first half of that case would go through: if $x \in U_{n-1}$, then by the same token as in the case of $\wedge$, $\mathfrak{U}_n \models_x \beta \vee \gamma$ iff $\mathfrak{U}_{n-1} \models_x \beta \vee \gamma$. The breakdown comes in the latter half: if $x \notin U_{n-1}$, then, if y_0 and y_1 are the immediate descendants of x, there is, in general, nothing to prevent that

$$\mathfrak{U}_n \models_{y_0} \beta\,, \quad \mathfrak{U}_n \not\models_{y_1} \beta\,, \quad \mathfrak{U}_n \not\models_{y_0} \gamma\,, \quad \mathfrak{U}_n \models_{y_1} \gamma\,,$$

whence

$$\mathfrak{U}_n \models_{y_0} \beta \vee \gamma\,, \quad \mathfrak{U}_n \models_{y_1} \beta \vee \gamma\,,$$

and

$$\mathfrak{U}_n \not\models_x \beta \vee \gamma\,.$$

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A fine hierarchy of partition cardinals

by

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Abstract. A modification is made to the definition of a partition cardinal; it is shown that this modified definition gives a finer hierarchy than that used, e.g., in Silver's thesis. Some properties are proved and an application to the constructible universe is given.

1. Introduction. Most of the definitions in use for large cardinals are properties which single out certain cardinals, which from the point of view of all cardinals are few and far between. This in the case, for example, with weakly or strongly compact cardinals, measurable cardinals, or the various sorts of indescribable cardinals — for all of these, if it seems natural to assume the existence of any of them, it would seem almost equally natural (but at the same time a definite extension) to assume the existence of many of them — perhaps even a proper class of them.

The partition cardinals introduced by Erdös and exploited by Rowbottom, Silver and others, stand out from the other large cardinals in this respect. The cardinal $\varkappa(\alpha)$, for limit ordinal α, is defined as the least cardinal $\varkappa$ such that $\varkappa \to (\alpha)_2^{<\omega}$ (the notation is explained in section 2), and for any $\varkappa' \geqslant \varkappa(\alpha)$, $\varkappa' \to (\alpha)_2^{<\omega}$ also holds. It is the purpose of this note to show how the definition of $\varkappa(\alpha)$ can be modified to yield a notion of "α-partition cardinal" which is more analogous to the other large cardinal definitions in that it is a definite extension to assume that there are many such cardinals, and it is relatively consistent to assume that there is only one (or indeed any fixed ordinal number of them). Of course, the usual definition gives the hierarchy $\{\varkappa(\alpha) \mid \alpha \text{ a limit ordinal}\}$; our hierarchy will be a refinement of this one. An application to the constructible universe is given.

An earlier parallel is provided by measurable cardinals: at first the question was asked, whether there could be a countably additive, 2-valued measure defined on all subsets of a set, and a cardinal was called measurable if it admitted such a measure. Today we would say that this definition applies to all cardinals $\geqslant$ the first measurable cardinal, and we call a cardinal $\varkappa$ measurable only if it carries a $\varkappa$-additive measure. Our procedure is suggested by this change.

2. Preliminaries. We shall use the usual set-theoretic notations as in [1]. In particular, the partition symbol $\varkappa \to (\alpha)^{<\omega}_{\lambda}$, where $\varkappa$, λ and α are any ordinals, means that for any function $f\colon [\varkappa]^{<\omega} \to \lambda$ (where $[\varkappa]^{<\omega}$ is the set of finite subsets of $\varkappa$), there is a set $X \subset \varkappa$, with order-type α, which is *homogeneous* for f, i.e. for any $x, y \subset X$ with $\bar{\bar{x}} = \bar{\bar{y}} < \omega$, $f(x) = f(y)$. We write $\varkappa \not\to (\alpha)^{<\omega}_{\lambda}$ if this fails. (We want this notion mainly for limit α, but the definition makes sense for any α.)

For a structure $\mathfrak{A}$ with universe A containing a linearly ordered subset Y, a set $X \subset Y$ is called *indiscernible* over $\mathfrak{A}$ if for any two finite subsets of X, $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$, and any formula φ of the (first order) language for $\mathfrak{A}$, with just n free variables, we have

$$\mathfrak{A} \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \mathfrak{A} \models \varphi(y_1, \ldots, y_n)\,.$$

Some basic results on these notions are the following (for proofs see [1]):

LEMMA 1 (Silver). *$\varkappa \to (\alpha)^{<\omega}_{2^\lambda}$ iff every structure $\mathfrak{A}$ with a well-ordered subset of order-type $\varkappa$, and with the length of $\mathfrak{A} \leqslant \lambda$, has a set of indiscernibles of order-type α.*

LEMMA 2 (Rowbottom). *If α is a limit ordinal and $\varkappa \to (\alpha)^{<\omega}_{\lambda}$, then also $\varkappa \to (\alpha)^{<\omega}_{\lambda^\omega}$.*

LEMMA 3 (Erdös, Hajnal). *If $\varkappa$ is the first cardinal such that $\varkappa \to (\alpha)^{<\omega}_{\lambda}$, then $\varkappa \not\to (\alpha+1)^{<\omega}_{\lambda}$.*

Another basic result is due to Jensen; a proof is given in [2], but we give a more direct proof here:

LEMMA 4 (Jensen). *If α is a limit ordinal and $\varkappa \to (\alpha)^{<\omega}_{\lambda}$ then also $\varkappa \to (\alpha)^{<\omega}_{2^\lambda}$.*

Proof. Let $f\colon [\varkappa]^{<\omega} \to {}^{\lambda}2$, and define $g\colon [\varkappa]^{<\omega} \to \lambda+1$ as follows: suppose $x \subset \varkappa$ has $2n$ members $x_1 < \ldots < x_n < x_{n+1} < \ldots < x_{2n}$; then $g(x)$ is the largest ordinal $\gamma \leqslant \lambda$ such that for all $\beta < \gamma$,

$$f(\{x_1, \ldots, x_n\})(\beta) = f(\{x_{n+1}, \ldots, x_{2n}\})(\beta)\,.$$

(i.e. γ is the first argument where $f(\{x_1, \ldots, x_n\})$ and $f(\{x_{n+1}, \ldots, x_{2n}\})$ differ, or λ if they don't differ).

For $x \subset \varkappa$ of odd cardinal, g is arbitrary (set $g(x) = 0$). Now by Lemma 2 we can assume $\lambda \geqslant \omega$, so $\lambda+1 \sim \lambda$, and since by assumption $\varkappa \to (\alpha)^{<\omega}_{\lambda}$, there must be a set $X \subset \varkappa$ of order type α, homogeneous for g. We show that X must be homogeneous for f also.

For let $x, y \subset X$ have $\bar{\bar{x}} = \bar{\bar{y}} < \omega$, and suppose first that x and y are disjoint and do not overlap in the ordering of $\varkappa$; suppose x comes before y. Since α is a limit ordinal, we can pick $z \subset X$ above both x and y with $\bar{\bar{x}} = \bar{\bar{z}}$, and we shall have

$$g(x \cup y) = g(x \cup z) = g(y \cup z) = \gamma\,, \quad \text{say}\,.$$

But if $\gamma < \lambda$, this means that $f(x)$, $f(y)$ and $f(z)$ are members of ${}^{\lambda}2$ which agree for $\beta < \gamma$ but all differ at γ; and this is impossible: so we must have $\gamma = \lambda$ and $f(x) = f(y) = f(z)$. Similarly if x and y overlap, we can take $z \subset X$ above both of them and we have $f(x) = f(z) = f(y)$; and hence X is homogeneous for f.

2. α-partition cardinals. We now give the basic definition of α-partition cardinals for a limit ordinal α:

DEFINITION. If α is a limit ordinal, $\varkappa$ is an *α-partition cardinal* if for some $\lambda \geqslant 2$, $\varkappa$ is the first cardinal such that $\varkappa \to (\alpha)^{<\omega}_{\lambda}$.

With this definition it is clear that $\varkappa(\alpha)$ as defined in [3] or [1], is the first α-partition cardinal, and the second will be the first cardinal μ such that $\mu \to (\alpha)^{<\omega}_{\varkappa(\alpha)}$. The main justification for this definition is the following list of properties:

THEOREM 5. *If α is a limit ordinal, and $\varkappa$ is an α-partition cardinal, then*

(i) *$\varkappa$ is strongly inaccessible;*

(ii) *$\varkappa \to (\alpha)^{<\omega}_{\lambda}$ holds for every $\lambda < \varkappa$;*

(iii) *If $\mathfrak{A}$ is any structure of length $< \varkappa$, with a well-ordered subset of order-type $\varkappa$, then $\mathfrak{A}$ has a set of indiscernibles of order-type α.*

Proof. (i) follows by exactly the same arguments which prove that the first α-partition cardinal, $\varkappa(\alpha)$ is strongly inaccessible: namely, one shows that if $\delta \not\to (\alpha)^{<\omega}_{\lambda}$ then also $2^\delta \not\to (\alpha)^{<\omega}_{\lambda}$ (when α is a limit ordinal); and if ζ is singular, say $\zeta = \bigcup_{\beta<\delta} \xi_\beta$, where each $\xi_\beta \not\to (\alpha)^{<\omega}_{\lambda}$ and also $\delta \not\to (\alpha)^{<\omega}_{\lambda}$, then $\zeta \not\to (\alpha)^{<\omega}_{\lambda}$. The proofs are standard (see e.g. [1]), and we shall not repeat them. (ii) is also proved by an extension of the proof (due to Silver) for the first α-partition cardinal: since this is a slight variation, we give it here.

So suppose that ζ is a cardinal $\geqslant 2$ such that $\varkappa$ is the first cardinal such that $\varkappa \to (\alpha)^{<\omega}_{\zeta}$, and that $\lambda < \varkappa$. Then $\lambda \not\to (\alpha)^{<\omega}_{\zeta}$, and so let $g\colon [\lambda]^{<\omega} \to \zeta$ be a function with no homogeneous subset of order-type α. Let f be any function $f\colon [\varkappa]^{<\omega} \to \lambda$.

For each $n < \omega$, let $f_n(a_1, \ldots, a_n) = f(\{a_1, \ldots, a_n\})$ if $a_1 < \ldots < a_n$, and 0 otherwise; and similarly let $g_n(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$ if $a_1 < \ldots < a_n < \lambda$, and 0 otherwise. Let $\mathfrak{A}$ be the structure

$$\langle \varkappa, <, (f_n)_{n<\omega}, (g_n)_{n<\omega}, (\xi)_{\xi<\zeta} \rangle\,.$$

Now by Lemmas 2 and 4, $\varkappa \to (\alpha)^{<\omega}_{2^\zeta}$ and $\varkappa \to (\alpha)^{<\omega}_{2^\omega}$; and so by Lemma 1, since the length of $\mathfrak{A}$ is $\max(\omega, \zeta)$, there is a subset of $\varkappa$, X say, which is a set of indiscernibles for $\mathfrak{A}$, and X has order-type α. (ii) will follow when we show that X is homogeneous for f.

So fix $n < \omega$, $n \neq 0$; since $n\alpha = \alpha$ (α is a limit ordinal), we can split X into a sequence $(x_\beta)_{\beta<\alpha}$ of non-overlapping subsets, each of cardinal n, such that if $\beta < \beta' < \alpha$, each member of x_β precedes each member of $x_{\beta'}$. By indiscernibility of X in $\mathfrak{A}$, using the operation f_n of $\mathfrak{A}$, we must have

(a) $f(x_\beta) < f(x_{\beta'})$ for all $\beta < \beta' < \alpha$, or

(b) $f(x_\beta) = f(x_{\beta'})$ for all $\beta < \beta' < \alpha$, or

(c) $f(x_\beta) > f(x_{\beta'})$ for all $\beta < \beta' < \alpha$.

(c) is clearly impossible since each $f(x_\beta)$ is an ordinal, and (c) would imply an infinite decreasing sequence. We want to show that (b) holds for each n: then a further application of indiscernibility shows that X is homogeneous for f, and (ii) will follow. So suppose that (a) holds, and let $\gamma_\beta = f(x_\beta)$, $Y = \{\gamma_\beta |\ \beta < \alpha\}$. Then Y will be a subset of λ of order-type α; we show that Y is homogeneous for g, which contradicts the assumption that g has no such homogeneous set. For let $z, z' \subset Y$ be finite subsets of the same cardinal, m say; suppose z has members $\gamma_{\beta_1} < \ldots < \gamma_{\beta_m}$, and z' has members $\gamma_{\beta'_1}, \ldots, \gamma_{\beta'_m}$. Then if $\xi = g(z)$, then

$$\mathfrak{A} \models g_m(f_n(\vec{x}_{\beta_1}), \ldots, f_n(\vec{x}_{\beta_m})) = \underline{\xi}\,,$$

where $\underline{\xi}$ is the constant symbol denoting ξ, and $\vec{x}_\beta$ is the list of members of x_β in ascending order.

But now by indiscernibility of X in $\mathfrak{A}$,

$$\mathfrak{A} \models g_m(f_n(\vec{x}_{\beta'_1}), \ldots, f_n(\vec{x}_{\beta'_m})) = \underline{\xi} \quad \text{also}\,,$$

i.e. $g(z') = \xi$, and Y is homogeneous for g, giving the required contradiction.

(iii) now follows from (ii) using Lemmas 1, 2 and 4.

Property (ii) or (iii) of Theorem 5 might also be used as a definition of an α-partition cardinal; but this seems less appropriate since it does not imply regularity. It is easy to see that any limit of cardinals satisfying (ii) will also satisfy (ii), but will not be an α-partition cardinal as we have defined it.

The following "transitivity" follows from Theorem 5; a direct proof does not seem to be immediate:

Corollary 6. *If α is a limit ordinal and if $\varkappa \nrightarrow (\alpha)^{<\omega}_\lambda$ and $\lambda \nrightarrow (\alpha)^{<\omega}_\delta$, then also $\varkappa \nrightarrow (\alpha)^{<\omega}_\delta$.*

Proof. If $\varkappa \to (\alpha)^{<\omega}_\delta$ and $\lambda \nrightarrow (\alpha)^{<\omega}_\delta$, then if $\varkappa'$ is the first cardinal such that $\varkappa' \to (\alpha)^{<\omega}_\delta$, we have $\varkappa \geqslant \varkappa > \lambda$. By (ii) of Theorem 5, we have $\varkappa' \to (\alpha)^{<\omega}_\lambda$, and hence also $\varkappa \to (\alpha)^{<\omega}_\lambda$.

The next result shows that we have a finer hierarchy of partition cardinals than the usual one given by $\varkappa(\alpha)$.

Theorem 7. *If α is a limit ordinal, and $\lambda = \varkappa(\alpha+\omega)$ is the first $(\alpha+\omega)$-partition cardinal, then there are λ α-partition cardinals $< \lambda$.*

More generally, if $\alpha < \alpha'$ are limit ordinals and μ is any α'-partition cardinal, then there are μ α-partition cardinals less than μ; and if $\nu > \mu$ is also an α'-partition cardinal, there are ν α-partition cardinals between μ and ν.

Proof. Let $\varkappa_0$ be $\varkappa(\alpha)$ (the first α-partition cardinal), and given $\varkappa_\beta$, let $\varkappa_{\beta+1}$ be the first cardinal $\varkappa$ such that $\varkappa \to (\alpha)^{<\omega}_{\varkappa_\beta}$. For limit ordinals δ, let $\varkappa_\delta = \bigcup_{\beta<\delta} \varkappa_\beta$. Since for any ζ, $\zeta \nrightarrow (\alpha)^{<\omega}_\zeta$, this gives a strictly increasing sequence of ordinals.

Now since $\lambda \to (\alpha+\omega)^{<\omega}_\delta$ for any $\delta < \lambda$, if $\varkappa_\beta$ is defined and $< \lambda$, then $\lambda \to (\alpha+\omega)^{<\omega}_{\varkappa_\beta}$. Hence $\varkappa_{\beta+1}$ is defined and since $\varkappa_{\beta+1} \nrightarrow (\alpha+1)^{<\omega}_{\varkappa_\beta}$ by Lemma 3, $\varkappa_{\beta+1} < \lambda$. So since λ is strongly inaccessible, $\varkappa_\beta < \lambda$ for $\beta < \lambda$; i.e. there are λ α-partition cardinals below λ.

The general statement follows similarly.

4. α-partition cardinals and indescribable cardinals. A cardinal β is *δ-indescribable* if for any formula φ of the language of set theory with one additional one-place predicate, and any subset $A \subset V_\beta$, if

$$\langle V_{\beta+\delta}, \varepsilon, A\rangle \models \varphi$$

then also for some $\beta' < \beta$,

$$\langle V_{\beta'+\delta}, \varepsilon, A \cap V_{\beta'}\rangle \models \varphi\,.$$

This notion was introduced by Silver to generalize the notion of Π^n_m- or Σ^n_m-indescribable cardinals: if $\delta \geqslant \omega$ and β is δ-indescribable, then β is totally indescribable, i.e. Π^n_m-indescribable for all $n, m < \omega$. It was shown that if $\varkappa$ is $\varkappa(\alpha)$ for limit ordinal α, then there are many $\varkappa$-indescribable cardinals below $\varkappa$; indeed, they form a stationary subset of $\varkappa$ (they intersect every closed unbounded subset of $\varkappa$). The same method proves the following extension:

Theorem 8. *If $\varkappa$ is an α-partition cardinal for limit α, then the $\varkappa$-indescribable cardinals form a stationary subset of $\varkappa$.*

Proof. Suppose that $\varkappa$ is the first cardinal such that $\varkappa \to (\alpha)^{<\omega}_\lambda$, and consider the structure

$$\mathfrak{B} = \langle V_\varkappa, \varepsilon, \pi, A, (\xi)_{\xi<\lambda}\rangle\,,$$

where A is any closed unbounded subset of $\varkappa$, and π is a 1-1 map from $\varkappa$ onto $V_\varkappa$ (so that using π we can choose from any non-empty subset of $V_\varkappa$; such a π exists since $\varkappa$ is strongly inaccessible.)

Now since $\mathfrak{B}$ has cardinal $\varkappa$ and length $< \varkappa$, there will be a subset X of $\varkappa$ of order-type α which is a set of indiscernibles for $\mathfrak{B}$. Now we repeat the proof exactly as in [1] that if X is chosen with smallest member as small as possible, then we must have $X \subset A$ and the members of X are all $\varkappa$-indescribable. Since the details are almost exactly as before, we shall not repeat them here.

Since the $\varkappa$-indescribable cardinals are weakly compact (since they are Π_1^1-indescribable), they are very highly Mahlo cardinals (e.g. they are hyper-Mahlo, and hyper-hyper Mahlo, etc.). So we have also proved:

Corollary 9. *If $\varkappa$ is an α-partition cardinal for limit α, then $\varkappa$ is hyper-Mahlo, and hyper-hyper Mahlo, etc.*

5. An application in the constructible universe. Silver [4] shows that if α is countable in L, then $\varkappa(\alpha)$ also satisfies $\varkappa \to (\alpha)_2^{<\omega}$ in L. This proof also generalizes to other α-partition cardinals:

Theorem 10. *If α is a limit ordinal which is countable in L, and $\varkappa \to (\alpha)_\lambda^{<\omega}$ holds (in V), then $\varkappa \to (\alpha)_\lambda^{<\omega}$ also holds when relativized to L.*

Proof. Exactly as in [4] (or [1]), we rewrite $\varkappa \to (\alpha)_\lambda^{<\omega}$ as the statement that for each function $f\colon [\varkappa]^{<\omega} \to \lambda$, a certain partial order on a certain set is not well-founded; the partial order and the set involved are defined absolutely from f and a given well-ordering $\underset{\sim}{r}$ of ω in the order-type α. This means that $\varkappa \to (\alpha)_\lambda^{<\omega}$ is a Π_1^{ZF} statement in $\underset{\sim}{r}$ and so (provided $\underset{\sim}{r} \in L$) is preserved from V to L.

(This is presumably one of the generalizations which Silver refers to in [4] [1].)

Corollary 11. *If there is a cardinal $\varkappa$ such that $\varkappa \to (\omega_1^L)_2^{<\omega}$, then the following is consistent with $V = L$:*

(*) *For each $\alpha < \omega_1$, there is a proper class of α-partition cardinals.*

Proof. By Theorems 6 and 9, this will hold in $L_\varkappa$ if $\varkappa$ is $\varkappa(\omega_1^L)$ (where ω_1^L is the first ordinal uncountable in L).

Note that (*) also implies that there will be a proper class of cardinals $\varkappa$ in L satisfying in L:

(**) $\qquad \varkappa \to (\alpha)_\lambda^{<\omega} \quad$ for all $\alpha < \omega_1$ and all $\lambda < \varkappa$,

so that (*) is a strengthening of the statements of [4] saying that there are large cardinals in L; (**) will be a formula of Silver's $0^\#$.

[1] Professor Silver confirms this; many of the results used here were known to him at the time he wrote his thesis (1966). (Theorem 8 is an exception.)

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