[19] Н. М. Суворов, Т. П. Уфнаровская, *Пример бесконечной квазигруппы, допускающей только дискретную топологию*, в сб. „Исследования по теории квазигрупп", Кишинев 1973.

[20] А. Д. Тайманов, *Пример полугруппы, допускающей только дискретную топологию* Алгебра и Логика 12 (1) (1973), стр. 114–116.

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Quasi-inverses of morphisms

by

Roman Sikorski (Warszawa)

Abstract. A morphism A is said to be *quasi-invertible*, if there exists a quasi-inverse of A, i.e. a morphism B such that $ABA = A$. A characteristic factorization for quasi-invertible morphisms is proved. A necessary and sufficient condition is given for the composition B_1B_2 of quasi-inverses B_1, B_2 of morphisms A_1, A_2 to be a quasi-inverse of the composition A_2A_1. A characteristic factorization for A_2A_1 is formulated. A general theorem is proved on the existence of quasi-invertible morphisms A_1, A_2 such that A_2A_1 is not quasi-invertible.

The notion of quasi-inverse which is the subject of this paper is a generalization of the notion of inverse, left-hand inverse and right-hand inverse of morphisms. Theorem 2.5 (and its modification 3.1) gives a simple characteristic factorization of quasi-invertible morphisms. Theorem 2.6 yields a simple necessary and sufficient condition for the composition of quasi-inverses of morphisms to be a quasi-inverse of the composition of the morphisms. If it is satisfied, then theorem 2.7 (and its modification 3.2) gives a characteristic factorization for the composition.

In many categories the composition of quasi-invertible morphisms is not always quasi-invertible. Two examples of this kind, which were communicated to me by A. Białynicki-Birula (in the case of the category of abelian groups) and P. Wojtaszczyk (in the case of the category of Banach spaces), suggested me a general theorem 4.4 which produces easily examples of this kind for many concrete categories. On the other hand, there are also non-trivial concrete categories such that every morphism is quasi-invertible.

The notion of quasi-inverse is closely related to that of projection. It is often supposed in this paper that every projection in the category under consideration can be split in a way explained in the first section.

§ 1. Splits of projections. Let $\mathfrak{C}$ be a fixed *category*. The letter O (with indices) will always denote *objects* in $\mathfrak{C}$. For any objects O_1, O_2 in $\mathfrak{C}$ the symbol $\mathrm{Hom}(O_1, O_2)$ will stand for the set of all *morphisms* from O_1 into O_2. If $A \in \mathrm{Hom}(O_1, O_2)$, we say that O_1 is the *domain* of A, and O_2 is the *co-domain* of A. The domain and the co-domain of a morphism A

are uniquely determined by A and will be denoted by $\mathrm{d}A$ and $\mathrm{cd}A$ respectively. If $A \in \mathrm{Hom}(O_1, O_2)$ and $B \in \mathrm{Hom}(O_2, O_3)$, then $BA \in \mathrm{Hom}(O_1, O_3)$ is the *composition* of A and B. The composition BA exists if and only if $\mathrm{cd}A = \mathrm{d}B$ and it is associative, i.e. $(CB)A = C(BA)$ for any $A \in \mathrm{Hom}(O_1, O_2)$, $B \in \mathrm{Hom}(O_2, O_3)$, $C \in \mathrm{Hom}(O_3, O_4)$.

For every object O in $\mathfrak{C}$ there exists exactly one *unit* I_O in $\mathrm{Hom}(O, O)$, i.e. such a morphism that $AI_O = A$ and $I_O B = B$ for any morphisms $A \in \mathrm{Hom}(O, O_1)$ and $B \in \mathrm{Hom}(O_2, O)$. A morphism $B \in \mathrm{Hom}(O_2, O_1)$ is said to be the *inverse* of a morphism $A \in \mathrm{Hom}(O_1, O_2)$ if $BA = I_{O_1}$ and $AB = I_{O_2}$. The inverse B of A, if it exists, is unique and is denoted by A^{-1}. A morphism A is said to be *invertible*, if A^{-1} exists.

A morphism P is said to be a *projection* if $PP = P$. Since the composition PP exists, we have necessarily $\mathrm{d}P = \mathrm{cd}P$.

Any pair $\boldsymbol{P} = (\boldsymbol{P}^>, \boldsymbol{P}^<)$ of morphisms is said to be a *split* of a morphism P if

$$(1) \qquad \boldsymbol{P}^>\boldsymbol{P}^< = P, \qquad \boldsymbol{P}^<\boldsymbol{P}^> = I_{|\boldsymbol{P}|},$$

where $|\boldsymbol{P}|$ is an object. This object is uniquely determined by the split $\boldsymbol{P}$, viz.,

$$|\boldsymbol{P}| = \mathrm{d}\boldsymbol{P}^> = \mathrm{cd}\boldsymbol{P}^<.$$

If a split of P exists, then P is a projection since

$$PP = (\boldsymbol{P}^>\boldsymbol{P}^<)(\boldsymbol{P}^>\boldsymbol{P}^<) = \boldsymbol{P}^>(\boldsymbol{P}^<\boldsymbol{P}^>)\boldsymbol{P}^< = \boldsymbol{P}^> I_{|\boldsymbol{P}|}\boldsymbol{P}^< = \boldsymbol{P}^>\boldsymbol{P}^< = P.$$

Therefore, in what follows, we shall discuss only splits of projections. The intuitive meaning of $\boldsymbol{P}^>, \boldsymbol{P}^<$ and $|\boldsymbol{P}|$ will be explained in § 4 (for concrete categories). It follows from (1) that

$$\mathrm{d}\boldsymbol{P}^< = \mathrm{d}P = \mathrm{cd}P = \mathrm{cd}\boldsymbol{P}^>.$$

We shall always apply the convention that if a symbol denotes a split of a projection, then the same symbol with signs $^>$, $^<$ denotes respectively the first and the second term of the split. As a rule, the letters $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S}$ will stand for splits of projections P, Q, R, S respectively.

1.1. *If $\boldsymbol{P}$ is a split of a projection P, then*

$$P\boldsymbol{P}^> = \boldsymbol{P}^>, \qquad \boldsymbol{P}^< P = \boldsymbol{P}^<.$$

Indeed,

$$P\boldsymbol{P}^> = (\boldsymbol{P}^>\boldsymbol{P}^<)\boldsymbol{P}^> = \boldsymbol{P}^>(\boldsymbol{P}^<\boldsymbol{P}^>) = \boldsymbol{P}^> I_{|\boldsymbol{P}|} = \boldsymbol{P}^>.$$

The proof of the second equation is analogous.

The next theorem states that the split, if it exists, is not unique in general. However it is unique up to isomorphisms.

1.2. *If $\boldsymbol{P}$ is a split of a projection P, A is an invertible morphism and $\mathrm{d}A = |\boldsymbol{P}|$, then the pair $\boldsymbol{P}_1 = (\boldsymbol{P}_1^>, \boldsymbol{P}_1^<)$ where*

$$(2) \qquad \boldsymbol{P}_1^> = \boldsymbol{P}^> A^{-1}, \qquad \boldsymbol{P}_1^< = A\boldsymbol{P}^<$$

is also a split of P, and $|\boldsymbol{P}_1| = \mathrm{cd}A$. Conversely, if $\boldsymbol{P} = (\boldsymbol{P}^>, \boldsymbol{P}^<)$ and $\boldsymbol{P}_1 = (\boldsymbol{P}_1^>, \boldsymbol{P}_1^<)$ are splits of a projection P, then the morphism

$$(3) \qquad A = \boldsymbol{P}_1^<\boldsymbol{P}^>$$

is invertible,

$$(4) \qquad A^{-1} = \boldsymbol{P}^<\boldsymbol{P}_1^>,$$

and equations (2) *hold.*

It follows directly from the definition that morphisms (2) satisfy the equations analogous to (1)

$$\boldsymbol{P}_1^>\boldsymbol{P}_1^< = P, \qquad \boldsymbol{P}_1^<\boldsymbol{P}_1^> = I_{\mathrm{cd}A}$$

what proves that $\boldsymbol{P}_1$ is a split of P and that $|\boldsymbol{P}_1| = \mathrm{cd}A$. Conversely, if $\boldsymbol{P}$ and $\boldsymbol{P}_1$ are splits of P, then (4) is an inverse of (3) since

$$(\boldsymbol{P}_1^<\boldsymbol{P}^>)(\boldsymbol{P}^<\boldsymbol{P}_1^>) = \boldsymbol{P}_1^<(\boldsymbol{P}^>\boldsymbol{P}^<)\boldsymbol{P}_1^> = \boldsymbol{P}_1^< P \boldsymbol{P}_1^> = \boldsymbol{P}_1^<\boldsymbol{P}_1^> = I_{|\boldsymbol{P}_1|}$$

by (1) and 1.1 (where $\boldsymbol{P}$ should be replaced by $\boldsymbol{P}_1$), and

$$(\boldsymbol{P}^<\boldsymbol{P}_1^>)(\boldsymbol{P}_1^<\boldsymbol{P}^>) = I_{|\boldsymbol{P}|}$$

by the same argument. Multiplying (3) by $\boldsymbol{P}^<$ from the right we get the second of equations (2). Multiplying (4) by $\boldsymbol{P}^>$ from the left, we get the first of equations (2).

1.3. *Let $\boldsymbol{P}$ be a split of a projection P. If C is such a morphism that $\boldsymbol{P}^> C = P$, then $C = \boldsymbol{P}^<$. If C is such a morphism that $C\boldsymbol{P}^< = P$, then $C = \boldsymbol{P}^>$.*

If $\boldsymbol{P}^> C = P$, then $C = \boldsymbol{P}^<\boldsymbol{P}^> C = \boldsymbol{P}^< P = \boldsymbol{P}^<$ by 1.1, what proves the first part of 1.3. The proof of the second part is analogous.

Theorem 1.3 states that if $(\boldsymbol{P}^>, \boldsymbol{P}^<)$ is a split of a projection P, then each of the morphisms $\boldsymbol{P}^>, \boldsymbol{P}^<$ determines uniquely the remaining one.

1.4. *If $\boldsymbol{P}$ is a split of a projection P, $\boldsymbol{Q}$ is a split of a projection Q and $\mathrm{d}Q = |\boldsymbol{P}|$, then the morphism*

$$R = \boldsymbol{P}^> Q \boldsymbol{P}^<$$

is a projection, $\mathrm{d}R = \mathrm{d}P$, *the pair* $\boldsymbol{R} = (\boldsymbol{R}^>, \boldsymbol{R}^<)$ where

$$(5) \qquad \boldsymbol{R}^> = \boldsymbol{P}^>\boldsymbol{Q}^>, \qquad \boldsymbol{R}^< = \boldsymbol{Q}^<\boldsymbol{P}^<,$$

is a split of R, and $|\boldsymbol{R}| = |\boldsymbol{Q}|$.

are uniquely determined by A and will be denoted by $\mathrm{d}A$ and $\mathrm{cd}A$ respectively. If $A \in \mathrm{Hom}(O_1, O_2)$ and $B \in \mathrm{Hom}(O_2, O_3)$, then $BA \in \mathrm{Hom}(O_1, O_3)$ is the *composition* of A and B. The composition BA exists if and only if $\mathrm{cd}A = \mathrm{d}B$ and it is associative, i.e. $(CB)A = C(BA)$ for any $A \in \mathrm{Hom}(O_1, O_2)$, $B \in \mathrm{Hom}(O_2, O_3)$, $C \in \mathrm{Hom}(O_3, O_4)$.

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is also a split of P, and $|\mathbf{P}_1| = \mathrm{cd}A$. Conversely, if $\mathbf{P} = (\mathbf{P}^>, \mathbf{P}^<)$ and $\mathbf{P}_1 = (\mathbf{P}_1^>, \mathbf{P}_1^<)$ are splits of a projection P, then the morphism

$$(3) \qquad A = \mathbf{P}_1^<\mathbf{P}^>$$

is invertible,

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and equations (2) *hold.*

It follows directly from the definition that morphisms (2) satisfy the equations analogous to (1)

$$\mathbf{P}_1^>\mathbf{P}_1^< = P, \qquad \mathbf{P}_1^<\mathbf{P}_1^> = I_{\mathrm{cd}A}$$

what proves that $\mathbf{P}_1$ is a split of P and that $|\mathbf{P}_1| = \mathrm{cd}A$. Conversely, if $\mathbf{P}$ and $\mathbf{P}_1$ are splits of P, then (4) is an inverse of (3) since

$$(\mathbf{P}_1^<\mathbf{P}^>)(\mathbf{P}^<\mathbf{P}_1^>) = \mathbf{P}_1^<(\mathbf{P}^>\mathbf{P}^<)\mathbf{P}_1^> = \mathbf{P}_1^< P\mathbf{P}_1^> = \mathbf{P}_1^<\mathbf{P}_1^> = I_{|\mathbf{P}_1|}$$

by (1) and 1.1 (where $\mathbf{P}$ should be replaced by $\mathbf{P}_1$), and

$$(\mathbf{P}^<\mathbf{P}_1^>)(\mathbf{P}_1^<\mathbf{P}^>) = I_{|\mathbf{P}|}$$

by the same argument. Multiplying (3) by $\mathbf{P}^<$ from the right we get the second of equations (2). Multiplying (4) by $\mathbf{P}^>$ from the left, we get the first of equations (2).

1.3. *Let $\mathbf{P}$ be a split of a projection P. If C is such a morphism that $\mathbf{P}^> C = P$, then $C = \mathbf{P}^<$. If C is such a morphism that $C\mathbf{P}^< = P$, then $C = \mathbf{P}^>$.*

If $\mathbf{P}^> C = P$, then $C = \mathbf{P}^<\mathbf{P}^> C = \mathbf{P}^< P = \mathbf{P}^<$ by 1.1, what proves the first part of 1.3. The proof of the second part is analogous.

Theorem 1.3 states that if $(\mathbf{P}^>, \mathbf{P}^<)$ is a split of a projection P, then each of the morphisms $\mathbf{P}^>, \mathbf{P}^<$ determines uniquely the remaining one.

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$$R = \mathbf{P}^> Q\mathbf{P}^<$$

is a projection, $\mathrm{d}R = \mathrm{d}P$, *the pair $\mathbf{R} = (\mathbf{R}^>, \mathbf{R}^<)$* where

$$(5) \qquad \mathbf{R}^> = \mathbf{P}^>\mathbf{Q}^>, \qquad \mathbf{R}^< = \mathbf{Q}^<\mathbf{P}^<,$$

is a split of R, and $|\mathbf{R}| = |\mathbf{Q}|$.

We have

$$(P^{>}Q^{>})(Q^{<}P^{<}) = P^{>}(Q^{>}Q^{<})P^{<} = P^{>}QP^{<} = R\,,$$
$$(Q^{<}P^{<})(P^{>}Q^{>}) = Q^{<}(P^{<}P^{>})Q^{>} = Q^{<}I_{|P|}Q^{>} = Q^{<}Q^{>} = I_{|Q|},$$

what proves theorem 1.4.

1.5. *For any projections P, Q, R and for any split $\boldsymbol{R}$ of R,*

(6) $$\text{if} \quad PR = R\,, \quad \text{then} \quad P\boldsymbol{R}^{>} = \boldsymbol{R}^{>},$$

(7) $$\text{if} \quad RQ = R\,, \quad \text{then} \quad \boldsymbol{R}^{<}Q = \boldsymbol{R}^{<}.$$

In fact, $P\boldsymbol{R}^{>} = P(R\boldsymbol{R}^{>}) = (PR)\boldsymbol{R}^{>} = R\boldsymbol{R}^{>} = \boldsymbol{R}^{>}$ by 1.1. The proof of (7) is similar.

In theorems 1.6 and 1.7 below we suppose that *every projection in the category $\mathfrak{C}$ under consideration has a split.*

1.6. *Let $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ be splits of projections*

$$P \in \mathrm{Hom}(O, O)\,, \quad Q \in \mathrm{Hom}(O', O') \quad \text{and} \quad R \in \mathrm{Hom}(O, O)$$

respectively and let

$$A \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{Q}|)$$

be an invertible morphism. If $PR = R$, then there exist a projection $Q_1 \in \mathrm{Hom}(O', O')$, a split $\boldsymbol{Q}_1$ of Q_1 and an invertible morphism $A_1 \in \mathrm{Hom}(|\boldsymbol{R}|, |\boldsymbol{Q}_1|)$ such that

(8) $$\boldsymbol{Q}^{>}A\boldsymbol{P}^{<}\boldsymbol{R}^{>} = \boldsymbol{Q}_1^{>}A_1\,.$$

More precisely, *if $PR = R$, then the morphism*

(9) $$S = A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1} \in \mathrm{Hom}(|\boldsymbol{Q}|, |\boldsymbol{Q}|)$$

is a projection and the morphism

(10) $$Q_1 = \boldsymbol{Q}^{>}S\boldsymbol{Q}^{<} \in \mathrm{Hom}(O', O')$$

is a projection. If $\boldsymbol{S}$ is a split of S, then $\boldsymbol{Q}_1 = (\boldsymbol{Q}^{>}\boldsymbol{S}^{>}, \boldsymbol{S}^{<}\boldsymbol{Q}^{<})$ is a split of Q_1, $|\boldsymbol{Q}_1| = |\boldsymbol{S}|$, and the morphism

(11) $$A_1 = \boldsymbol{S}^{<}A\boldsymbol{P}^{<}\boldsymbol{R}^{>} \in \mathrm{Hom}(|\boldsymbol{R}|, |\boldsymbol{Q}_1|)$$

is invertible, viz.

(12) $$A_1^{-1} = \boldsymbol{R}^{<}\boldsymbol{P}^{>}A^{-1}\boldsymbol{S}^{>} \in \mathrm{Hom}(|\boldsymbol{Q}|, |\boldsymbol{R}|)\,.$$

Moreover, equation (8) *holds.*

We have

$$SS = A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1}A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1} = A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1} = A\boldsymbol{P}^{<}RPR\boldsymbol{P}^{>}A^{-1}$$
$$= A\boldsymbol{P}^{<}RR\boldsymbol{P}^{>}A^{-1} = A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1} = S$$

which proves that S is a projection. It follows from 1.4 that Q_1 is a projection, $\boldsymbol{Q}_1 = (\boldsymbol{Q}^{>}\boldsymbol{S}^{>}, \boldsymbol{S}^{<}\boldsymbol{Q}^{<})$ is a split of Q_1 and $|\boldsymbol{Q}_1| = |\boldsymbol{S}|$. We verify that

$$(\boldsymbol{S}^{<}A\boldsymbol{P}^{<}\boldsymbol{R}^{>})(\boldsymbol{R}^{<}\boldsymbol{P}^{>}A^{-1}\boldsymbol{S}^{>}) = \boldsymbol{S}^{<}(A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1})\boldsymbol{S}^{>} = \boldsymbol{S}^{<}S\boldsymbol{S}^{>} = \boldsymbol{S}^{<}\boldsymbol{S}^{>} = I_{|\boldsymbol{S}|}$$

and

$$(\boldsymbol{R}^{<}\boldsymbol{P}^{>}A^{-1}\boldsymbol{S}^{>})(\boldsymbol{S}^{<}A\boldsymbol{P}^{<}\boldsymbol{R}^{>}) = \boldsymbol{R}^{<}\boldsymbol{P}^{>}A^{-1}SA\boldsymbol{P}^{<}\boldsymbol{R}^{>}$$
$$= \boldsymbol{R}^{<}\boldsymbol{P}^{>}A^{-1}(A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1})A\boldsymbol{P}^{<}\boldsymbol{R}^{>}$$
$$= \boldsymbol{R}^{<}\boldsymbol{P}^{>}\boldsymbol{P}^{<}R\boldsymbol{P}^{>}\boldsymbol{P}^{<}\boldsymbol{R}^{>} = \boldsymbol{R}^{<}PRP\boldsymbol{R}^{>}$$
$$= \boldsymbol{R}^{<}RP\boldsymbol{R}^{>} = \boldsymbol{R}^{<}R\boldsymbol{R}^{>} = \boldsymbol{R}^{<}\boldsymbol{R}^{>} = I_{|\boldsymbol{R}|}$$

by (6) and by 1.1. This proves that (12) is the inverse of (11). To prove (8) let us calculate that

$$\boldsymbol{Q}_1^{>}A_1 = (\boldsymbol{Q}^{>}\boldsymbol{S}^{>})(\boldsymbol{S}^{<}A\boldsymbol{P}^{<}\boldsymbol{R}^{>}) = \boldsymbol{Q}^{>}\boldsymbol{S}^{>}\boldsymbol{S}^{<}A\boldsymbol{P}^{<}\boldsymbol{R}^{>} = \boldsymbol{Q}^{>}SA\boldsymbol{P}^{<}\boldsymbol{R}^{>}$$
$$= \boldsymbol{Q}^{>}A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}A^{-1}A\boldsymbol{P}^{<}\boldsymbol{R}^{>} = \boldsymbol{Q}^{>}A\boldsymbol{P}^{<}R\boldsymbol{P}^{>}\boldsymbol{P}^{<}\boldsymbol{R}^{>}$$
$$= \boldsymbol{Q}^{>}A\boldsymbol{P}^{<}RP\boldsymbol{R}^{>} = \boldsymbol{Q}^{>}A\boldsymbol{P}^{<}R\boldsymbol{R}^{>} = \boldsymbol{Q}^{>}A\boldsymbol{P}^{<}\boldsymbol{R}^{>}$$

by (6) and 1.1.

1.7. *Let $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ be splits of projections*

$$P \in \mathrm{Hom}(O, O)\,, \quad Q \in \mathrm{Hom}(O', O')\,, \quad R \in \mathrm{Hom}(O', O')$$

respectively, and let

$$A \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{Q}|)$$

be an invertible morphism. If $RQ = R$, then there exist a projection $P_1 \in \mathrm{Hom}(O, O)$, a split $\boldsymbol{P}_1$ of P_1, and an invertible morphism $A_1 \in \mathrm{Hom}(|P_1|, |R|)$ such that

(8′) $$\boldsymbol{R}^{<}\boldsymbol{Q}^{>}A\boldsymbol{P}^{<} = A_1\boldsymbol{P}_1^{<}.$$

More precisely, *if $RQ = R$, then the morphism*

(9′) $$S = A^{-1}\boldsymbol{Q}^{<}R\boldsymbol{Q}^{>}A \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{P}|)$$

is a projection, and the morphism

(10′) $$P_1 = \boldsymbol{P}^{>}S\boldsymbol{P}^{<} \in \mathrm{Hom}(O, O)$$

is a projection. If $\boldsymbol{S}$ *is a split of* S, *then* $\boldsymbol{P}_1 = (P^{>}S^{>}, S^{<}P^{<})$ *is a split of* P_1, $|\boldsymbol{P}_1| = |\boldsymbol{S}|$, *and the morphism*

(11′) $$A_1 = \boldsymbol{R}^{<}\boldsymbol{Q}^{>}A\boldsymbol{S}^{>} \in \mathrm{Hom}(|\boldsymbol{P}_1|, |\boldsymbol{R}|)$$

is invertible, viz.

(12′) $$A_1^{-1} = \boldsymbol{S}^{<}A^{-1}\boldsymbol{Q}^{<}\boldsymbol{R}^{>} \in \mathrm{Hom}(|\boldsymbol{R}|, |\boldsymbol{P}_1|).$$

Moreover, equation (8′) *holds.*

Theorem 1.7 is dual to theorem 1.6. The proof of 1.7 is dual to that of 1.6.

§ 2. Quasi-inverses. Let

(1) $$A \in \mathrm{Hom}(O_1, O_2), \quad B \in \mathrm{Hom}(O_2, O_1)$$

be any morphisms in the category $\mathfrak{C}$. By hypothesis,

$$BA \in \mathrm{Hom}(O_1, O_1), \quad AB \in \mathrm{Hom}(O_2, O_2).$$

We recall that B is said to be a *left-hand inverse* of A if $BA = I_{O_1}$. Similarly B is said to be a *right-hand* inverse of A if $AB = I_{O_2}$. Thus B is an inverse of A if B is simultaneously a left-hand inverse of A and a right-hand inverse of A.

B is said to be a *quasi-inverse* of A if

(2) $$ABA = A.$$

B is said to be a *reciprocal quasi-inverse* of A if B is a quasi-inverse of A, and A is a quasi-inverse of B, that is, if the following equations hold

(3) $$ABA = A, \quad BAB = B.$$

2.1. *If B is a quasi-inverse of A, then BAB is a reciprocal quasi-inverse of A.*

In fact,

$$A(BAB)A = AB(ABA) = ABA = A,$$

$$(BAB)A(BAB) = B(ABA)BAB = BABAB = BAB.$$

If A has a quasi-inverse (or, equivalently, if A has a reciprocal quasi-inverse), then A is said to be *quasi-invertible*.

2.2. *If $A \in \mathrm{Hom}(O_1, O_2)$ has a left-hand (right-hand) inverse, then for every $B \in \mathrm{Hom}(O_2, O_1)$ the following conditions are equivalent:*

(i) *B is a quasi-inverse of A,*

(ii) *B is a left-hand (right-hand) inverse of A,*

(iii) *B is a reciprocal quasi-inverse of A.*

Consequently, if A is invertible, then A^{-1} is the only quasi-inverse of A and the only reciprocal quasi-inverse of A.

Suppose A has a left-hand inverse C, i.e. $CA = I_{O_1}$. If B is a quasi-inverse of A, i.e. if (2) holds, then multiplying (2) by C from the left we get $BA = I_{O_1}$. Thus B is a left-hand inverse of A.

If B is a left-hand inverse of A, i.e. if $BA = I_{O_1}$, then multiplying the equation by A from the left, or by B from the right, we get equations (3). Thus B is a reciprocal quasi-inverse.

Since every reciprocal quasi-inverse is a quasi-inverse, this proves the first part of 2.2 when a left-hand inverse of A exists. If a right-hand inverse exists, the proof is analogous. The second part of 2.2 directly follows from the first part.

In general, the notion of a quasi-inverse does not coincide with the notion of a reciprocal quasi-inverse. For instant, every projection $P \in \mathrm{Hom}(O, O)$ is quasi-invertible, viz. P is a reciprocal quasi-inverse of P. The unit I_O is also a quasi-inverse of P, but it is not a reciprocal quasi-inverse of P, except the case where $P = I_O$. By § 1 (1) and 1.1, if $\boldsymbol{P}$ is a split of a projection P, then $\boldsymbol{P}^{>}$ is a reciprocal quasi-inverse of $\boldsymbol{P}^{<}$, and conversely.

2.3. *If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are splits of projections $P \in \mathrm{Hom}(O_1, O_1)$ and $Q \in \mathrm{Hom}(O_2, O_2)$ respectively, and $A_0 \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{Q}|)$ is invertible, then the morphism*

(4) $$A = \boldsymbol{Q}^{>}A_0\boldsymbol{P}^{<} \in \mathrm{Hom}(O_1, O_2)$$

is quasi-invertible, viz. the morphism

(5) $$B = \boldsymbol{P}^{>}A_0^{-1}\boldsymbol{Q}^{<} \in \mathrm{Hom}(O_2, O_1)$$

is a reciprocal quasi-inverse of A. Moreover

(6) $$A_0 = \boldsymbol{Q}^{<}A\boldsymbol{P}^{>}, \quad A_0^{-1} = \boldsymbol{P}^{<}B\boldsymbol{Q}^{>},$$

(7) $$P = BA, \quad Q = AB,$$

(8) $$AP = A, \quad QA = A,$$

(9) $$PB = B, \quad BQ = B.$$

Equations (7), (8), (9) directly follow from § 1 (1) and 1.1 It follows from (7) and (8) that $ABA = AP = A$. Similarly we infer from (7) and (9) that $BAB = B$. Thus B is a reciprocal quasi-inverse of A. Multiplying (4) by $\boldsymbol{Q}^{<}$ from the left and by $\boldsymbol{P}^{>}$ from the right, and using § 1 (1) again, we get the first of equations (6). Similarly we deduce from (5) the second of equations (6).

2.4. *If* $B \in \mathrm{Hom}(O_2, O_1)$ *is a quasi-inverse of* $A \in \mathrm{Hom}(O_1, O_2)$, *then the morphisms*

$$P = BA \in \mathrm{Hom}(O_1, O_2), \quad Q = AB \in \mathrm{Hom}(O_2, O_2)$$

are projections. If $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are splits of* P *and* Q *respectively, then the morphism*

$$A_0 = \boldsymbol{Q}^{<} A \boldsymbol{P}^{>} \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{Q}|)$$

is invertible, viz.

$$A_0^{-1} = \boldsymbol{P}^{<} B \boldsymbol{Q}^{>} \in \mathrm{Hom}(|\boldsymbol{Q}|, |\boldsymbol{P}|). \tag{10}$$

Moreover equations (4) *and* (8) *hold, and*

$$\boldsymbol{P}^{>} A_0^{-1} \boldsymbol{Q}^{<} = BAB.$$

If B *is a reciprocal quasi-inverse of* A, *then also equations* (5) *and* (9) *hold.*

Since $(BA)(BA) = B(ABA) = BA$, the morphism P is a projection. Similarly we verify that Q is a projection. Equations (8) coincide with $ABA = A$, and equations (9) coincide with $BAB = B$.

It follows from (8) and from § 1 (1) and 1.1 that

$$(\boldsymbol{P}^{<} B \boldsymbol{Q}^{>})(\boldsymbol{Q}^{<} A \boldsymbol{P}^{>}) = \boldsymbol{P}^{<} B (\boldsymbol{Q}^{>} \boldsymbol{Q}^{<}) A \boldsymbol{P}^{>} = \boldsymbol{P}^{<} BQA \boldsymbol{P}^{>}$$
$$= \boldsymbol{P}^{<} BA \boldsymbol{P}^{>} = \boldsymbol{P}^{<} P \boldsymbol{P}^{>} = \boldsymbol{P}^{<} \boldsymbol{P}^{>} = I_{O_1}.$$

Similarly we prove that $(\boldsymbol{Q}^{<} A \boldsymbol{P}^{>})(\boldsymbol{P}^{<} B \boldsymbol{Q}^{>}) = I_{O_2}$. This proves (10).

Multiplying the equation $A_0 = \boldsymbol{Q}^{<} A \boldsymbol{P}^{>}$ by $\boldsymbol{Q}^{>}$ from the left and by $\boldsymbol{P}^{<}$ from the right we get (4). Multiplying equation (10) by $\boldsymbol{P}^{>}$ from the left and by $\boldsymbol{Q}^{<}$ from the right we get

$$\boldsymbol{P}^{>} A_0^{-1} \boldsymbol{Q}^{<} = \boldsymbol{P}^{>} \boldsymbol{P}^{<} B \boldsymbol{Q}^{>} \boldsymbol{Q}^{<} = PBQ = BABAB = BAB.$$

Thus (5) holds if B is a reciprocal quasi-inverse.

Till the end of this section we shall suppose that *every projection in the category* $\mathfrak{C}$ in question *has a split.*

2.5. *A morphism* $A \in \mathrm{Hom}(O_1, O_2)$ *is quasi-invertible if and only if it is of the form* (4), *that is,*

$$A = \boldsymbol{Q}^{>} A_0 \boldsymbol{P}^{<}$$

where $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are splits of projections* $P \in \mathrm{Hom}(O_1, O_1)$ *and* $Q \in \mathrm{Hom}(O_2, O_2)$, *and* $A_0 \in \mathrm{Hom}(|\boldsymbol{P}|, |\boldsymbol{Q}|)$ *is invertible.*

This directly follows from 2.3 and 2.4.

The right-hand side of (4) is said to be a *characteristic factorization* of A. Characteristic factorization (4) of A is not uniquely determined by A. Every characteristic factorization (4) of a quasi-invertible morphism A determines uniquely a reciprocal quasi-inverse B and a characteristic factorization (5) of B. Characteristic factorization (5) will be called the *factorization dual to factorization* (4). It is easy to see that the factorization dual to factorization (5) is again factorization (4). The value B of the dual characteristic factorization (5) is called the *quasi-inverse* of A *dual to characteristic factorization* (4) *of* A. It is always a reciprocal quasi-inverse of A. However it is not uniquely determined by A itself.

If characteristic factorization (4) of a morphism A is constructed as in 2.4 by means of a given quasi-inverse B of A, then the quasi-inverse of A dual to the characteristic factorization (4) is equal to BAB. Consequently it coincides with B if and only if the quasi-inverse B is reciprocal.

If B_1 is a quasi-inverse of A_1, and B_2 is a quasi-inverse of A_2 and the composition A_2A_1 is feasible, then the composition B_1B_2 is also feasible. However it may happen that B_1B_2 is not a quasi-inverse of A_2A_1, even in the case where B_1, B_2 are reciprocal quasi-inverses. For instance, if $P, Q \in \mathrm{Hom}(O, O)$ are projections, then P is a reciprocal quasi-inverse of P, Q is a reciprocal quasi-inverse of Q, but QP is a quasi-inverse of PQ if and only if the composition PQ is a projection. In fact,

$$(PQ)(QP)(PQ) = (PQ)(PQ).$$

Thus the left-hand side of the equation is equal to PQ if and only if the right-hand side of the equation is equal to PQ, i.e. if PQ is a projection. Since the composition of two projections is not always a projection, we see that the composition of quasi-inverses is not always a quasi-inverse of the composition. A necessary and sufficient condition for the composition of reciprocal quasi-inverses of morphisms to be a reciprocal quasi-inverse of the composition of the morphisms is given in the following theorem.

2.6. *Let*

$$A_1 \in \mathrm{Hom}(O_1, O_2), \quad A_2 \in \mathrm{Hom}(O_2, O_3) \tag{11}$$

be quasi-invertible morphisms, and let

$$A_1 = \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{P}_1^{<}, \quad A_2 = \boldsymbol{Q}_2^{>} A_{0,2} \boldsymbol{P}_2^{<} \tag{12}$$

are given characteristic factorizations of morphisms (11), *that is,* $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{Q}_1, \boldsymbol{Q}_2$ *are splits of projections*

$$P_1 \in \mathrm{Hom}(O_1, O_1), \quad P_2, Q_1 \in \mathrm{Hom}(O_2, O_2), \quad Q_2 \in \mathrm{Hom}(O_3, O_3)$$

respectively, and

$$A_{0,1} \in \mathrm{Hom}(|\boldsymbol{P}_1|, |\boldsymbol{Q}_1|), \quad A_{0,2} \in \mathrm{Hom}(|\boldsymbol{P}_2|, |\boldsymbol{Q}_2|)$$

are invertible morphisms. Let

(13) $$B_1 \in \operatorname{Hom}(O_2, O_1), \quad B_2 \in \operatorname{Hom}(O_3, O_2)$$

be the quasi-inverses of (11) *dual to characteristic factorizations* (12) *respectively, that is,*

(14) $$B_1 = P_1^{>} A_{0,1}^{-1} Q_1^{<}, \quad B_2 = P_2^{>} A_{0,2}^{-1} Q_2^{<}.$$

Then the morphism B_1B_2 is a quasi-inverse of the morphism A_2A_1 if and only if

(15) *the morphism P_2Q_1 is a projection.*

The morphism B_1B_2 is a reciprocal quasi-inverse of the morphism A_2A_1 if and only if

(16) *the morphisms P_2Q_1 and Q_1P_2 are projections.*

B_1B_2 is a quasi-inverse of A_2A_1 if and only if

(17) $$A_2A_1B_1B_2A_2A_1 = A_2A_1.$$

Since $A_1B_1 = Q_1$ and $B_2A_2 = P_2$ by (7), equation (17) is equivalent to the equation

$$A_2Q_1P_2A_1 = A_2A_1,$$

i.e. to the equation

(18) $$\boldsymbol{Q}_2^{>} A_{0,2} \boldsymbol{P}_2^{<} Q_1 P_2 \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{P}_1^{<} = \boldsymbol{Q}_2^{>} A_{0,2} \boldsymbol{P}_2^{<} \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{P}_1^{<}.$$

Multiplying (18) by $\boldsymbol{Q}_2^{<}$ from the left and by $\boldsymbol{P}_1^{>}$ from the right we get the equation

(19) $$A_{0,2} \boldsymbol{P}_2^{<} Q_1 P_2 \boldsymbol{Q}_1^{>} A_{0,1} = A_{0,2} \boldsymbol{P}_2^{<} \boldsymbol{Q}_1^{>} A_{0,1}.$$

Multiplying (19) by $A_{0,2}^{-1}$ from the left and by $A_{0,1}^{-1}$ from the right, we get the equation

(20) $$\boldsymbol{P}_2^{<} Q_1 P_2 \boldsymbol{Q}_1^{>} = \boldsymbol{P}_2^{<} \boldsymbol{Q}_1^{>}.$$

Multiplying (20) by $\boldsymbol{P}_2^{>}$ from the left and by $\boldsymbol{Q}_1^{<}$ from the right we get the equation

(21) $$P_2Q_1P_2Q_1 = P_2Q_1,$$

i.e. the condition for P_2Q_1 to be a projection. Conversely, multiplying (21) by $\boldsymbol{P}_2^{<}$ from the left and by $\boldsymbol{Q}_1^{>}$ from the right we get (20). Multiplying (20) by $A_{0,2}$ from the left and by $A_{0,1}$ from the right we get (19). Multiplying (19) by $\boldsymbol{Q}_2^{>}$ from the left and by $\boldsymbol{P}_1^{<}$ from the right we get (18). This proves that (17) is equivalent to (21), i.e. that (15) is a necessary and sufficient condition for B_1B_2 to be a quasi-inverse of A_2A_1.

Similarly we prove that A_2A_1 is a quasi-inverse of B_1B_2 is and only if Q_1P_2 is a projection. This completes the proof of 2.6.

We recall that if $P_2Q_1 = Q_1P_2$, then condition (16) is satisfied.

The representations (12) of morphisms (11) are not unique. Therefore it can happen that for some representations (12) of (11) conditions (15) or (16) are satisfied, and for other representations they are not satisfied.

2.7. *If quasi-invertible morphisms* (11) *can be represented in the form* (12) *in such a way that the morphism $R = P_2Q_1$ is a projection, then the morphism $A = A_2A_1$ is quasi-invertible and its characteristic factorization* (4) *can be obtained as follows.*

Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be splits of the projections

(22) $$S_1 = A_{0,1}^{-1} \boldsymbol{Q}_1^{<} R \boldsymbol{Q}_1^{>} A_{0,1}, \quad S_2 = A_{0,2} \boldsymbol{P}_2^{<} R \boldsymbol{P}_2^{>} A_{0,2}^{-1},$$

respectively, let

(23) $$P = \boldsymbol{P}_1^{>} S_1 \boldsymbol{P}_1^{<}, \quad Q = \boldsymbol{Q}_2^{>} S_2 \boldsymbol{Q}_2^{<},$$

(24) $$P^{>} = \boldsymbol{P}_1^{>} \boldsymbol{S}_1^{>}, \quad P^{<} = \boldsymbol{S}_1^{<} \boldsymbol{P}_1^{<}, \quad Q^{>} = \boldsymbol{Q}_2^{>} \boldsymbol{S}_2^{>}, \quad Q^{<} = \boldsymbol{S}_2^{<} \boldsymbol{Q}_2^{<},$$

and let

(25) $$A_0 = \boldsymbol{S}_2^{<} A_{0,2} \boldsymbol{P}_2^{<} \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{S}_1^{>}.$$

Then

(i) *P is a projection and $\boldsymbol{P} = (\boldsymbol{P}^{>}, \boldsymbol{P}^{<})$ is a split of P,*

(ii) *Q is a projection and $\boldsymbol{Q} = (\boldsymbol{Q}^{>}, \boldsymbol{Q}^{<})$ is a split of Q,*

(iii) *A_0 is invertible and $A = \boldsymbol{Q}^{>} A_0 \boldsymbol{P}^{<}$.*

Let $\boldsymbol{R}$ be a split of R. Since $RQ_1 = R$, it follows from 1.7 that S_1 is a projection, (i) is true and that

$$\boldsymbol{R}^{<} \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{P}_1^{<} = A_{1,1} \boldsymbol{P}^{<}$$

where

$$A_{1,1} = \boldsymbol{R}^{<} \boldsymbol{Q}_1^{>} A_{0,1} \boldsymbol{S}_1^{>}$$

is invertible. Since $P_2R = R$, it follows from 1.6 that S_2 is a projection, (ii) is true and

$$\boldsymbol{Q}_2^{>} A_{0,2} \boldsymbol{P}_2^{<} \boldsymbol{R}^{>} = \boldsymbol{Q}^{>} A_{1,2}$$

where

$$A_{1,2} = \boldsymbol{S}_2^{<} A_{0,2} \boldsymbol{P}_2^{<} \boldsymbol{R}^{>}$$

is invertible. Since

$$\boldsymbol{P}_2^{<} \boldsymbol{Q}_1^{>} = (\boldsymbol{P}_2^{<} P_2)(Q_1 \boldsymbol{Q}_1^{>}) = \boldsymbol{P}_2^{<} R \boldsymbol{Q}_1^{>} = \boldsymbol{P}_2^{<} \boldsymbol{R}^{>} \boldsymbol{R}^{<} \boldsymbol{Q}_1^{>}$$

by 1.1 and § 1 (1), we infer that

$$A_{1,2}A_{1,1} = \boldsymbol{S}_2^{<}A_{0,2}\boldsymbol{P}_2^{<}\boldsymbol{R}^{>}\boldsymbol{R}^{<}\boldsymbol{Q}_1^{>}A_{0,1}\boldsymbol{S}_1^{>} = \boldsymbol{S}_2^{<}A_{0,2}\boldsymbol{P}_2^{<}\boldsymbol{Q}_1^{>}A_{0,1}\boldsymbol{S}_1^{>} = A_0$$

and consequently, by (12),

$$\begin{aligned} A &= \boldsymbol{Q}_2^{>}A_{0,2}\boldsymbol{P}_2^{<}\boldsymbol{Q}_1^{>}A_{0,1}\boldsymbol{P}_1^{<} = (Q_2^{>}A_{0,2}\boldsymbol{P}_2^{<}\boldsymbol{R}^{>})(\boldsymbol{R}^{<}\boldsymbol{Q}_1^{>}A_{0,1}\boldsymbol{P}_1^{<}) \\ &= \boldsymbol{Q}^{>}A_{1,2}A_{1,1}\boldsymbol{P}^{<} = \boldsymbol{Q}^{>}A_0\boldsymbol{P}^{<}. \end{aligned}$$

This proves (iii).

§ 3. Selected splits. In this section we suppose that the category $\mathfrak{C}$ in question has the following property: There is defined a function which assigns to every projection P in $\mathfrak{C}$ a pair $(P^{>}, P^{<})$ of morphisms in such a way that

1) $\boldsymbol{P} = (P^{>}, P^{<})$ is a split of P, called the *selected split* of P, the object $|\boldsymbol{P}|$ of this split being denoted by $\mathrm{ra}P$,

2) if P, Q are projections and $\mathrm{d}Q = \mathrm{ra}P$, then

$$(P^{>}QP^{<})^{>} = P^{>}Q^{>}, \quad (P^{>}QP^{<})^{<} = P^{<}Q^{<}. \tag{1}$$

We recall that, by 1.4, the morphism $R = P^{>}QP^{<}$ is necessarily a projection and that $(P^{>}Q^{>}, Q^{<}P^{<})$ is a split of R. Condition 2) requires that this split is the selected split of R.

In the proofs of all theorems in sections 1 and 2 we have dealt either with arbitrary splits of projections under considerations, or with splits formed from given splits by means of theorems 1.4. It follows from 2) that if the given splits are selected, the splits formed by means of 1.4 are also selected. Thus all theorems in sections 1 and 2 remains true if we restrict all splits under consideration to selected splits. In such a way we get the following modification of theorem 2.5.

3.1. *A morphism $A \in \mathrm{Hom}(O_1, O_2)$ is quasi-invertible if and only if it is of the form*

$$A = Q^{>}A_0P^{<} \tag{2}$$

where $P \in \mathrm{Hom}(O_1, O_1)$ and $Q \in \mathrm{Hom}(O_2, O_2)$ are projections and $A_0 \in \mathrm{Hom}(\mathrm{ra}P, \mathrm{ra}Q)$ is invertible.

The characteristic factorization (2) of an invertible morphism A is not unique, in general.

If we restrict our consideration only to selected splits, we get the following modification of 2.7.

3.2. *If quasi-invertible morphisms A_1, A_2 have characteristic factorizations*

$$A_1 = Q_1^{>}A_{0,1}P_1^{<}, \quad A_2 = Q_2^{>}A_{0,2}P_2^{<} \tag{3}$$

of form (2), such that P_2Q_1 is a projection, then the composition A_2A_1 has the following characteristic factorization of form (2):

$$\begin{aligned} A_2A_1 = {} & (Q_2^{>}A_{0,2}P_2^{<}Q_1P_2^{>}A_{0,2}^{-1}Q_2^{<})^{>} \\ & \big((Q_{0,2}P_2^{<}Q_1P_2^{>}A_{0,2}^{-1})^{<}A_{0,2}P_2^{<}Q_1^{>}A_{0,1}(A_{0,1}^{-1}Q_1^{<}P_2Q_1^{>}A_{0,1})^{>}\big) \\ & (P_1^{>}A_{0,1}^{-1}Q_1^{<}P_2Q_1^{>}A_{0,1}P_1^{<})^{<}. \end{aligned} \tag{4}$$

In other words, if $P_1, P_2, Q_1, Q_2, P_2Q_1$ are projections, $A_{0,1}, A_{0,2}$ are invertible, and (3) hold, then the expressions in the first line and in the third line of the right-hand side of (4) are of the form $Q^{>}$ and $P^{<}$ respectively, where P and Q are projections, and the expression in the second line of (4) is an invertible morphism A_0. Moreover (4) holds i.e., $A_2A_1 = Q^{>}A_0P^{<}$.

§ 4. The case of conrete categories. For any mapping f from a set X into a set Y the symbol $\mathrm{d}f$ will denote the domain of f, i.e. the set X, and the symbol $\mathrm{ra}f$ will denote the range of f, i.e. the set of all $f(x)$, $x \in X$. If $Z \subset X$, the symbol $f|Z$ denotes the restriction of f to the set Z. If f is one-to-one, then f^{-1} denotes the inverse of f. By definition, a mapping g is the inverse of a mapping f if $\mathrm{d}g = \mathrm{ra}f$, $\mathrm{ra}g = \mathrm{d}f$ and

$$y = f(x) \quad \text{if and only if} \quad x = g(y).$$

If f, g are mappings and $\mathrm{ra}f \subset \mathrm{d}g$, then the symbol gf stands for the composition of f and g. By definitions, $gf(x) = g(f(x))$ for $x \in \mathrm{d}f$.

A mapping g is said to be a *quasi-inverse* of a mapping f if $fgf = f$. This equation implies that $\mathrm{ra}f \subset \mathrm{d}g$ and $\mathrm{ra}g \subset \mathrm{d}f$. A mapping g is said to be a *reciprocal quasi-inverse* of a mapping f if g is a quasi-inverse of f, and f is a quasi-inverse of g, i.e. if simultaneously $fgf = f$ and $gfg = g$.

4.1. *A mapping g is a quasi-inverse of a mapping f if and only if the mapping $g|\mathrm{ra}f$ is the inverse of the mapping $f|\mathrm{ra}(g|\mathrm{ra}f)$. A mapping g is a reciprocal quasi-inverse of a mapping f if and only if $g|\mathrm{ra}f$ is the inverse of the mapping $f|\mathrm{ra}g$.*

In this section we assume that the category $\mathfrak{C}$ in question is *concrete*. Thus all objects are sets (with additional structures, in general), and morphisms $A \in \mathrm{Hom}(O_1, O_2)$ are triples $A = (\bar{A}, O_1, O_2)$ where $\bar{A}$ is a mapping from O_1 into O_2. We shall always apply the convention that if a symbol denotes a morphism, then the same symbol with the dash denotes the corresponding mapping, i.e. the first term of the morphism. In particular, if O is an object, then $\bar{I}_O$ denotes the identity mapping of the set O onto O.

Observe that

$$\overline{BA} = \bar{B}\bar{A}$$

for any morphisms A, B with $\mathrm{cd}A = \mathrm{d}B$.

More generally, if $A_1, \dots, A_n$ are morphisms and $\overline{A_1 \dots A_n}$ exists, then $\overline{A}_1 \dots \overline{A}_n$ exists and is equal to $\overline{A_1 \dots A_n}$. Moreover, if B is a morphism and $\mathrm{d}B = \mathrm{d}A_1$, $\mathrm{cd}B = \mathrm{cd}A_n$, then

$$B = A_1 \dots A_n \quad \text{if and only if} \quad \overline{B} = \overline{A}_1 \dots \overline{A}_n .$$

Observe else that if A is an invertible morphism, then the inverse $\overline{A}^{-1}$ of $\overline{A}$ exists and is equal to $\overline{A^{-1}}$.

For any morphism A, by the *range* $\mathrm{ra}A$ of A we mean the range $\mathrm{ra}\overline{A}$ of the corresponding mapping $\overline{A}$. By definition,

$$\mathrm{d}A = \mathrm{d}\overline{A}, \quad \mathrm{ra}A = \mathrm{ra}\overline{A} \subset \mathrm{cd}A .$$

4.2. *A morphism B is a quasi-inverse of a morphism A if and only if the mapping $\overline{B}$ is a quasi-inverse of the mapping $\overline{A}$. A morphism B is a reciprocal quasi-inverse of a morphism A if and only if the mapping $\overline{B}$ is a reciprocal quasi-inverse of the mapping $\overline{A}$.*

In the sequel of the section we shall suppose that the category $\mathcal{C}$ has the following property: for every projection P the set $\mathrm{ra}P$ is an object in $\mathcal{C}$, and the triples

(2) $$P^{>} = (\overline{I}_{\mathrm{ra}P}, \mathrm{ra}P, \mathrm{d}P), \quad P^{<} = (\overline{P}, \mathrm{d}P, \mathrm{ra}P)$$

are morphisms in $\mathcal{C}$.

It follows directly from (2) that the pair $\boldsymbol{P} = (P^{>}, P^{<})$ is a split of the projection P and $|\boldsymbol{P}| = \mathrm{ra}P$. The split $\boldsymbol{P}$ will be called the *natural split* of P.

It is easy to see that natural splits satisfy conditions 1) and 2) from section 3, and that for every projection P the symbol $\mathrm{ra}P$ introduced in section 3 coincides with the symbol $\mathrm{ra}P$ introduced in section 4. Let us assume natural splits as selected splits in $\mathcal{C}$ in what follows.

The characterization 3.1 of quasi-invertible morphisms can now be formulated as follows:

4.3. *A morphism A is quasi-invertible if and only if the following two conditions are satisfied:*

(i) *there exists a projection P such that $A_0 = (\overline{A}|\mathrm{ra}P, \mathrm{ra}P, \mathrm{ra}A)$ is an invertible morphism (in particular, the mapping $\overline{A}|\mathrm{ra}P$ is one-to-one and onto* $\mathrm{ra}A$),

(ii) *there exists a projection Q such that $\mathrm{ra}A = \mathrm{ra}Q$.*

If conditions (i), (ii) are satisfied, then $A = Q^{>}A_0P^{<}$ and consequently $\overline{A} = \overline{A}_0\overline{P}$. If B is the reciprocal quasi-inverse dual to the characteristic factorization $A = Q^{>}A_0P^{<}$, then $\overline{B} = \overline{A_0^{-1}}\overline{Q}$.

Suppose now that the concrete category $\mathcal{C}$ is *closed with respect to cartesian products*, i.e. that the following conditions are satisfied:

1) if O_1 and O_2 are objects, then the set $O_1 \times O_2$ is an object, and the triples

$$P_1 = (\overline{P}_1, O_1 \times O_2, O_1), \quad P_2 = (\overline{P}_2, O_1 \times O_2, O_2),$$

where

$$\overline{P}_1(x, y) = x \quad \text{and} \quad \overline{P}_2(x, y) = y \quad \text{for} \quad x \in O_1 \text{ and } y \in O_2,$$

are morphisms,

2) if O, O_1, O_2 are objects, $A \in \mathrm{Hom}(O, O_1)$, $B \in \mathrm{Hom}(O, O_2)$, then the triple $C = (\overline{C}, O, O_1 \times O_2)$, where

$$\overline{C}(x) = (\overline{A}(x), \overline{B}(x)) \quad \text{for} \quad x \in O,$$

is a morphism,

3) for any objects O_1, O_2 there exists an element $o \in O_2$ such that the triple $\boldsymbol{o} = (\overline{\boldsymbol{o}}, O_1, O_2)$, where

$$\overline{\boldsymbol{o}}(x) = o \quad \text{for every } x \in O_1,$$

is a morphism.

The morphism C defined in 2) will be denoted by (A, B).

4.4. *If O_1, O_2 are such objects that*

(a) *O_2 is a subset of O_1 and the injection $I = (\overline{I}, O_2, O_1)$, where $\overline{I}(x) = x$ for every $x \in O_2$, is a morphism,*

(b) *there exists no projection $P \in \mathrm{Hom}(O_1, O_1)$ such that $\mathrm{ra}P = O_2$, then the morphisms*

$$A_1 = (P_2, P_2) \in \mathrm{Hom}(O_1 \times O_2, O_1 \times O_2),$$

$$A_2 = (P_1, \boldsymbol{o}P_1) \in \mathrm{Hom}(O_1 \times O_2, O_1 \times O_2),$$

where P_1, P_2 and $\boldsymbol{o}$ are defined as in 1) *and* 3), *are projections and therefore quasi-invertible. However the composition $A = A_2A_1$ is not quasi-invertible.*

By definition, for any $x \in O_1$ and $y \in O_2$,

$$\overline{A}_1(x, y) = (y, y), \quad \overline{A}_2(x, y) = (x, o),$$

and consequently $A(x, y) = (y, o)$. Thus $\mathrm{ra}A$ is the set Z of all (y, o) where $y \in O_2$. By 4.3, in order to prove that A is not quasi-invertible, it suffices to show that there exists no projection $Q \in \mathrm{Hom}(O_1 \times O_2, O_1 \times O_2)$ such that $\mathrm{ra}Q = Z$.

Suppose Q is such a projection. Then the composition $P = P_1Q(I, \boldsymbol{o}) \in \mathrm{Hom}(O_1, O_1)$ is a projection and $\mathrm{ra}P = O_2$ which contradicts (b).

It follows from 4.4 that in many concrete categories there exist quasi-invertible morphisms A_1, A_2 (and even projections) such that their composition is not quasi-invertible. It is so e.g. in the case of the category of groups, of abelian groups, of finite abelian groups, and also in the category of topological spaces, of metric spaces, of compact spaces and of Banach spaces. It is less evident that it is so in the category of all totally disconnected compact spaces (and, therefore, also in the category of Boolean algebras) and in the category of dyadic spaces. To prove it, let O_1 be a non-metrizable Cantor space (i.e. an uncountable product of two-point Hausdorff spaces). Take two copies of O_1, choose a point in each of them and identify the points. The space so obtained is homeomorphic to a subspace O_2 of O_1. The spaces O_1 and O_2 are compact, dyadic and totally disconnected, and satisfy conditions (a) and (b) in 4.4 (for a proof of (b), see R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. 57, 1965, pp. 287–304, Theorem 16; the above example of spaces O_1 and O_2 was communicated to me by R. Engelking).

On the other hand, there are concrete categories with the property that every morphism is quasi-invertible. It is so e.g. in the case of the category of all sets, the category of all linear spaces (over a fixed field), and in the case of the Fredholm category, i.e. the category whose objects are Banach spaces and morphisms are triples (f, O_1, O_2) where f is a bounded linear mapping from O_1 into O_2 that satisfies the well known Fredholm theorem.

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