

$$U(k, G) = \bigcup \{U(k, G, V)\colon \emptyset \neq V \subseteq M^C,\ V \text{ open}\}.$$

Hence $U(k, G)$ is open and dense in M^C and for every $f \in U(k, G)$ there exists an $m(f) \geqslant k$ such that

(4) $f[C(a_1)] \times \ldots \times f[C(a_r)] \subset G$ for every sequence $a_1, \ldots, a_r$ of distinct elements of $\{0,1\}^{m(f)}$.

Let now

$$U(G) = \bigcap \{U(k, G)\colon k < \omega\}.$$

Hence

(5) $$U(G) \text{ is residual in } M^C,$$

and by (4) for every $f \in U(G)$ and every sequence $x_1, \ldots, x_r$ of distinct elements of C we have

(6) $$\big(f(x_1), \ldots, f(x_r)\big) \in G.$$

By the supposition of the theorem each R_n is meagre in $M^{r(n)}$. Hence the complement of R_n includes an intersection $\bigcap \{G_{n,i}\colon i < \omega\}$, where each $G_{n,i}$ is open and dense in $M^{r(n)}$. By (6) each $f \in \bigcap \{U(G_{n,i})\colon n, i < \omega\}$ is independent in $\langle M, R_n\rangle_{n<\omega}$ and by (5) this last intersection is residual in M^C. Q.E.D.

Our next theorem is an easy refinement of Theorem 1. Let $(M^C)^\omega$ be the direct product of ω copies of M^C with the usual product topology. Thus $(M^C)^\omega$ is still metrisable and an absolute G_δ. The points of $(M^C)^\omega$ can be interpreted as maps of $C \times \omega$ into M. Thus it makes sense to say that an $f \in (M^C)^\omega$ is or is not independent in $\langle M, R_n\rangle_{n<\omega}$.

THEOREM 2. *If each R_n is meagre in $M^{r(n)}$ then the set of all $f \in (M^C)^\omega$ which are independent in $\langle M, R_n\rangle_{n<\omega}$ is residual in $(M^C)^\omega$.*

Proof. Let S_k for $k < \omega$ be the set of all $f \in (M^C)^\omega$ such that f restricted to $C \times \{0, 1, \ldots, k\}$ is independent in $\langle M, R_n\rangle_{n<\omega}$. It follows from Theorem 1 that S_k is residual in $(M^C)^\omega$. It is clear that the set of all $f \in (M^C)^\omega$ which are independent in $\langle M, R_n\rangle_{n<\omega}$ equals $\bigcap \{S_k\colon k < \omega\}$. Hence Theorem 2 follows.

Remarks. It is easy to check that if M has a countable basis of open sets then the set of all $f \in (M^C)^\omega$ such that the range of f is everywhere dense in M is residual in $(M^C)^\omega$. And again, if M has no isolated points, then the set of all $f \in (M^C)^\omega$ which are one-to-one is residual.

AN OPEN PROBLEM. Let us consider the simplest application of Theorem 1: If $R \subseteq [0,1]^2$ is meagre then the set of all continuous functions $f\colon C \to [0,1]$ such that

(7) $$\big(f(x), f(y)\big) \notin R \quad \text{for every } x, y \in C \text{ with } x \neq y$$

is residual. Now replace the supposition that R is meagre by the supposition that R is of 2-dimensional Lebesgue measure 0. It follows from the theorem of [13] that there are one-to-one continuous f's satisfying (7). Can one make precise a statement that almost all f's satisfy (7)? (A partial solution of this problem is given in [6].)

An application to topological groups. Let G be a topological group and put

$$D = \{(\alpha, \beta) \in G^2\colon \alpha \text{ and } \beta \text{ generate a subgroup everywhere dense in } G\},$$

$$F_n = \{(\alpha_1, \ldots, \alpha_n) \in G^n\colon \alpha_1, \ldots, \alpha_n \text{ are free generators of a free subgroup of } G \text{ of rank } n\}.$$

Schreier and Ulam proved [19] that if G is a metric connected compact group then D is residual in G^2. By the theorems of [10] and [11] (see also [4]) if G is a connected locally compact non-solvable group then F_n is residual in G^n. It follows that if G is a non-abelian connected compact metric group then D and all F_n are residual. Thus Theorem 2 and Remarks yield the following corollary.

COROLLARY 1. *Let G be a non-abelian connected compact metric group and S be the set of all $f \in (G^C)^\omega$ satisfying the following conditions*

(i) *f is one-to-one,*

(ii) *for any* $a, b \in C \times \omega$, *with* $a \neq b$, *the subgroup of* G *generated by* $f(a)$ *and* $f(b)$ *is dense in* G,

(iii) *the range of* f *is a set of free generators of a free subgroup of* G *and is everywhere dense in* G.

Then S *is residual in* $(G^C)^\omega$.

PROBLEMS. Can one prove the same for all non-solvable connected locally compact metric groups? (M. Kuranishi has proved ([10], Lemma 3) that if G is a perfect (i.e., the commutator of G equals G) connected Lie group then D is open in G^2.)

It would be interesting to refine clause (ii) of Corollary 1 in the following way

(ii*) every non-abelian subgroup of the group generated by the range of f is everywhere dense in G.

But I do not know how to prove this except in the special case when G is the group of rotations of $\boldsymbol{R}^3$ around the origin (since in this case every free non-abelian subgroup of G is everywhere dense in G).

Let H_n, $n = 1, 2, \ldots$ be the group of orientation preserving homeomorphisms of the sphere $S^n_\pm = \{(x_0, \ldots, x_n) \in \boldsymbol{R}^{n+1}\colon x_0^2 + \ldots + x_n^2 = 1\}$ with the metric

$$d(h_1, h_2) = \max\{\|h_1(x) - h_2(x)\| + \|h_1^{-1}(x) - h_2^{-1}(x)\|\colon x \in S^n\}.$$

Clearly H_n is a complete metric group (it was recently proved [8], [9] that H_n is arcwise connected for all n except possibly 4, and locally connected for all n [3].)

We can easily prove that if w is a non-trivial group word in the variables $x_1, \ldots, x_r$ then the relation $\{x \in H_n^r\colon w(x) = e\}$ is meagre in H_n^r (in fact it is closed and its complement is dense, see [5], proof of Lemma 9).

COROLLARY 2. *Corollary* 1 *with clause* (ii) *deleted is valid for all the groups* H_n.

(Schreier and Ulam [20] found a subgroup of H_n dense in H_n and generated by 5 elements. But I do not know if full Corollary 1 (or perhaps with 2 in clause (ii) replaced by 5 is valid for H_n.)

An application in general algebra. Let $S_0, S_1, \ldots$ be discrete spaces such that for every $k < \omega$ there is an $m < \omega$ with $m \geqslant k$ and $\mathrm{card}(S_m) \geqslant k$. Let $M = \boldsymbol{P}\langle S_m\colon m < \omega\rangle$ be the direct product of those spaces. Thus M has a complete metrisation. We put for every $n < \omega$

$$R_n = \{(x_1, \ldots, x_n) \in M^n\colon \mathrm{card}\{x_1(m), \ldots, x_n(m)\} < n \text{ for every } m < \omega\}.$$

It is easy to see that all R_n are closed and nowhere dense in M^n. Hence by Theorem 1 we get the following proposition.

PROPOSITION. *The set of all* $f \in M^C$ *such that for every distinct elements* $x_1, \ldots, x_n \in C$ *there exists an* $m < \omega$ *such that* $\mathrm{card}\{f(x_1)(m), \ldots, f(x_n)(m)\} = n$ *is residual.*

Let now $F_0, F_1, \ldots$ be free algebras in a variety (i.e., equational class) V such that S_m is the set of free generators of F_m and let there be for every $k < \omega$ and $m < \omega$ such that $m \geqslant k$ and $\mathrm{card}(S_m) \geqslant k$. Let again $M = \boldsymbol{P}\langle S_m\colon m < \omega\rangle$ each S_m having a discrete topology. By Proposition we get the following corollary.

COROLLARY 3. *The set of all* $f \in M^C$ *such that the range of* f *is a set of free generators of a subalgebra of* $\boldsymbol{P}\langle F_m\colon m < \omega\rangle$ *free in* V *is residual.*

Let S be the group of all order preserving permutations of the set of rational numbers. It is easy to see (see again [5], proof of Lemma 9) that S has free subgroups of rank two. Such subgroups have free subgroups of rank $\aleph_0$. Also the full direct power S^ω is isomorphic to a subgroup of S. Hence, by Corollary 3, S *has free subgroups of power* $2^{\aleph_0}$ (1).

For other possibilities of applications see [12].

Bibliographical note. I wish to complete here the bibliography collected in [12]. [12] and the present paper is a continuation of the work which began with Hausdorff's lemma ([7]) that the free product $Z_2 * Z_3$, where Z_n is the n-element cyclic group, is imbeddable in the group of rotations of $\boldsymbol{R}^3$ around the origin (which he used in his proof of the non-existence of finitely additive universal invariant measures in $\boldsymbol{R}^3$), and with the paper of von Neumann [15]. Later papers of related character were [0], [1], [2], [4], [10], [12], [13], [16], [17], [18], [19], and [20] (see also [21] and the bibliography collected there and in [12]). When we wrote [2] we were not aware of the existence of [16] and the overlap is considerable.

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A problem on series of ordinals

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Abstract. Since ordinal addition is not commutative, the sum of a series of ordinals will in general depend upon the order of the terms. Thus the question arises, how many different sums can we obtain from a given sequence simply by permuting the terms? This seems to be a most difficult question to answer in detail, and in this paper we concentrate on formulating conditions under which the number of different sums is finite. One of the principal techniques for obtaining results of this kind seems to be the temporary "elimination" of terms that seem likely to cause trouble: we prove a general result validating this method.

It is of course well-known that the sum of a series of ordinals is in general dependent upon the order of the terms. Thus the following problem presents itself: given a certain series of ordinals, how many different sums can we obtain by rearranging the terms? In order to make this precise, we introduce the following terminology and notation.

DEFINITION 1. Let $s = (s_\xi)_{\xi<\alpha}$ be an α-sequence of ordinals, where α is a given ordinal, and let $t = (t_\xi)_{\xi<\alpha}$ be another α-sequence of ordinals. We say that t is an *arrangement of* s if there exists a permutation $\mathfrak{p}$ of α such that $t_\xi = s_{\mathfrak{p}(\xi)}$ for $\xi < \alpha$. Clearly $\mathfrak{p}$, if it exists, is unique, and in this case we shall denote t by "$\mathfrak{p}[s]$". For any α-sequence $s = (s_\xi)$ of ordinals, we denote by "$\Sigma(s)$" the sum of the associated series: $\Sigma(s) = \sum_{\xi<\alpha} s_\xi$. Finally, given an α-sequence s, we define the ordinal set $S(s)$ by $S(s) = \{\Sigma(\mathfrak{p}[s]);\ \mathfrak{p}$ is a permutation of $\alpha\}$.

Couched in this terminology, the preceding problem becomes: "Given an α-sequence s of ordinals, what is the cardinality of $S(s)$?".

Now Sierpiński in [1] showed that if s is any ω-sequence, then $S(s)$ is finite. It can be shown, in a relatively straightforward manner, that this result of Sierpiński's characterizes ω amongst the transfinite ordinals: for any $\alpha > \omega$, there exists an α-sequence s for which $S(s)$ is infinite. This result still holds even if we demand that the s-terms be positive and pairwise distinct; indeed, it would be somewhat surprising if this