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Forcing in a general setting (1)

by

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Abstract. Abstract topological notions of forcing and generic set are presented. These notions are independent of the general notions of language and structure. Most particular notions of forcing in the literature are subsumed under this notion. The abstract notion is used to construct notions of forcing for languages containing the equi-cardinality quantifier, infinitary languages containing dependent quantifiers, and second-order languages.

The method of *forcing* was first invented by Cohen [Coh 1], [Coh 2] to solve questions regarding the logical independence of the axiom of choice and the continuum hypothesis with regard to the axioms of Zermelo-Fraenkel set theory. Subsequently Feferman [Fe] transferred the method to the settings of number theory and analysis and Robinson [Ro 1], [Ro 2] extended if the setting of general first order model theory.

Takenti realized that the existence of generic sets in set-theoretic forcing could be derived from the Baire Category Theorem and developed this point of view in [Ta] and lectures at the University of Illinois during 1965-66. This point of view was further developed in [Bo 3] and its extension to first order model theory was announced in [Bo 1]. The extension to second order logic was presented in [Bo 2].

In this paper we develop extremely abstract topological notions of forcing and generic objects which are entirely independent of the notions of language and structure. This development is presented in § 2. That it apparently subsums a great many of the forcing notions already extant in the literature is sketched in § 3. The extension of the notion of forcing to languages involving the equicardinality quantifier Q and to infinitary languages involving dependent quantifiers in the sense of [Ma] is presented in § 4.

The formulation of abstract forcing as given in § 2 is more general than necessary in that in §§ 3-5 we always take the sets X and X_0 to be $X = \{0, 1\}$ and $X_0 = \{0\}$. We hope to use this generality to extend the forcing concept to continuous model theory in the sense of [C/K] in a future publication.

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§ 1. Preliminaries. Let θ be a fixed infinite regular cardinal number. An arbitrary set is said to be *θ-finite* if it has power $<\theta$. A topological space S is said to be a *θ-space* provided that every θ-finite intersection of open sets of S is again open in S. A base $\mathcal{U}$ for S is a *θ-base* for S if every θ-finite intersection of elements of $\mathcal{U}$ is expressible as a union of elements of $\mathcal{U}$.

As usual, we say that $T\subseteq S$ is nowhere dense in S if $S-\overline{T}=S$. T is *θ-meager in S* if T is the union of θ nowhere dense subsets of S; otherwise, T is *θ-non-meager* in S. If $S-T$ is θ-meager, then T is called *θ-co-meager in S*. Clearly, if S is θ-non-meager in itself, then every θ-co-meager subset of S is non-empty, and is in fact θ-non-meager in S. If S is T_1 with no isolated points, then all subsets of power $\leqslant\theta$ are θ-meager since then points are nowhere dense. We say that S is a *θ-Baire space* if each θ-co-meager subset of S is everywhere dense in S. A θ-space S is a *θ-Platek space* if for every $\mathcal{C}\subseteq\mathcal{U}$, where $\mathcal{U}$ is a θ-base for S, if $\mathcal{C}$ is linearly ordered by $\subseteq$, has power $\leqslant\theta$, and contains no empty sets, then $\bigcap\mathcal{C}\neq 0$.

Lemma 1.1. (cf. [Pl]) *Any θ-Platek space is a θ-Baire space.*

Proof. As indicated in [Pl], one can easily imitate the classical proof that the space of real numbers is an ω-Baire space (even though this space is not an ω-Platek space; cf. [Ku], p. 414).

For any product $S=\bigtimes_{i\in I}S_i$ of topological spaces, the *θ-product topology* is the smallest refinement of the usual product topology which makes S a *θ-space*. A θ-base for this topology consists of all sets of the form $\bigtimes_{i\in I}U_i$, where for each i, U_i is open in S_i, and $\mathrm{card}(\{i\in I\colon U_i\neq S_i\})<\theta$. Then if $\mathrm{card}(X)\geqslant\theta$, X is given the discrete topology, and ${}^{\varkappa}X$ is given the θ-product topology, it is easy to check that ${}^{\varkappa}X$ is a θ-Platek space and hence a θ-Baire space. Similarly, ${}^{\varkappa}2$ becomes a θ-Baire space.

If $\mathcal{C}$ is a collection of subsets of S such that every basic open subset of S is a union of elements of $\mathcal{C}$, then $\mathcal{C}$ is called a *covering system for S*.

The *θ-Borel subsets* of S constitute the smallest θ-complete set algebra of subsets of S which contains the open subsets of S. A subset $T\subseteq S$ has the *θ-property of Baire* if there is an open set O of S such that $T\Delta O\big(=(T-O)\cup(O-T)\big)$ is θ-meager. It is easy to check that in a θ-space, the collection of sets possessing the θ-property of Baire forms a θ-complete set algebra containing all the open sets; hence every θ-Borel set possesses the θ-property of Baire.

We write $\theta^{\underline{\mu}}=\sum_{\alpha<\mu}\mathrm{card}({}^{\alpha}\theta)$. If $\theta^{\underline{\theta}}=\theta$, then for any $\gamma<\theta$, $\mathrm{card}({}^{\gamma}\theta)=\theta$. Clearly $\omega^{\underline{\omega}}=\omega$, if θ is inaccessible, $\theta^{\underline{\theta}}=\theta$, and the GCH implies that for any regular θ, $\theta^{\underline{\theta}}=\theta$. We write $S_\theta(X)$ for the set of θ-finite subsets of X, and θ^{+} for the smallest cardinal greater than θ.

§ 2. Abstract forcing. Let θ be a fixed regular cardinal, let S be a θ-Baire space, and let $\mathcal{C}$ be a covering system of power $\leqslant\theta$ for S. Let X and Y be fixed with $X\neq 0$ and $\mathrm{card}(Y)\leqslant\theta$. Given $X_0\subseteq X$, a map $\Phi\colon S\times Y\to X$ is called *(X_0,θ)-Baire* if for every $y\in Y$, the set $\{s\in S\colon \Phi(s,y)\in X_0\}$ has the θ-property of Baire in S. In the remainder of this section $X_0\neq 0$ is fixed and Φ is a fixed (X_0,θ)-Baire map. For $y\in Y$, we set $Z(y)=\{s\in S\colon \Phi(s,y)\in X_0\}$ and $Z_*(y)=S-Z(y)$.

In the applications below the points of S will correspond to structures for a certain language L and the elements of Y will be sentences from a certain extension L' of L. The space X will be the space of truth values and X_0 will be the set of designated truth values. In all of the applications here, we will take $X=\{T,F\}$ and $X_0=\{T\}$. The map Φ corresponds to the satisfaction relation for sentences from Y in structures from S. $Z(y)$ is thus the set of points of X at which y holds (i.e., has a designated truth value), while $Z_*(y)$ is the set of points of S at which y fails. The predicate $p\models_S^{\mathrm{na}}y$ defined below indicates that y holds at all but a "few" points of p. When the language L contains a usual negation sign $\neg$, the predicate $p\models_S^{\mathrm{na}^*}y$ to be defined below would correspond to $p\models_S^{\mathrm{na}}\neg y$.

Definition 2.1. Let $p\in\mathcal{C}$ and $y\in Y$. Then $p\models_S^{\mathrm{na}}y$ iff $p\cap Z_*(y)$ is θ-meager in S, and $p\models_S^{\mathrm{na}^*}y$ iff $p\cap Z(y)$ is θ-meager in S. For $s\in S$, write $s\models_S^{\mathrm{na}}y$ ($s\models_S^{\mathrm{na}^*}y$) iff for some $p\in\mathcal{C}$, $s\in p$ and $p\models_S^{\mathrm{na}}y$ ($p\models_S^{\mathrm{na}^*}y$).

When no confusion is likely, we will omit the subscript S above. Set $\mathcal{C}(y)=\{s\in S\colon s\models^{\mathrm{na}}y\}$.

Lemma 2.2. (cf. [Ta] and [Bo 3]). *Let $y\in Y$. For nearly all $s\in S$, $s\models^{\mathrm{na}}y$ or $s\models^{\mathrm{na}^*}y$.*

Proof. Since both $Z(y)$ and $Z_*(y)$ have the θ-property of Baire, there are open sets O_1 and O_2 such that both $Z(y)\Delta O_1$ and $Z_*(y)\Delta O_2$ are θ-meager. Since $O_1\cap O_2\subseteq\big(Z(y)\Delta O_1\big)\cup\big(Z_*(y)\Delta O_2\big)$ and $O_1\cap O_2$ is open, it follows that $O_1\cap O_2=0$. If $s\in Z(y)\cap O_1$, then for some $p\in\mathcal{C}$, $s\in p\subseteq O_1$. Then $p\cap Z_*(y)\subseteq O_1\Delta Z(y)$, so that $p\models^{\mathrm{na}}y$ and hence $s\models^{\mathrm{na}}y$. Similarly, if $s\in Z_*(y)\cap O_2$, then $s\models^{\mathrm{na}^*}y$. The lemma now follows, since $\big(Z(y)\cap O_1\big)\cup\big(Z_*(y)\cap O_2\big)$ is θ-co-meager.

Set $\mathcal{K}=\{s\in S\colon \text{for all } y\in Y,\ s\models^{\mathrm{na}}y \text{ or } s\models^{\mathrm{na}^*}y\}$. Then since $\mathrm{card}(Y)\leqslant\theta$ and S is a θ-Baire space, it follows that $\mathcal{K}$ is θ-co-meager and hence dense in S. $\mathcal{K}$ may still contain some points with unpleasant properties: there appears to be no reason why we cannot find an $s\in\mathcal{K}$ and a $y\in Y$ such that $s\models^{\mathrm{na}}y$, yet $\Phi(s,y)\notin X_0$. We can remove these points as follows.

Definition 2.3. For each $y\in Y$ and $p\in\mathcal{C}$, set

$$\mathcal{E}(y,p)=\{s\in p\colon s\in Z_*(y)\ \&\ p\models^{\mathrm{na}}y\}\cup\{s\in p\colon s\in Z(y)\ \&\ p\models^{\mathrm{na}^*}y\}.$$

Then set

$$\mathfrak{E}=\mathcal{K}-\bigcup_{y\in Y}\bigcup_{p\in\mathcal{C}}\mathcal{E}(y,p).$$

By definition, each $\mathcal{E}(y, p)$ is θ-meager in S. Hence, since both Y and $\mathcal{C}$ are of power $\leqslant \theta$, $\mathfrak{E}$ is θ-co-meager in S and hence is dense in S since S is a θ-Baire space.

We will say that $\mathfrak{F} \subseteq S$ is *generic* if $\mathfrak{F} \subseteq \mathfrak{E}$ and $\mathfrak{F}$ is θ-co-meager in S. Clearly $\mathfrak{E}$ is generic.

DEFINITION 2.4. For $\mathfrak{F} \subseteq S$, $p \in \mathcal{C}$, and $y \in Y$, define $p \Vdash^{S}_{\mathfrak{F}} y$ (*p forces y with respect to $\mathfrak{F}$ and S*) to mean that $p \cap \mathfrak{F} \subseteq Z(y)$. For $s \in S$, we write $s \Vdash^{S}_{\mathfrak{F}} y$ if there is a $p \in \mathcal{C}$ with $s \in p$ such that $p \Vdash^{S}_{\mathfrak{F}} y$.

When no confusion is likely, we will omit the superscripts S above. Set $\mathfrak{F}(y) = \{s \in S\colon s \Vdash_{\mathfrak{F}} y\}$. Clearly $\mathfrak{F}(y) \cap \mathfrak{F} \subseteq Z(y)$ and $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ implies $\mathfrak{F}_1(y) \supseteq \mathfrak{F}_2(y)$. The theorem below is the general form of what is usually known as the truth lemma, even though it is the heart of all forcing constructions. The present general point of view has led to the statement as given. A restatement in slightly more perspicacious notation appears at the end of the proof.

THEOREM 2.5 (Truth lemma). *If $\mathfrak{F}$ is generic and $y \in Y$, then*

$$\mathfrak{E}(y) \cap \mathfrak{F} = \mathfrak{F}(y) \cap \mathfrak{F} = \mathcal{C}(y) \cap \mathfrak{F} = Z(y) \cap \mathfrak{F}\,.$$

Proof. First note that $s \in p \cap \mathfrak{F}$ and $p \vDash^{\mathrm{na}} y$ implies $s \notin \mathcal{E}(y, p)$, since $\mathfrak{F} \subseteq \mathfrak{E}$, and hence $\Phi(s, y) \in X_0$. Thus $\mathcal{C}(y) \cap \mathfrak{F} \subseteq Z(y) \cap \mathfrak{F}$. Also, if $s \in p \cap \mathfrak{F} \subseteq Z(y)$, then $p \cap Z^*(y) \subseteq S - \mathfrak{F}$ and so $p \vDash^{\mathrm{na}} y$ since $\mathfrak{F}$ is θ-co-meager. Thus $\mathfrak{F}(y) \cap \mathfrak{F} \subseteq \mathcal{C}(y) \cap \mathfrak{F}$. Thus, by remarks above, we now have:

$$\mathfrak{E}(y) \cap \mathfrak{F} \subseteq \mathfrak{F}(y) \cap \mathfrak{F} \subseteq \mathcal{C}(y) \cap \mathfrak{F} \subseteq Z(y) \cap \mathfrak{F}\,.$$

Now suppose that $s \in Z(y) \cap p \cap \mathfrak{F}$ and $p \vDash^{\mathrm{na}^*} y$. Then $s \in \mathcal{E}(y, p)$, contradicting $\mathfrak{F} \subseteq \mathfrak{E}$. Thus $s \in Z(y) \cap \mathfrak{F}$ implies not-$s \vDash^{\mathrm{na}^*} y$, and so, since $s \in \mathfrak{F} \subseteq \mathcal{H}$, $s \vDash^{\mathrm{na}} y$. Thus $Z(y) \cap \mathfrak{F} \subseteq \mathcal{C}(y) \cap \mathfrak{F}$. Finally, suppose $s \in p \cap Z_*(y)$ and $p \vDash^{\mathrm{na}} y$. Then $s \in \mathcal{E}(y, P)$, so that $s \notin \mathfrak{E}$. Thus if $p \vDash^{\mathrm{na}} y$, then $p \cap \mathfrak{E} \subseteq Z(y)$, and so it follows that $\mathcal{C}(y) \cap \mathfrak{F} \subseteq \mathfrak{E}(y) \cap \mathfrak{F}$. Hence we have

$$Z(y) \cap \mathfrak{F} \subseteq \mathfrak{E}(y) \cap \mathfrak{F}\,,$$

completing the proof. Thus if $\mathfrak{F}$ is generic, $s \in \mathfrak{F}$, and $y \in Y$,

$$s \Vdash_{\mathfrak{E}} y \quad \text{iff } s \Vdash_{\mathfrak{F}} y \quad \text{iff } s \vDash^{\mathrm{na}} y \quad \text{iff } \Phi(s, y) \in X_0\,.$$

A map $\Phi\colon S \times Y \to X$ is an (X_0, θ)-Borel map if for every $y \in Y$, $\{s \in S\colon \Phi(s, y) \in X_0\}$ is a θ-Borel subset of S. Since every θ-Borel set has the θ-property of Baire, every (X_0, θ)-Borel map is an (X_0, θ)-Baire map, and hence the foregoing results hold for such a Φ. Most of the maps we deal with in the sequel will be (X_0, θ)-Borel maps.

Above, we began with a θ-Baire space S and assumed that our map Φ was an (X_0, θ)-Baire map. Next we will consider a somewhat special situation in which we begin with a *set* S and a map $\tilde{\Phi}$ and look for a topology on S which will make S a θ-Baire space and $\tilde{\Phi}$ an (X_0, θ)-Baire map.

Let $0 \neq Y_0 \subseteq Y$ be fixed, let $S \subseteq^{Y_0} X$, and let $\tilde{\Phi}\colon S \times Y_0 \to X$. Let S be given the topology $\mathcal{T}_{\tilde{\Phi}}$ whose base is

$$\{0\} \cup \big\{\{s \in S\colon (\lambda x)\tilde{\Phi}(s, x) \restriction T = g\}\colon T \in S_\theta(Y_0) \wedge g \in {}^{T}X\big\}\,.$$

Let $\beta < \theta$ and let $O_\alpha = \{s \in S\colon (\lambda x)\tilde{\Phi}(s, x) \restriction T_\alpha = g_\alpha\}$ be a basic open set for each $\alpha < \beta$. Suppose that there are $\alpha, \alpha' < \beta$ such that $g_\alpha \restriction T_\alpha \cap T_{\alpha'} \neq g_{\alpha'} \restriction T_\alpha \cap T_{\alpha'}$. Then $\bigcap_{\alpha<\beta} O_\alpha = 0$ which is again a basic open set. On the other hand, if for all $\alpha, \alpha' < \beta$, $g_\alpha \restriction T_\alpha \cap T_{\alpha'} = g_{\alpha'} \restriction T_\alpha \cap T_{\alpha'}$, then let $T = \bigcup_{\alpha<\beta} T_\alpha$ and $g = \bigcup_{\alpha<\beta} T_\alpha$. Then $g \in {}^{T}X$ and since $\operatorname{card}(T_\alpha) < \theta$ for each $\alpha < \theta$, then by the regularity of θ, $\operatorname{card}(T) < \theta$. Thus $\bigcap_{\alpha<\beta} O_\alpha = \{s \in S\colon (\lambda x)\tilde{\Phi}(s, x) \restriction T = g\}$ is again a basic open set. Hence $(S, \mathcal{T}_{\tilde{\Phi}})$ is a θ-space.

DEFINITION 2.6. The map $\tilde{\Phi}\colon S \times Y_0 \to X$ is *(S, θ)-compact* iff

$$\forall U \in S_{\theta+}(Y_0) \forall h \in {}^{U}X \big[\forall T \in S_\theta(U) \exists s \in S \forall y \in T[\tilde{\Phi}(s, y) = h(y)]$$
$$\to \exists s \in S \forall y \in U[\tilde{\Phi}(s, y) = h(y)]\big]\,.$$

LEMMA 2.7. *If $\tilde{\Phi}\colon S \times Y_0 \to X$ is (S, θ)-compact, then the space $(S, \mathcal{T}_{\tilde{\Phi}})$ is a θ-Baire space, and for each $y \in Y_0$, $\{s \in S\colon \tilde{\Phi}(s, y) \in X_0\}$ is a θ-Borel subset of S and hence has the θ-property of Baire.*

Proof. Let $\langle O_\alpha\colon \alpha < \theta\rangle$ be a descending sequence of non-empty basic open subsets of S, say $O_\alpha = \{s \in S\colon (\lambda x)\tilde{\Phi}(s, x) \restriction T_\alpha = g_\alpha\}$ for each $\alpha < \theta$. Then $\alpha < \beta$ implies that $T_\alpha \subseteq T_\beta$, $g_\alpha \subseteq g_\beta$, and $g_\beta \restriction T_\alpha = g_\alpha$. Let $U = \bigcup_{\alpha<\theta} T_\alpha$ and $h = \bigcup_{\alpha<\theta} g_\alpha$, so that $h \in {}^{U}X$. Since θ is regular and $\operatorname{card}(T_\alpha) < \theta$ for each $\alpha < \theta$, then $U \in S_{\theta^+}(Y_0)$. Now let $T \in S_\theta(U)$. Since θ is regular, there is an $\alpha_0 < \theta$ such that $T \subseteq T_{\alpha_0}$, and so for $y \in T$, if $s \in O_{\alpha_0} \neq 0$,

$$\tilde{\Phi}(s, y) = g_{\alpha_0}(y) = h(y)\,.$$

Hence by the (S, θ)-compactness of $\tilde{\Phi}$, there is an $s \in S$ such that for any $y \in U$, $\tilde{\Phi}(s, y) = h(y)$. Thus $(\lambda x)\tilde{\Phi}(s, x) \restriction U = h$, and so $s \in \bigcap_{\alpha<\theta} O_\alpha$. Thus $(S, \mathcal{T}_{\tilde{\Phi}})$ is a θ-Platek space and so by Lemma 1.1, $(S, \mathcal{T}_{\tilde{\Phi}})$ is a θ-Baire space. Now if $y \in X_0$ and $x_0 \in X_0$, $\{s \in S\colon \tilde{\Phi}(s, y) = x_0\}$ is a basic open set of S, and hence $\{s \in S\colon \tilde{\Phi}(s, y) \in X_0\} = \bigcup_{x_0 \in X_0} \{s \in S\colon \tilde{\Phi}(s, y) = x_0\}$ is a θ-Borel subset of S since $\operatorname{card}(X_0) \leqslant \theta$.

§ 3. Relations with previous notions. We examine the relation between the abstract formulation of forcing given in § 2 and some examples of forcing from the literature.

Let M be a fixed countable transitive model of ZFC and let C and $\leqslant$ be elements of M such that $(C, \leqslant)$ is a partially ordered set with largest element 1. Let ${}^{C}2$ have the product topology (2 has the discrete topology), and set $S = \{G \in {}^{C}2\colon G$ meets (G1)-(G2) of § 3 of [Sh 2]}. Then S is closed in ${}^{C}2$, so it is a compact regular space and therefore an ω-Baire space (Thm. 34, p. 200, [Kel]). Let L be the language of ZF and let $L(M)$ be the extension of L containing a constant for each element of M (cf. [Sh 1]). Take Y to be the set of sentences A (closed formulas) of $L(M)$ and set $X = \{0, 1\}$ and $X_0 = \{0\}$. (We interpret 0 as truth and 1 as falsity.) Finally, simply let $\mathcal{C}$ be the usual countable open base for the topology on S. For an arbitrary subset $G \subseteq C$, let $M[G]$ be as defined in § 4 of [Sh 2]. Then define $\Phi\colon S \times Y \to X$ by $\Phi(G, A) = 0$ iff $M[G] \models A$, where if $a \in M$ and $\boldsymbol{i}$ is the name of a in $L(M)$ (cf. [Sh 1]), then in $M[G]$, $\boldsymbol{i}$ is interpreted as a constant denoting $K_G(a) \in M[G]$ (cf. § 4 of [Sh 2]).

LEMMA 3.1. *Φ is an ω-Borel map* (cf. [Sa 2], [Ta] and [Bo 3]).

Proof. We must show that for any $A \in Y$, $\{G \in S\colon M[G] \models A\}$ is an ω-Borel set. We proceed by induction on the structure of A. If A is atomic, $Z(A)$ is open since it is either empty or all of S. If A is $\neg B$ or $B \vee C$, the induction is obvious. Let D be the set of constants in $L(M)$; D is clearly countable. Then if A is $\exists x B$,

$$Z(\exists x B) = \{G \in S\colon M[G] \models \exists x B\}$$
$$= \{G \in S\colon \text{for some } \boldsymbol{i} \in D,\ M[G] \models B_x[\boldsymbol{i}]\}$$
$$= \bigcup_{i \in D} Z(B_x[\boldsymbol{i}])\,.$$

Hence by induction, $Z(\exists x B)$ is ω-Borel.

Thus Φ is θ-Borel, hence also θ-Baire, and thus the conclusions of § 2 apply. In particular, $\mathfrak{E}$ is ω-co-meager in S. Moreover, if $\mathfrak{F} \subset \mathfrak{E}$ is generic, $G \in \mathfrak{F}$, and $A \in Y$, then

$$G \Vdash_{\mathfrak{F}} A \quad \text{iff } G \Vdash_{\mathfrak{G}} A \quad \text{iff } G \models^{\mathrm{na}} A \quad \text{iff } M[G] \models A\,.$$

This is the Truth Lemma for elements of $\mathfrak{F}$. Given $p \in C$, let $\check{p} = \{G \in S\colon p \in G\}$. Then $\check{p} \in \mathcal{C}$. Clearly $q \leqslant p \to \check{q} \subseteq \check{p}$. Thus if we define $p \Vdash_{\mathfrak{F}} A$ to mean that $\check{p} \Vdash_{\mathfrak{F}} A$ as in § 2, then the Extension Lemma holds. Finally, one shows, that $p \Vdash_{\mathfrak{G}} A$ iff $p \Vdash^{*} \neg\neg A$, where $\Vdash^{*}$ is as defined in [Sh 2], by methods similar to those used to prove Theorems 1.3.24 and 1.3.25 of [Bo 3].

Next we consider the notion of forcing in model theory as introduced in [Ro 1]. Again we take $X = \{0, 1\}$ and $X_0 = \{0\}$. Let L be a first-order language containing denumerably infinitely many individual constants and let K be a fixed non-empty consistent set of sentences of L. Let Y be the set of all sentences of L. Let At be the set of all atomic sentences of L, let $B = At \cup \{\neg A\colon A \in At\}$ (the sentences in B are called *basic*), let $\bar{S} = {}^{At}2$ with the product topology, and let $\bar{\mathcal{C}}$ be the usual countable base for this topology. An element $s \in \bar{S}$ is *consistent with* K if the set of sentences $K_s = K \cup s^{-1}(0) \cup \{\neg A\colon A \in s^{-1}(1)\}$ is consistent. Since s is inconsistent with K iff some finite subset of K_s is inconsistent, it follows that the set S of all $s \in \bar{S}$ which are consistent with K is a closed subspace of $\bar{S}$. Since $\bar{S}$ was a compact complete Hausdorff space with the property of Baire, so is S. Let $\mathcal{C} = \{C \cap S\colon C \in \bar{\mathcal{C}}\}$. Given $s \in S$, define the structure M_s to be the canonical structure for the theory K_s in the sense of [Sh]. Finally, define the map Φ by $\Phi(s, A) = 0$ iff $M_s \models A$. It is easy to verify that Φ is again an ω-Borel map. Hence the results of § 2 apply. For any finite $p \subseteq B$, we say that p is a *set of conditions* if $K \cup p$ is consistent. Let p^{+} be the set of atomic sentences in p and p^{-} be the set of negations in p. Set $\check{p} = \{s \in S\colon p^{+} \subseteq s^{-1}(0)$ and $p^{-1} \subseteq s^{-1}(1)\}$, so that $\check{p} \in \mathcal{C}$.

THEOREM 3.2. *Let L have $\neg$, $\wedge$, and $\forall$ as its logical primitives, let $p \Vdash \forall x A$ be defined to mean that for all constants c of L, $p \Vdash A_x[c]$, and for all other formulas A of L, let $p \Vdash A$ be defined as in* [Ro 1]. *Then for all sets of conditions p and all formulas A,*

$$p \Vdash A \quad \textit{iff} \quad p \Vdash_{\mathfrak{G}} A\,.$$

Proof. We proceed by induction on the number of logical symbols in A. For atomic A it is immediate since the theory K_s is complete with respect to basic sentences, and for conjunctions and universal sentences, the proof is straightforward. Now suppose that $\check{p} \Vdash_{\mathfrak{G}} \neg A$, so that $\forall s \in \mathfrak{E}[s \in \check{p} \to \neg M_s \models A]$. Let q be a set of conditions such that $p \subseteq q \subseteq B$. Since $\check{q} \cap \mathfrak{E} \neq 0$ it follows that $\neg \forall s \in \mathfrak{E}[s \in \check{q} \to M_s \models A]$ and so $\neg q \Vdash_{\mathfrak{G}} A$. By induction, we now have $\neg q \Vdash A$. Since q was arbitrary, it follows that $p \Vdash \neg A$. Finally, suppose that $p \Vdash \neg A$, and let $s \in \mathfrak{E} \cap \check{p}$. Then $s \models^{\mathrm{na}} A \vee s \models^{\mathrm{na}*} A$. Suppose $M_s \models A$. Since $s \in \mathfrak{E}$, then $s \notin \mathfrak{S}(A, \check{q})$ for any q; note that any element of $\mathcal{C}$ is of the form $\check{q}$ for some set of conditions q. Thus since $M_s \models A$, it follows that $s \models^{\mathrm{na}} A$. Let $\check{q} \in \mathcal{C}$ be such that $s \in \check{q}$ and $\check{q} \models^{\mathrm{na}} A$. Now $s \in \check{p} \cap \check{q}$ implies that $p \cup q$ is consistent and hence is a set of conditions. Moreover, if $s' \in \mathfrak{E} \cap \check{p} \cap \check{q}$, then $M_{s'} \models A$, so that $\check{p} \cap \check{q} \Vdash_{\mathfrak{G}} A$. Then by induction, $p \cup q \Vdash A$, which contradicts the assumption that $p \Vdash \neg A$. Thus given $s \in \check{p} \cap \mathfrak{E}$ we must have $M_s \models \neg A$, and it follows that $\check{p} \Vdash_{\mathfrak{G}} \neg A$.

The notion of infinite forcing as introduced in [Ro 2] can also be treated within the present framework. Let K be a consistent set of sentences in a countable language L and let Σ_K be as in [Ro 2]. Let $S = \Sigma_K$ with the *elementary topology*; i.e., the open sets are the EC_Δ subclasses of S. Since these classes are easily seen to also be closed, S is regular. Let $\bar{K}$ be the set of universal closures of formulas A such that A is a disjunction of negations of open formulas and $\vdash_K A$. Then by the Model Extension Theorem ([Sh 1], p. 75), a structure M belongs to Σ_K iff M is a model of $\bar{K}$. If $\mathfrak{M}$ is the class of all structures for L and is also equipped with the elementary topology, it thus follows that Σ_K is a closed subspace of $\mathfrak{M}$. Since $\mathfrak{M}$ is compact by the Compactness Theorem, it follows that S is also compact. Hence ([Kel], p. 200), S is an ω-Baire space. Let $\overline{Y}$ be the set of sentences of L and let $X = \{0, 1\}$ and $X_0 = \{0\}$. Finally, for $M \in S$ and $A \in Y$, define $\Phi(M, A) = 0$ iff $M \models A$. Then as in Lemma 3.1 it is easy to verify that Φ is an ω-Borel map. Consequently, taking $\mathfrak{C}$ to be the collection of EC subclasses of Σ_K, it follows that the results of § 2 apply and in particular, just as above, for any $M \in \Sigma_K$, $M \mid\mid\!= A$ iff $M \Vdash_{\mathfrak{G}} A$ where $\mid\mid\!=$ is defined in [Ro 2].

Whether or not the construction in § 1 of [Sa 1] can be subsummed under the present point of view is unclear. That such forcing constructions as [Lé] and [Ea] can be treated from the present point of view can be seen by examining [Bo 3].

As our next example, we consider the construction in [Kei] with which we assume familiarity. Let the fragment L_A, the set K_A, and the forcing property $\mathcal{F} = \langle P, \leqslant, f\rangle$ be fixed. Given $s \in {}^P2$, let $\hat{s} \subseteq P$ be $\hat{s} = \{p \in P\colon s(p) = 0\}$. Let B be the set of basic sentences of K_A and for $s \in {}^P2$, set

$$T_s = \bigcup \{f(p)\colon p \in s\} \cup \{\neg A\colon A \in B \text{ and } A \notin \bigcup \{f(p)\colon p \in s\}\}.$$

Let $S = \{s \in {}^P2\colon T_s$ is consistent$\}$. Since each T_s is an $L_{\omega\omega}$-theory, we can use the Compactness Theorem for $L_{\omega\omega}$ to conclude that S is a closed subspace of ${}^P2$, where ${}^P2$ is given the product topology and 2 the discrete topology. Since ${}^P2$ is compact and regular, so is S and hence S is an ω-Baire space. Now let $Y = K_A$, $X = 2$ and $X_0 = 1$. For $s \in S$, let M_s be the canonical model for the theory T_s in the sense of [Sh 1]. Since T_s is an $L_{\omega\omega}$-theory and is complete for basic sentences, M_s is a model of T_s. Finally, define Φ as usual by $\Phi(s, A) = 0$ off $M_s \models A$. Then as above, it is easy to verify that Φ is ω-Borel, and hence the conclusions of § 2 apply.

As our final example we consider the construction of § 2 of [Fe] with which we assume familiarity. Let δ and L^* be fixed. Let K be the set of true basic sentences of L (i.e., true in the standard model of arithmetic), and let At and B be the sets of atomic and basic sentences of L^*, respectively. For $s \in {}^{At}2$, let $K_s = K \cup s^{-1}(0) \cup \{\neg A\colon A \in s^{-1}(1)\}$, and let $S = \{s \in {}^{At}2\colon K_s$ is consistent$\}$. As usual, equipping 2 with the discrete topology and ${}^{At}2$ with the product topology, it follows by use of the Compactness Theorem that S is a closed subspace of ${}^{At}2$. Hence S is compact and regular and as such, S is an ω-Baire space. Let Y be the set of sentences of L^* and let $X = 2$ and $X_0 = 1$. For $s \in S$, let $\mathfrak{N}_s$ be that structure for L^* such that the restriction of $\mathfrak{N}_s$ to L is just the standard model of arithmetic and for each $n < \delta$, the interpretation of S_n in $\mathfrak{N}_s$ is

$$S_n^{\mathfrak{N}_s} = \{k \in \omega\colon \text{the formula } \bar{k} \in S_n \text{ occurs in } K_s\}.$$

Finally define Φ by $\Phi(s, A) = 0$ iff $\mathfrak{N}_s \models A$. As usual, it is easy to verify

that Φ is ω-Borel, and hence the results of § 2 apply. For finite $p \subseteq B$, let $\check{p} = \{s \in S\colon p \subseteq K_s\}$, and define $p \Vdash_{\mathfrak{G}} A$ iff $\check{p} \Vdash_{\mathfrak{G}} A$. Then it is easy to verify that

$$p \Vdash_{\mathfrak{G}} A \quad \text{iff} \quad p \Vdash A,$$

where the latter is as defined in [Fe]. The construction in § 3 of [Fe] can be treated similarly.

§ 4. Forcing for extended first order languages. Forcing for infinitary languages has been considered in [Kei], where primarily countable fragments of $L_{\omega_1\omega}$ are considered, and [Cov], where Robinson's notion of infinite forcing is extended to the languages $L_{\theta\omega}$ with θ regular. Regarding forcing as a method for constructing extensions of structures, we will use the machinery of § 2 to construct an analogue of finite forcing for some of the languages $L_{\theta\omega}$ which extends to the languages $L_{\theta\theta}$ under suitable assumptions on θ.

Let σ be a fixed first order similarity type of power $\leqslant \theta$ where θ is regular and $\theta^{\underline{\omega}} = \theta$, and let L be the $L_{\theta\omega}$-language of similarity type σ. Let K be a fixed consistent set of basic sentences of L and let C be a set of new individual constants with power $\leqslant \theta$. Let L' be the language obtained by adding the elements of C to L and let At and B be the sets of atomic and basic sentences of L', respectively. Equip ${}^{At}2$ with the θ-product topology: the basic open sets are those of the form $\{s \in {}^{At}2\colon s \restriction D = t \restriction D\}$, where $D \in S_\theta({}^{At}2)$ and $t \in {}^{At}2$. For $s \in {}^{At}2$, set

$$K_s = K \cup s^{-1}(0) \cup \{\neg A\colon A \in s^{-1}(1)\}.$$

Let $S = \{s \in {}^{At}2\colon K_s$ has a model$\}$. Since the theories K_s all consist of basic sentences, the ordinary Compactness Theorem for $L_{\omega\omega}$ applies to them and it follows that S is a closed subspace of ${}^{At}2$. From this it is not difficult to see that S is a θ-Platek space and hence by Lemma 1.1 is a θ-Baire space. Let $X = 2$ and $X_0 = 1$, and let Y be the set of sentences of L'. For each $s \in S$, let M_s be the canonical structure for K_s as defined in [Sh 1]. Since K_s is an $L_{\omega\omega}$-theory which is complete for basic sentences, then M_s is a model of K_s. Finally, define the map Φ by $\Phi(s, A) = 0$ iff $M_s \models A$. Note that $\operatorname{card}(Y) = \theta$ since $\theta^{\underline{\omega}} = \theta$.

LEMMA 4.1. *Φ is a θ-Borel map.*

Proof. By induction on the number of logical symbols in A. By definition, if A is atomic, $Z(A)$ is open, and for any A, $Z(\neg A) = S - Z(A)$. Also $Z(\bigwedge_{\alpha<\beta} A_\alpha) = \bigcap_{\alpha<\beta} Z(A_\alpha)$ and by definition of $L_{\theta\omega}$, we must have $\beta < \theta$. Finally, $Z(\forall x A) = \bigcap_{c \in C'} Z(A_x[c])$, where C' is the set of all individual constants of L'. This follows since each M_s with $s \in S$ is a canonical model so that each element of M_s is denoted by some element of C'. But since the cardinality of the similarity type of L is $\leqslant \theta$ and $\text{card}(C) \leqslant \theta$, then $\text{card}(C') \leqslant \theta$, and we are done.

Thus the results of § 2 apply. Now if M is a fixed structure and L includes as constants names for each element of M and K is the diagram of M (i.e., the set of basic sentences true in M), then each M_s with $s \in S$ is an extension of M. Note that M may have cardinality as large as θ, and that $\text{card}(M) \leqslant \text{card}(M_s) \leqslant \theta$. The only place we have made use of the hypothesis $\theta^{\underset{\smile}{\theta}} = \theta$ is to guarantee that $\text{card}(Y) = \theta$. Thus an alternative to the assumption that $\theta^{\underset{\smile}{\theta}} = \theta$ would be that Y was a set of formulas of $L_{\theta\omega}$ of power θ which contained all basic formulas, was closed under $\neg$, finite $\wedge$, $\forall$, and substitution of terms for variables, and contained all subformulas of its members.

A *fragment* of $L_{\theta^+\omega}$ is a set of formulas of $L_{\theta^+\omega}$ which contains all of $L_{\theta\omega}$, is closed under $\neg$, $\forall \boldsymbol{x}$, $\bigwedge$ of length $< \theta$, and substitution of terms for variables, and contains all subformulas of its elements. Then the development above goes through if Y is a fragment of $L_{\theta^+\omega}$ with $\text{card}(Y) = \theta$. (The existence of such a fragment implies $\theta^{\underset{\smile}{\theta}} = \theta$.) For the only alteration necessary in the proof of Lemma 4.1 is to observe that $\beta \leqslant \theta$ instead of $\beta < \theta$.

Finally, assume that $\theta^{\underset{\smile}{\theta}} = \theta$ and that instead of confining ourselves to $L_{\theta\omega}$, we allow Y to consist of all sentences which can be formulated in $L_{\theta\theta}$ using the constants of L'. Then we still have $\text{card}(Y) = \theta$ and Lemma 4.1 still holds. The proof proceeds as before with the addition of a new case. If $f \in {}^{\beta}C'$, let $A_{\langle x_\alpha : \alpha < \beta\rangle}[f]$ indicate the simultaneous substitution of $f(\alpha)$ for $\boldsymbol{x}_\alpha$ for all $\alpha < \beta$. Then as before, we have

$$Z(\forall \langle \boldsymbol{x}_\alpha\colon \alpha < \beta\rangle A) = \bigcap_{f \in {}^{\beta}C'} Z(A_{\langle x_\alpha : \alpha<\beta\rangle}[f]),$$

and $\text{card}({}^{\beta}C') = \theta^{\beta} \leqslant \theta^{\underset{\smile}{\theta}} = \theta$.

Two further extensions are possible. If θ is inaccessible and $\text{card}(C') < \theta$, then we can allow the language to contain arbitrary dependent quantifier expressions $\langle \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}\rangle$ in the sense of [Ma] with $\text{card}(\boldsymbol{X} \cup \boldsymbol{Y}) < \theta$. For then in the proof of Lemma 4.1 we have

$$Z(\langle \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{f}\rangle \boldsymbol{B}) = \bigcup_{\varrho \in Q} \bigcap_{h \in R} \Big(\bigcup_{y \in Y} \{s \in \boldsymbol{S}\colon h(y) \neq \varrho(\boldsymbol{y})(h \restriction \boldsymbol{f}(\boldsymbol{y}))\} \cap Z(\boldsymbol{B}_h)\Big),$$

where

$$Q = {}^{Y}\big(\bigcup_{y \in Y} {}^{(f(y)}{}^{C')}C'\big),$$

$R = {}^{(X \cup Y)}C'$, and $\boldsymbol{B}_h$ is the result of simultaneously substituting $h(\boldsymbol{x})$ for $\boldsymbol{x}$ and $h(\boldsymbol{y})$ for $\boldsymbol{y}$ in $\boldsymbol{B}$, for all $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{y} \in \boldsymbol{Y}$. By induction, $Z(\boldsymbol{B}_h)$ is θ-Borel. Moreover,

$$\{s \in \boldsymbol{S}\colon h(\boldsymbol{y}) \neq \varrho(\boldsymbol{y})(h \restriction \boldsymbol{f}(\boldsymbol{y}))\}$$

is either empty or all of $\boldsymbol{S}$, and hence is θ-Borel. Since $\text{card}(\boldsymbol{X} \cup \boldsymbol{Y}) < \theta$ and $\text{card}(C') < \theta$, then $\text{card}(R) < \theta$ since θ is inaccessible. Since for each $\boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{f}(\boldsymbol{y}) \subseteq \boldsymbol{X}$, it follows similarly that $\text{card}(Q) < \theta$, and so we are done.

For the second extension, we simply assume that θ is regular with $\theta^{\underset{\smile}{\theta}} = \theta$. Then we can allow our underlying language to contain the equicardinality quantifier Q:

$$M \models Q\boldsymbol{x}A \quad \text{iff} \quad \text{card}\{m \in M\colon M \models A_x[\boldsymbol{m}]\} = \text{card}(M).$$

For, once more, in the proof of Lemma 4.1, we would have

$$Z(Qx B) = \bigcup_{\mu<\theta} (\{s \in \boldsymbol{S}\colon \text{card}\, m \in \boldsymbol{M}_s\colon \boldsymbol{M}_s \models \boldsymbol{B}[\boldsymbol{m}] = \mu\} \cap \{s \in \boldsymbol{S}\colon \text{card}(\boldsymbol{M}_s) = \mu\}),$$

where $\boldsymbol{B}[\boldsymbol{m}]$ is $\boldsymbol{B}_x[\boldsymbol{m}]$. Now for $\mu < \theta$,

$$\{s \in \boldsymbol{S}\colon \text{card}(M_s) = \mu\}$$
$$= \bigcup_{h \in {}^{\mu}C'} \Big(\bigcap_{\alpha<\mu} \bigcap_{\beta<\mu} \big(\{s \in \boldsymbol{S}\colon \alpha = \beta\} \cup Z(h(\alpha) \neq h(\beta))\big) \cap \bigcap_{c \in C'} \bigcup_{\alpha<\mu} Z(c = h(\alpha))\Big),$$

while

$$\{s \in \boldsymbol{S}\colon \text{card}(M_s) = \theta\} = \bigcap_{\mu<\theta} \bigcap_{h \in {}^{\mu}C'} \bigcup_{c \in C'} \bigcap_{\alpha<\mu} Z(c \neq h(\alpha)).$$

It follows from $\text{card}(C') \leqslant \theta$ and $\theta^{\underset{\smile}{\theta}} = \theta$ that both of these are θ-Borel. Again, for $\mu < \theta$

$$\{s \in \boldsymbol{S}\colon \text{card}\{m \in M_s\colon M_s \models B[\boldsymbol{m}]\} = \mu\}$$
$$= \bigcup_{h \in {}^{\mu}C'} \Big(\bigcap_{\alpha<\mu} Z(\boldsymbol{B}[h(\alpha)]) \cap \bigcap_{\alpha<\mu} \bigcap_{\beta<\mu} P_{\alpha,\beta} \cap \bigcap_{c \in C'} N_c^{\mu}\Big),$$

where

$$P_{\alpha,\beta} = \{s \in S\colon \alpha = \beta\} \cup Z(h(\alpha) \neq h(\beta)),$$

and

$$N_c^{\mu} = \big(\boldsymbol{S} - Z(\boldsymbol{B}[c]) \cup \bigcup_{\alpha<\mu} Z(c = h(\alpha))\big).$$

As above, these can be seen to be θ-Borel. And finally,

$$\{s \in \boldsymbol{S}\colon \text{card}\{m \in M_s\colon M_s \models \boldsymbol{B}[\boldsymbol{m}]\} = \theta\}$$
$$= \bigcap_{\mu<\theta} \bigcap_{h \in {}^{\mu}C'} \bigcap_{c \in C'} \Big(Z(\boldsymbol{B}[c]) \cap \bigcap_{\alpha<\mu} Z(c \neq h(\alpha))\Big),$$

and this is also θ-Borel. It now follows that $Z(Qx\boldsymbol{B})$ is θ-Borel. Thus the results of § 2 will still apply.

§ 5. Forcing for second order logic. Given any fixed first order language L in the sense of [Sh 1], the corresponding second order language L^2 is obtained from L by adding variables

$$X, Y, Z, X', Y', Z', X'' \ldots$$

for sets (or unary predicates) together with the following additional formation rules (using X as a metavariable for the new second order variables).

o) If a is any term and X any set variable, then $a \in X$ is an atomic formula.

v) If u is a formula, then $\exists X u$ is a formula.

The system of *basic second order logic* is obtained from first order logic (cf. [Sh 1]) by adding the following axioms and rules:

Second order substitution axioms:

$$A_X[U] \to \exists X A .$$

Second order $\exists$-Introduction rule:

If X is not free in B, infer $\exists X A \to B$ from $A \to B$.

A *second order structure* $\mathfrak{A} = \langle A, \mathcal{A}, \ldots \rangle$ for L^2 consists of a first order structure $\langle A, \ldots \rangle$ for L and a set $\mathcal{A}$ of subsets of A.

We will consider an extension of the foregoing notion of second order logic. Specifically, in addition to the nonlogical symbols already permitted, we will permit symbols whose intended interpretations are certain relations and functionals which may take either first or second order entities as arguments, and the functionals may yield first or second order entities as values. In specifying a second order language L^2 we associate with each relation symbol (functional symbol) an integer n and an n-tuple ($(n+1)$-tuple) of 0's and 1's called its *index*. The formation rules for terms are then as follows:

(i) any variable $x(X)$ is a term of type 0 (1),

(ii) if f is n-ary with associated index $\langle i_1, \ldots, i_n, i_{n+1} \rangle$ and $u_1, \ldots, u_n$ are terms of types $i_1, \ldots, i_n$, respectively, then $fu_1 \ldots u_n$ is a term of type i_{n+1}.

The formation rules for *atomic formulas* are then:

(i) if u and v are terms of the same type, then $= uv$ (written $u = v$) is an atomic formula,

(ii) if u and v are terms of types 0 and 1, respectively, then $\in uv$ (written $u \in v$) is an atomic formula,

(iii) if p is an n-ary relation symbol with associated index $\langle i_1, \ldots, i_n \rangle$ and $u_1, \ldots, u_n$ are terms of types $i_1, \ldots, i_n$, respectively, then $pu_1 \ldots u_n$ is an atomic formula.

The formation rules for formulas are as before. In the second order substitution axioms, U is now regarded as ranging over terms of type 1. We also add all the appropriate instances of *identity* and *equality* axioms (cf. [Sh 1]). A second order theory T is specified wheni ts language $L^2(T)$ is specified and its nonlogical axioms are specified. Structures for such enlarged languages are as before except that as in the first order case,

they must provide interpretations of the relation and functional symbols of the obvious sort; equality between entities of type 1 is interpreted extensionally.

The obvious analogues of the major theorems of first order logic are known to extend to the usual second order logic and they are easily shown to extend to the present context. In particular, analogues of the following (cf. [Sh 1]) hold: the Validity, Tantology, Deduction, Equivalence, and Equality Theorems, as well as the Theorem on Constants. The canonical structure for a given theory T is defined just as for the first order case, as are the notions of extension and conservative extension for second order theories and expansion and restriction for second order structures. Using the model-theoretic characterization of the notions of extension and conservative extension afforded by the analogues of Exercise 3, p. 65 of [Sh 1], it is easy to prove that the canonical structure for a complete second order Henkin theory T is a model of T, that any second order theory T has a conservative Henkin extension, and hence that any T can be extended to a complete Henkin theory, thus yielding the Henkin Completeness Theorem. Similarly one proves the analogue of the Theorem on Functional Extensions ([Sh 1], p. 55) by model-theoretic means and thus obtains a version of Skolem's Theorem: Any second order theory T has an open conservative extension in basic second order logic. This turns out to be pivotal in extending forcing to second order theories.

Many formulations of second order logic explicitly or implicitly contain various *comprehension schemata* as logical axioms. For our purposes it is necessary to regard these as nonlogical axioms. In particular, we say that a theory T formulated in a second order language L is *impredicative* if all instances of the schema

$$(*) \qquad \exists X \forall y [y \in X \leftrightarrow A]$$

are provable from T in basic second order logic, where A ranges over all formulas of L not containing the variable X.

Now let T be a fixed countable second order theory formulated in a countable second order language L with equality, and let $\mathfrak{A}$ be a fixed countable second order model of T. Using Skolem's Theorem, let T' be an open conservative extension of T and let $\mathfrak{A}'$ be an expansion of to a model of T' (using the second order analogue of Exercise 3b, p. 65, [Sh 1]). Let $I = \{c_n: n < \omega\}$ be a countable set of new individual constants and let $S = \{S_n: n < \omega\}$ be a countable set of new set (i.e., second order) constants. Let L^* be the language obtained from $L(T')$ by adding the following:

(i) distinct constants for each individual and set in the structure $\mathfrak{A}'$.

(ii) the elements of $I \cup S$ (it is assumed that $I \cap S = 0$). Let Ax^*

consist of all closed instances of axioms of T' in L^* together with all closed quantifier-free formulas of $L(T')$ true in $\mathfrak{A}'$, and let At be the set of closed atomic formulas of L^*. Giving 2 the discrete topology, it follows that ${}^{At}2$ equipped with the product topology is compact and regular. For $s \in {}^{At}2$, let $T'_s = Ax^* \cup s^{-1}(0) \cup \{\neg A\colon A \in s^{-1}(1)\}$, and let S be the collection of $s \in {}^{At}2$ for which T_s is consistent in basic second order logic. By the Henkin Completeness Theorem it follows that S is a closed subset of ${}^{At}2$, and hence with the subspace topology S is itself compact and regular. Thus S is an ω-Baire space and if $\mathfrak{C}$ is the usual basis for the subspace topology on S, then $\operatorname{card}(\mathfrak{C}) = \aleph_0$. Let $X = 2$, $X_0 = 1$, and let Y be the set of all sentences of L^*. For $s \in S$, let M_s be the structure for L^* obtained by converting the canonical structure for T_s in the sense described above (cf. [Sh 1]) to an extensional structure in which $=$ and $\in$ have their standard interpretations. The main step in this consists in replacing the equivalence class $U^0 = \{V\colon \vdash_{T'_s} U = V\}$, where U is a variable-free second-order term, by the set of all equivalence classes a^0 of first order terms a such that for some $b \in a^0$ and $V \in U^0$, $\vdash_{T'_s} b \in V$. Then define the map Φ by $\Phi(s, A) = 0$ iff $M_s \models A$. Then by using arguments similar to those used in earlier sections it follows that Φ is an ω-Borel map, and thus all the results of § 2 apply in this setting. Let $\mathfrak{E}$ be as defined in § 2.

Now for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, let A_i and B_j be closed atomic formulas of L^* and let C be the sentence

$$A_1 \vee \ldots \vee A_n \vee \neg B_1 \vee \ldots \vee \neg B_m .$$

Let $s \in \mathfrak{E}$. It is easy to see that if D is an atomic sentence or the negation of an atomic sentence of L^* and $\vdash_{T'_s} D$, then $M_s \models D$. We claim that if $\vdash_{T'_s} C$, then $M_s \models C$. By the remark above it would suffice to show that either for some i, $\vdash_{T'_s} A_i$, or for some j, $\vdash_{T'_s} \neg B_j$. Suppose that for no i do we have $\vdash_{T'_s} A_i$. Then since T'_s is complete with respect to the basic formulas of L^*, it follows that for $i = 1, \ldots, n$, we have $\vdash_{T'_s} \neg A_i$. Using $\vdash_{T'_s} C$, we have $\vdash_{T'_s} \neg B_1 \vee \ldots \vee \neg B_n$. Now since T'_s is consistent (since $\mathfrak{E} \subseteq S$), it follows that for some j, B_j is not provable in T'_s. But then again, since T'_s is consistent and complete with regard to basic formulas, we have $\vdash_{T'_s} \neg B_j$, as desired.

Now let D be an element of Ax^* and let $s \in \mathfrak{E}$. Let D' be logically equivalent to D where D' is in conjunctive normal form, say D' is $C_1 \wedge \ldots \wedge C_p$. Now by definition of T'_s, $\vdash_{T'_s} D$ and so $\vdash_{T'_s} D'$. Hence for $k = 1, \ldots, p$, $\vdash_{T'_s} C_k$. But the remarks above apply to each C_k, and hence it follows that $M_s \models D$. Thus $M_s | L(T')$ is a model of T' and since T' is a conservative extension of T, then $M_s | L(T)$ is a model of T. Moreover, since the diagram of $\mathfrak{A}'$ is included in Ax^*, it follows in the usual way that $M_s | L(T)$ can be regarded as an extension of $\mathfrak{A}$. Thus we see that for each $s \in \mathfrak{E}$, $M_s | L(T)$ is an extension of $\mathfrak{A}$ which is a model of T.

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