Weak decidability of some modal extension of trigonometry

by

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Abstract. A modal extension MT_n of the elementary trigonometry T_n is constructed for which there is an algorithm R, remaking any closed formula A of T_n into 0 or 1 so that the following conditions are fulfilled:

$$\text{if} \quad RA = 0, \quad \text{then} \quad \vdash \Delta A;$$

$$\text{if} \quad RA = 1, \quad \text{then} \quad \vdash \Delta \neg A.$$

The new general notion of a weak decidability is introduced and discussed.

It is known [1] the most mathematical theories are undecidable. This is why we are often compelled to use in the applied science experimental decision methods for problems, which have mathematical formulations. Instead of real experiments "quasi-experiments" are useful too. They are approximate calculations with help of computers. But in what sense can an experiment resolve a mathematical problem? In the present article we are trying to investigate this question taking the trigonometry as an example.

A result of experiment is a truth of fact, hence it is a modal truth [2]. A result of mathematics is a necessary truth, hence it is a modal truth also. But these two modalities are distinct: the first depends on the chosen experimental procedure, the second depends on the chosen class of mathematical models. Avoiding the detailed logical analysis, we accept the fact is a little stronger than the mathematical necessity of the experimental possibility. We pass to the exact formulations.

Let G be a theory in a standard formalization [1]. We suppose, all the axioms and rules of inference of a classical first-order predicate logic are in G. A modal extension MG of G can be obtained in the following way. We enrich the alphabet of G with the symbol Δ (possible) and we enrich the rules of formation with the following rule: If A is a formula, then ΔA is a formula too. At last we enrich the list of axioms and rules of inference. The symbol $\vdash$ will mean the derivability in MG. We introduce a new sign ∇ (necessary) by means of an abbreviative definition

$$\nabla A \underset{\mathrm{df}}{=} \neg \Delta \neg A$$

for any formula A.

DEFINITION 1. We call a modal extension MG of a theory G *weakly decidable* if there is an algorithm R, which remakes any closed formula A of G into 0 or 1, such that

(1) $$\text{if} \quad RA = 0\,, \quad \text{then} \quad \vdash \Delta A\,,$$

and

(2) $$\text{if} \quad \vdash \Delta A\,, \quad \text{then} \quad RA = 0 \text{ or } \vdash \Delta \neg A\,.$$

Here RA is the result of the using of R to A. $RA = 0$ means intuitively: it is an experimental fact, that A.

Of course, any decidable modal extension is weakly decidable.

DEFINITION 2. We call a modal extension MG of G *weakly complete* if

$$\vdash \Delta A \quad \text{or} \quad \vdash \Delta \neg A$$

for any closed formula A of G.

One can see, we obtain the definition of the weakly complete and at the same time weakly decidable modal extension if we replace in Definition 1 the condition (2) by:

(2*) $$\text{if} \quad RA = 1\,, \quad \text{then} \quad \vdash \Delta \neg A\,.$$

PROPOSITION 1. *Any axiomatizable and weakly complete modal extension MG of G is weakly decidable.*

Indeed, we can generate all the derivable formulae of MG and, sooner or later, we obtain $\vdash \Delta A$ or $\vdash \Delta \neg A$. This is just the required decision procedure.

DEFINITION 3. We call a modal extension MG of G *inwardly decidable* if there is a function I which maps the set of formulae of G into the set of formulae of MG and

(3) $$\vdash (IA \Rightarrow \Delta A)\,,$$

(4) $$\vdash (\Delta A \Rightarrow IA \vee \Delta \neg A)\,,$$

(5) $$\vdash IA \quad \text{or} \quad \vdash \neg IA\,,$$

for any closed formula A of G.

DEFINITION 4. We call a modal extension MG of G *inwardly complete* if

$$\vdash \Delta A \vee \Delta \neg A$$

for any closed formula A of G.

One can see, if we replace in Definition 3 the condition (4) by

(4*) $$\vdash (\nabla A \Rightarrow IA)\,,$$

then we obtain the definition of the inwardly complete and at the same time inwardly decidable modal extension.

PROPOSITION 2. *Any inwardly decidable and inwardly complete modal extension is weakly complete.*

Now take as an example the trigonometry T_n. In our example n is a fixed integer, $n \geq 1$. The alphabet of T_n contains four 3-ary predicate symbols P_k, where $k = 1, 2, 3, 4$, in addition to an infinite list of variables, brackets, comma and usual logical symbols $\& \vee \neg \forall \exists$. The interpretation of the predicate symbols is:

$$\begin{aligned}
&P_1(x, y, z) && \text{means} && z = x + y\,,\\
&P_2(x, y, z) && \text{means} && z = xy\,,\\
&P_3(x, y, z) && \text{means} && z = \sin(xy)\,,\\
&P_4(x, y, z) && \text{means} && z = \cos(xy)\,.
\end{aligned}$$

In the following we shall use the ordinary mathematical symbolics for abbreviation. The extralogical axioms of T_n are these of elementary algebra and the following trigonometrical axioms:

(6) $$|\sin(xy)| \leq 1\,,$$

(7) $$|\cos(xy)| \leq 1\,,$$

(8) $$|z| \leq 1 \Rightarrow \exists x \exists y \big(7|xy - u| < 22 \;\&\; z = \sin(xy)\big)\,,$$

(9) $$|z| \leq 1 \Rightarrow \exists x \exists y \big(7|xy - u| < 22 \;\&\; z = \cos(xy)\big)\,,$$

(10) $$|xy| < 5n^2 \Rightarrow 2n|\sin(xy) - t_3(xy)| < 1\,,$$

(11) $$|xy| < 5n^2 \Rightarrow 2n|\cos(xy) - t_4(xy)| < 1\,.$$

There we introduce the designations

$$t_3(u) \underset{\mathrm{df}}{=} \sum_{k=0}^{m} (-1)^k \frac{u^{2k+1}}{(2k+1)!}\,,$$

$$t_4(u) \underset{\mathrm{df}}{=} \sum_{k=0}^{m} (-1)^k \frac{u^{2k}}{(2k)!}\,,$$

where m is the least integer for which

$$(5n^2)^{2m+1}\frac{1}{(2m+1)!}<\frac{1}{2n}.$$

It is evident, the trigonometry T_n is axiomatizable, but undecidable.

We obtain the modal extension MT_n of T_n in the usual way adding the following axioms:

$$\Diamond(A*B)\Leftrightarrow\Diamond A*\Diamond B,$$

$$\Diamond WxA\Leftrightarrow Wx\Diamond A,$$

$$\Diamond P_i(x,y,z)\Leftrightarrow P_i(x,y,z),$$

$$\Delta P_j(x,y,z)$$

$$\Leftrightarrow\exists u\exists v\exists w\big(n|x-u|<1\ \&\ n|y-v|<1\ \&\ n|z-w|<1\ \&\ P_j(u,v,w)\big),$$

$$\nabla P_j(x,y,z)\Leftrightarrow P_j(x,y,z).$$

Here $*$ means & or $\vee$, W means $\forall$ or $\exists$, $\Diamond$ means Δ or ∇, $i=1,2$, $j=3,4$, A and B are formulae.

One can understand the meaning of our modalities by taking into account that the elementary trigonometrical predicates P_3 and P_4 can occur in any proposition either in the character of a "condition" (negative occurences), when the weakening of the predicate leads to the strengthening of the proposition or in the character of a "consequence" (positive occurences) when the result of the weakening of the predicates is the opposite. Then ΔA means A whose trigonometrical "consequences" are to be fulfilled only approximately with the exactness $1/n$. Analogously ∇A means that A will be true even if all the trigonometrical conditions are fulfilled no more than approximately.

Our modalities are so to say those of a physicist, who is manipulating with devices, giving only approximate results. Note, that we obtain another trigonometry T_n^* and another modal extension MT_n^* of it, taking predicates $\sin u=z$ and $\cos u=z$ instead of $\sin(xy)=z$ and $\cos(xy)=z$. In case of MT_n^* our physicist has devices for measuring phase u and amplitude z, but he has the devices for measuring frequency x, time y and amplitude z in case of MT_n.

Though the trigonometries T_n^* and T_n are equivalent, MT_n^* and MT_n are not. As we shall see, MT_n is weakly decidable, but author knows nothing about the weak decidability of MT_n^*.

PROPOSITION 3. *The modal extension MT_n of trigonometry T_n is inwardly complete and inwardly decidable.*

We define IA to be the result of substitution of

$$(2n|z-t_j(xy)|<1\ \&\ |xy|<5n^2)\vee(|z|\leqslant 1\ \&\ |xy|\geqslant 5n^2)$$

in A instead of all occurences of $P_j(x,y,z)$, for $j=3,4$.

Investigating all the cases one can state the following implications

(12) $$P_j(x,y,z)\Rightarrow IP_j(x,y,z),$$

(13) $$IP_j(x,y,z)\Rightarrow \Delta P_j(x,y,z).$$

As an example we prove (13) when $j=3$ and $|xy|\geqslant 5n^2$. In this case the antecedent of (13) is equivalent to $|z|\leqslant 1$ and its succedent is equivalent to

(14) $$\exists q\exists p[\exists u\exists v(n|u-x|<1\ \&\ n|v-y|<1\ \&\ qp=uv)\ \&\ \&\ n|z-\sin(qp)|<1].$$

Let $7|qp-xy|<22$. We have the identity

$$qp=(x+a\operatorname{sign}y)(y+a\operatorname{sign}x),$$

where

$$a=\tfrac{1}{2}\operatorname{sign}(xy)\big(-|x|-|y|+\sqrt{(|x|+|y|)^2+4(qp-xy)\operatorname{sign}(xy)}\big).$$

So far as in our case $n|a|<1$, we have

$$\exists u\exists v(n|u-x|<1\ \&\ n|v-y|<1\ \&\ qp=uv).$$

This means that instead of (14) one may prove

$$\exists q\exists p(7|qp-xy|<22\ \&\ n|z-\sin(qp)|<1),$$

but the last is evident by the axiom (8).

Turning to the proof of (3) we note that IA is obtained from A by substitutions $IP_j(x,y,z)$ instead of all occurrences of $P_j(x,y,z)$ and ΔA is obtained from A by substitutions $\Delta P_j(x,y,z)$ instead of all positive occurrences of $P_j(x,y,z)$. Thus, by (12) and (13) all the "conditions" of IA are weaker than the "conditions" of ΔA and all the "consequences" of IA are stronger than the "consequences" of ΔA. This proves (3), the proof of (4*) is analogous. Since IA is a sentence of the elementary algebra, which is a complete theory [3], we have (5).

COROLLARY. *The modal extension MT_n of trigonometry is weakly complete and weakly decidable.*

One may get many modal extensions of a theory replacing in it the predicate calculus by the one of the pure logical modal systems.

For this purpose one may take as an example one of the well known modal systems, described in [4]: system M of Wright, system $S4$ of Lewis or system Br of Brower. But any extension MG of a theory G, received in this way, will have the following property: for any formula A of G

$\vdash \Delta A$ if and only if A is derivable in G.

One can see, the weak decidability of MG coincides with the decidability of G for any such extension. Hence, such modal extensions are not interesting.

References

[1] A. Tarski, A. Mostowski and R. Robinson, *Undecidable Theories*, Amsterdam 1953.

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[3] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, Berkeley and Los Angeles 1951.

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Some remarks on set theory XI

by

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Abstract. Let $\varkappa, \lambda$ be infinite cardinals, $F \subset P(\varkappa)$, $A \not\subset B$ for $A \neq B \in F$; $|A| < \varkappa$ for $A \in F$. We give a necessary and sufficient condition (in ZFC) for the existence of an $F' \subset F$ with $|F'| = \varkappa$

$$|\varkappa - \bigcup F'| \geqslant \lambda .$$

§ 1. Let $\varkappa, \lambda$ be infinite cardinals, $F \subset P(\varkappa)$, $|F| = \varkappa$. Problems of the following type were considered in quite a few papers.

(1) Under what conditions for F does there exist $F' \subset F$, $|F'| = \varkappa$ such that $|\varkappa - \bigcup F'| \geqslant \lambda$?

(2) Assume f is a one-to-one mapping with domain $\varkappa$ and range F, $\xi \notin f(\xi)$. Under what conditions for F does the set mapping f have a free subset of cardinality λ, i.e. a subset $R \subset \varkappa$, $|R| = \lambda$ such that $\xi \notin f(\eta)$ for all $\xi, \eta \in R$?

It was proved in [3] that (1) holds with $\varkappa = \lambda$ provided there is a cardinal τ with $|A| < \tau < \varkappa$ for all $A \in F$. In [4] it was proved that the same condition also implies the stronger statement (2) with $\lambda = \varkappa$. It is obvious that if we only assume

(3) $$|A| < \varkappa \quad \text{for} \quad A \in F$$

we have to impose further conditions on F to obtain results of type (1) and (2).

The aim of this short note is to study the answer to (1) under the following simple condition

(4) $$A \not\subset B \quad \text{for all } A \neq B \in F .$$

Here we get a complete discussion without using G.C.H. and we give the solution of Problem 73 proposed in our paper [1] as well.

We mention that in a paper with A. Máté [2] we are going to study the answer to (2) under condition (3) and under some additional and more sophisticated conditions.