For this purpose one may take as an example one of the well known modal systems, described in [4]: system $M$ of Wright, system $S4$ of Lewis or system $Br$ of Brouwer. But any extension $MG$ of a theory $G$, received in this way, will have the following property: for any formula $A$ of $G$

\[ \vdash AA \text{ if and only if } A \text{ is derivable in } G. \]

One can see, the weak decidability of $MG$ coincides with the decidability of $G$ for any such extension. Hence, such modal extensions are not interesting.

References


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Some remarks on set theory XI

by

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Abstract. Let $\kappa, \lambda$ be infinite cardinals, $F \subset P(\omega)$, $A \subset B$ for $A \neq B \in F$; $|A| < \kappa$ for $A \in F$. We give a necessary and sufficient condition (in ZFC) for the existence of an $F' \subset F$ with $|F'| = \kappa$

\[ |\kappa - \bigcup F'| \geq \lambda. \]

§ 1. Let $\kappa, \lambda$ be infinite cardinals, $F \subset P(\omega)$, $|F| = \kappa$. Problems of the following type were considered in quite a few papers.

(1) Under what conditions for $F$ does there exist $F' \subset F$, $|F'| = \kappa$ such that $|\kappa - \bigcup F'| \geq \lambda$?

(2) Assume $f$ is a one-to-one mapping with domain $\kappa$ and range $F$, $\xi \neq f(\xi)$. Under what conditions for $F$ does the set mapping $f$ have a free subset of cardinality $\lambda$, i.e. a subset $G \subset F$, $|G| = \lambda$ such that $\xi \neq f(\eta)$ for all $\xi, \eta \in G$?

It was proved in [3] that (1) holds with $\kappa = \lambda$ provided there is a cardinal $\tau$ with $|A| < \tau < \kappa$ for all $A \in F$. In [4] it was proved that the same condition also implies the stronger statement (2) with $\lambda = \kappa$. It is obvious that if we only assume

\[ |A| < \kappa \text{ for } A \in F \]

we have to impose further conditions on $F$ to obtain results of type (1) and (2).

The aim of this short note is to study the answer to (1) under the following simple condition

\[ A \notin B \text{ for all } A \neq B \in F. \]

Here we get a complete discussion without using G.C.H. and we give the solution of Problem 73 proposed in our paper [1] as well.

We mention that in a paper with A. Máthé [2] we are going to study the answer to (2) under condition (3) and under some additional and more sophisticated conditions.
To have a short notation we say that \( P(x, \lambda) \) is true if (1) holds for all \( F \subseteq P(x) \), \(|F| = \kappa\), satisfying (3) and (4).

\section{2}

**Theorem 1.** Let \( x \) be regular. Then \( P(x, \lambda) \) holds if either \( \lambda < \kappa \) and \( \lambda = \kappa \) or \( \lambda < \kappa \) and \( x \) is weakly inaccessible.

**Theorem 2.** If \( x \) is singular then \( P(x, \lambda) \) is false.

**Proof of Theorem 1.**

First we prove

\( 5 \) If \( \lambda > \kappa \) for some \( \nu < \kappa \), \( \lambda > \kappa \) then \( P(x, \lambda) \) is false.

Proof. Let \( \lambda_0 \) be minimal such that there is \( \nu < \kappa \) with \( \nu^\lambda > \kappa \), and let \( \nu_0 \) be minimal such that \( \nu_0^\lambda > \kappa \). Then \( \kappa = \nu_0^\lambda < \kappa \), it is well known that then there are \( X \), \(|X| = \nu_0^\lambda \) and \( G \subseteq P(X) \), \(|G| = \nu_0^\lambda \) such that

\( 6 \) \(|A| = \lambda_0 \) for \( A \in G \) and \( |A \cap B| < \lambda_0 \) for \( A \neq B \in G \).

Let \( H = \text{Co}(G) = \{X - A: A \in G\} \). We may assume \( X \cap \kappa = \emptyset \). Let \( \{\xi: \xi < \kappa \} \subseteq H \) be one-to-one, and put \( A_\xi = B_\xi \cup \xi \) for \( \xi < \kappa \). Then \( |A_\xi| < \kappa \) for \( \xi < \kappa \), \(|X \cap \kappa| = \kappa \), \(|F| = \kappa \), \( A_\xi \subseteq A_\zeta \) for \( \xi < \zeta < \kappa \) since \( |B_\xi - B_\zeta| = \lambda_0 \). On the other hand if \( E \subseteq F \subseteq G \), \(|E| = \kappa \), then, by (6),

\[ |X \cap \kappa - \bigcup E| < \lambda_0 \leq \lambda. \]

This proves (5).

Now we prove

\( 7 \) Assume \( \lambda < \kappa \), \( \nu < \kappa \) for all \( \nu < \kappa \) then \( P(x, \lambda) \) holds.

Proof. Let \( F \) be a system satisfying (3) and (4). Let \( \xi < \kappa \). Put \( E_\xi = \{A \in E: |E_\xi| = \xi \} \). If \( |F| = \kappa \) for some \( \xi \) then by the regularity of \( \kappa \) and by (5), \( \lambda > \kappa \), (1) holds. We assume \( |F| = \kappa \) for all \( \xi < \kappa \) and we obtain a contradiction. Pick \( A_{\xi} \in E_\xi \) for each \( \xi < \kappa \). Put \( g(\xi) = \xi - A_{\xi} \). Then \( \lambda = \sup \{ g(\xi): \xi < \kappa \} \). We can choose a regular cardinal \( \tau \) such that \( \lambda < \tau < \kappa \) and otherwise \( \lambda = \kappa \), \( \lambda > \kappa \).

The set \( \mathcal{K}_\zeta = \{x < \kappa: c(x) = \tau \} \) is stationary in \( \kappa \) and \( h(\xi) < \xi \) for \( \xi \in \mathcal{K}_\lambda \). By Fodor's theorem there are \( \zeta < \kappa \) and a stationary set \( \mathcal{K} \subseteq \mathcal{K}_\zeta \) such that \( h(\xi) < \xi \) for \( \xi \in G \). By (5), \( |G| < \kappa \), there is \( C \subseteq \mathcal{K} \) cofinal in \( \kappa \) such that \( g(\xi) = \xi(\xi) \) for \( \xi \in C \). Choose \( \xi < \eta \in C \) such that \( A_\xi \subseteq A_\zeta \). Then \( A_\xi \subseteq A_\eta \) is a contradiction.

(5) and (7) prove the first part of our theorem.

We now prove

\( 8 \) Assume \( P(x, \kappa) \). Then \( x \) is weakly compact.

Proof. By the assumption \( P(x, \lambda) \) holds for \( \lambda < \kappa \) hence, by (5), \( \kappa^\kappa < \kappa \) for \( \kappa < \kappa \) if \( \kappa < \kappa \) is strongly inaccessible. Assume \( x \) is not weakly compact.

Then there is an Aronszajn tree \( \langle \xi, \xi^+ \rangle \) on \( x \). Let \( F_\xi \) denote the set of elements of rank \( \xi \) in the tree and put \( S_\xi = \bigcup F_\xi \). \( F_\xi \) is said to be a path of length \( \xi \) if \( F_\xi \) is a chain \( C \subseteq S_\xi \) and \( F \cap T_\xi \neq \emptyset \) for \( \eta < \xi \). It is well-known that there is a set \( K \subseteq \kappa \), \(|K| = \kappa \) such that there is a maximal path \( P_\xi \) of length \( \xi \) for each \( \xi \in \mathcal{K} \).

Let \( E = \{ (S_\xi - P_\xi): \xi \in \mathcal{K} \} \). Assume \( \xi < \eta < \xi \) then by the maximality of \( P_\xi \subseteq P_\xi \subseteq S_\xi \subseteq S_\xi - P_\xi \) and obviously \( S_\xi - P_\xi \subseteq S_\xi - P_\xi \).

On the other hand let \( L \subseteq \mathcal{K} \), \(|L| = \kappa \), \( \xi \notin \bigcup \{ (S_\xi - P_\xi): \xi \in \mathcal{K} \} \). Then there is \( \xi \in \mathcal{K} \) such that the ranks of \( x \) and \( y \) are less than \( \xi \), hence \( \xi, \gamma < x \) and \( \gamma < \kappa \).

It follows that \( \kappa = \bigcup \{ (S_\xi - P_\xi): \xi \in \mathcal{K} \} \) is a chain and thus it has cardinality less than \( \kappa \).

Thus \( E \) establishes not \( P(x, \kappa) \), hence if \( x \) holds \( \kappa \) must be weakly compact. This proves (8) (see Problem 73 of [1]).

Finally we have to prove

\( 9 \) If \( x \) is weakly compact then \( P(x, \kappa) \) is true.

Proof. Let \( F \) be a system of sets satisfying (3) and (4). It is well known that then there are \( A \subseteq \mathcal{K} \) and \( A_\xi \subseteq \mathcal{K} \) such that \( A \cap \xi = \emptyset \) and \( A_\xi \cap \xi \). First we claim that \( \kappa - \mathcal{K} \) is cofinal in \( \kappa \). Otherwise there is \( \xi \in \mathcal{K} \) such that \( \kappa - \mathcal{K} \subseteq \mathcal{K} \). Then there is \( \xi < \eta \) such that \( \eta \subseteq \mathcal{K} \) and then because of \( \eta \subseteq \mathcal{K} \subseteq \mathcal{K} \), \( A_\xi \subseteq A_\eta \).

Then by transfinite induction one can easily choose two increasing sequences \( \sigma_\xi, \tau_\xi \subseteq \kappa \) such that \( \sigma_\xi < \kappa \) and \( \tau_\xi > \sigma_\xi \) for \( \xi < \eta \). Then

\[ \{ \sigma_\xi: \eta < \kappa \} \supseteq \bigcup \{ A_\eta : \eta < \kappa \}. \]

This proves (9) and Theorem 1.

Proof of Theorem 2. Assume \( c(x) < \kappa \). Let \( \{ \sigma_\xi: \nu < c(x) \} \) be a normal sequence of type \( \kappa \) of cardinals less than \( x \), tending to \( x \) such that \( \sigma_\xi = c(x) \).

Then

\[ x = \kappa \cup \bigcup \{ \kappa_{\alpha+1} - \kappa_\alpha: \alpha < \omega \}. \]

For \( \kappa = \xi < \kappa \) let \( \nu(\xi) \) be the unique \( \nu \) for which \( \xi < \kappa_{\alpha+2} - \kappa_\alpha \). Put \( A_\xi = \kappa_{\alpha+2} - \nu(\xi) \) for \( \xi \in \mathcal{K} \) and \( F = \{ A_\xi: \xi < \kappa \} \).

Assume \( \kappa \notin \mathcal{K} \). If \( \nu(\xi) = \nu(\eta) \) then \( \nu(\xi) = \nu(\eta) \). Hence \( A_\xi \subseteq A_\eta \). On the other hand if \( \mathcal{K} \subseteq \kappa - \kappa \) is cofinal in \( \kappa \) then obviously

\[ \bigcup \{ A_\xi : \xi \in \mathcal{K} \} = \kappa. \]
§ 3. Remarks.

1) First we mention that the weak assumption (4) is insufficient to obtain set mapping theorems of type (2) as is shown by the following example.

For \( n \in \omega \) define

\[
f(n) = \{ m < n : \text{m is even} \} \cup \{ n+1 \} \quad \text{if } n \text{ is even}
\]

and

\[
f(n) = \{ m < n : \text{m is odd} \} \cup \{ n+1 \} \quad \text{if } n \text{ is odd}.
\]

Then \( f(n) \not\subseteq f(m) \) if \( n \neq m \) and there is no free set of three elements. (Two independent points obviously exist.)

2) The following would be a Ramsey-type generalization of the positive part of Theorem 1.

(10) Let \( 2 < k < \omega \) and let \( F : [\omega]^k \rightarrow [\omega]^\omega \) be such that \( F(X) \not\subseteq F(Y) \) for \( X \neq Y \in [\omega]^k \). Then there is a \( A \subseteq \omega \), \( |A| = \omega \) such that

\[
|\omega| - \Big| \bigcup \{ F(X) : X \in [A]^k \} \Big| > \omega.
\]

We have examples to show that (10) is false for \( k = 2 \) even if we assume that \( F = \{ F(X) : X \in [\omega]^2 \} \) satisfies the following stronger condition.

(11) No member of \( F \) is contained in the union of \( l \) others for some \( 2 < l < \omega \).

We suppress the proof.

3) We also mention that some of the counterexamples can be obtained with set-systems \( F \) satisfying the stronger condition (11).

Using the fact that for each \( 1 < l < \omega \) there is \( G \subseteq P(\omega) \) such that the intersection of \( l \) members of \( G \) is infinite and the intersection of \( l+1 \) members of \( G \) is finite one can strengthen the counterexample of Theorem 1 to

(12) For \( \omega_1 < \xi < 2^\omega \) there is \( F \subseteq P(\xi) \), \( |F| = \omega \) satisfying (11) and such that

\[
|\omega| - \big| \bigcup F' \big| < \omega \quad \text{for} \quad F' \subseteq F, \quad |F'| = \omega.
\]

The existence of the required \( G \) was pointed out to us by L. Réss. Assuming CH, we know that there is an \( F \) satisfying (12) and the following condition stronger than (11). No member of \( F \) is contained in the union of finitely many others. We did not investigate how far these results can be generalized.

4) Finally we mention a rather technical problem. Let \( F : [\omega]^3 \rightarrow [\omega]^\omega \) be such that \( F(X) \not\subseteq F(Y) \) for \( X \neq Y \in [\omega]^3 \). Does there exist an infinite path \( I \subseteq [\omega]^3 \) such that \( |\omega| - \big| \bigcup \{ F(X) : X \in I \} \big| > \omega \)