Low cardinality models for a type of infinitary theory

by

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Abstract. The infinitary language dealt with in this paper is like the usual language for first-order quantification theory with the sole difference being that the predicate symbols can be denumerably long, like $P(x_1, x_2, x_3, \ldots)$. It is shown that in this language certain types of finite sets of formulas must have denumerable models if they have models at all, while some finite sets of formulas may have only non-denumerable models. Similarly, certain types of sets of formulas with cardinality ω_1 must have models with cardinality ω_1 if they have any models, while some sets of formulas with cardinality ω_1 have only models with higher cardinality.

In this paper we present two theorems and four counter-examples which compose a solution to a problem posed by Leon Henkin in [1]. The theorems deal with the problem of how certain infinitary theories can force their models to have high orders of infinity. The type of infinitary language dealt with contains a denumerable set of variables $x_1, x_2, x_3, \ldots$ and a not necessarily denumerable set of constants a_i and predicates P_i and differs from the usual language of quantification theory in that a predicate symbol may be followed by a sequence of constants and variables of any type equal or less than ω_0, i.e. infinitely long predicates are allowed. But we emphasize that infinite conjunctions and disjunctions and infinite strings of quantifiers are not allowed. Such notions of quantification theory as "closed formula", "model for a theory", etc. carry over to this language in an obvious way. For theories with equality (E) as a predicate, we only speak of models such that $E(a, b)$ never holds for distinct elements a and b.

Before the theorems can be stated, two concepts must be defined. If S is a set of formulas, then by $\sum a_n/x_n(S)$ we mean the result of substituting a_n for each free x_n in S for all n. We say that S is a *set of comparable formulas* if each pair of infinite sequences of constants in $\sum a_n/x_n(S)$

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has the property that all but a finite number of constants appear in exactly the same places in the two sequences.

THEOREM 1. *If S is a finite set of comparable formulas and S has a model, then S has a denumerable model.*

THEOREM 2. *If S is a set of ω_1 (first uncountable cardinal) comparable formulas, all of which are either closed or constant-free and contain no finitely long predicates other than equality, and S has a model, then S has a model with cardinality equal or less than ω_1.*

Note that if a theory has no more than ω_1 predicates and constants and the axioms are comparable, there can be no more than ω_1 axioms.

Before proving these theorems, we give examples that show the impossibility of certain stronger theorems.

EXAMPLE 1. Let S consist of $(\exists x_1)A(x_1, x_2, x_3, \ldots)$ and all formulas of the form $\sim A(x_n, x_2, x_3, \ldots)$ where n ranges over all positive integers. S has as a model any structure where the set of all ordinals less than ω_1 is the universe and A is the set of all sequences of type ω_0 such that the first element in the sequence is greater than the least upper bound of the rest of the sequence. But S can not have a denumerable model because the theory requires that for every denumerable sequence of elements in the universe there are elements in the universe differing from each element in the sequence. So a comparable theory need not have a denumerable model.

EXAMPLE 2. Let S consist of

$$(\exists x_1)(\exists x_2)(x_1 \neq x_2) \quad \text{and} \quad (\exists x_1)A(x_1, x_2, x_3, \ldots)$$

and all formulas of the form

$$\big(A(x_1, x_3, x_5, \ldots, x_{2n-1}, \ldots) \wedge A(x_2, x_4, x_6, \ldots, x_{2n}, \ldots) \wedge x_{2n-1} \neq x_{2n}\big) \supset x_1 \neq x_2$$

where n ranges over all positive integers. In any model of this theory, there are at least 2^{ω_0} sequences of elements, $e_2, e_3, e_4, \ldots$ that can be chosen, and for each such sequence an element e_1 can be chosen such that $A(e_1, e_2, e_3, e_4, \ldots)$ is satisfied. In this way 2^{ω_0} sequences with 2^{ω_0} different first elements are produced. So any model of the theory has at least 2^{ω_0} elements. Thus a denumerable, non-comparable theory need not have models with cardinality less than 2^{ω_0}.

EXAMPLE 3. Using the same idea as in Example 1, we can construct a finite non-comparable theory which has models but no denumerable models. Let S consist of $(\exists x_1)A(x_1, x_2, x_3, \ldots)$ and $\sim A(x_1, x_1, x_3, x_4, \ldots)$ and $A(x_1, x_2, x_3, \ldots) \to A(x_1, x_3, x_4, \ldots)$.

EXAMPLE 4. To see the necessity of the requirements in Theorem 2 consider the theory containing the axioms $(\exists x_1)P(x_1, x_2, x_3, \ldots)$, $0 \neq 1$ and all formulas $P(x_1, \sigma(1), \sigma(2), \ldots, \sigma(k), x_{k+2}, x_{k+3}, \ldots) \to R\sigma(x_1)$, where σ runs through all mappings of finite initial segments of the natural numbers into $\{0, 1\}$, and $\sim\big(R_\sigma(x_1) \wedge R_\tau(x_1)\big)$ for each pair (σ, τ) such that σ and τ have the same domain and $\sigma \neq \tau$.

Thus all finite strings of the constants 0 and 1 appear in the axioms. The theory has models but all have power $\geqslant 2^{\omega_0}$ since in any model $P\big(a, f(1), f(2), f(3), \ldots\big) \wedge P\big(b, g(1), g(2), g(3), \ldots\big) \to a \neq b$ for any elements a, b and any mappings f and g from the natural numbers to $\{0, 1\}$ such that $f \neq g$.

Proof of Theorem 1. Since S is finite and comparable, all but a finite number of constants appear in exactly the same argument places in $\sum a_n/x_n(S)$. Call this set of constants E and call the remaining finite set of constants and variables F. The set of argument places that have members of F appearing in them somewhere in S can be partitioned into a finite set of sets of argument places, $\{F_i\colon 1 \leqslant i \leqslant n\}$, such that each infinitely long atomic formula appearing in S has only one constant or variable appearing throughout each F_i. Now expand S by adding, for each infinitely long predicate P, the axiom $P(i_1, i_2, i_3, \ldots) \leftrightarrow P'(x_1, x_2, x_3, \ldots, x_n)$ where $P(i_1, i_2, i_3, \ldots)$ has the same constants appearing in the argument places in E as the formulas in $\sum a_n/x_n(S)$ and x_i, for $1 \leqslant i \leqslant n$, appears in the argument places in F_i in $P(i_1, i_2, i_3, \ldots)$ as well as appearing in $P'(x_1, x_2, x_3, \ldots, x_n)$. Call the resulting theory S' clearly it has models. Now if we replace each atomic formula $P(j_1, j_2, j_3, \ldots)$ which appears in S by $P'(e_1, e_2, e_3, \ldots, e_n)$, where e_i, for each $1 \leqslant i \leqslant n$, is the constant or variable which appears in F_i in $P(j_1, j_2, j_3, \ldots)$, we get a theory $R(S)$ such that any model of S' is a model of $R(S)$. But since $R(S)$ is a finitary theory, it has a denumerable model M. We could expand M to a model of S by defining an infinitary predicate P corresponding to each finitary predicate P' that was introduced into $R(S)$. We do this according to the rule $P(a_1, a_2, a_3, \ldots) \leftrightarrow P'(b_1, b_2, b_3, \ldots, b_n)$ where b_j for $1 \leqslant j \leqslant n$ is the first element a_i which appears in F_j in $P(a_1, a_2, a_3, \ldots)$. Q.E.D.

Proof of Theorem 2. Call the set of constant-free formulas in S "Δ" and call the set of closed formulas "Γ".

Since $\Delta \cup \Gamma$ has a model therefore it either has a model of cardinality less than ω_1 or else there exists a set S', which has a model, of all the substitution instances of all formulas in $\Delta \cup \Gamma$ that can be made from the constants in $\{a_i\colon i < \omega_1\}$. The set S' could be divided into 2^{ω_0} equivalence classes formed by the relation of "comparability." Each equivalence class has ω_1 formulas in it. Denote the set of equivalence classes by $\{S'_i\colon i < 2^{\omega_0}\}$. Note that for each $i < 2^{\omega_0}$, $\Delta \cup S'_i$ has a model. We now define a set of sequences of sets of formulas. First note that each S'_i

could be arranged and denoted as $\{f'_{1,i,k}: k<\omega_1\}$, where each $f'_{1,i,k}$ (triple subscript) is a formula. Let $g'_1(S'_i)$ be the set formed by adding $h'_{1,i,k}(c'_{1,k})$ to S'_i for each formula $f'_{1,i,k}$ in S'_i which is of the form $(\exists x)h'_{1,i,k}(x)$ and adding $h'_{1,i,k}(c)$ for all constants c which appear in S'_i for each formula $f_{1,i,k}$ of the form $(x)h'_{1,i,k}(x)$.

And let $g'_{N+1}(S'_i)$ be the set formed by arranging $g'_N(S'_i)$ as $\{f'_{N+1,i,k}: k<\omega_1\}$ and adding $h'_{N+1,i,k}(c'_{N+1,k})$ to $g'_N(S'_i)$ for each formula $f'_{N+1,i,k}$ of the form $(\exists x)h'_{N+1,i,k}(x)$ and adding $h'_{N+1,i,k}(c)$ for all constants c which appear in S' if $f'_{N+1,i,k}$ is of the form $(x)h'_{N+1,i,k}(x)$.

Let S^2 be $\bigcup_{i<2^{\omega_0}}\bigcup_{N<\omega_0} g'_N(S'_i)$. We inductively define a sequence $\{S^\beta: \beta<\omega_1\}$, using the axiom of choice. If S^α has been chosen, then $S^{\alpha+1}$ is chosen as follows: let Δ^α be the set of all substitution instances of Δ whose constants are in S^α. Then divide $S^\alpha\cup\Delta^\alpha$ into equivalence classes based on the comparability relation. Call the set of classes $\{S^\alpha_i: i<2^{\omega_0}\}$. Each set S^α_i can be arranged and denoted as $\{f^\alpha_{1,i,k}: k<\omega_1\}$. Now, let $g^\alpha_1(S^\alpha_i)$ be the set formed by adding $h^\alpha_{1,i,k}(c^\alpha_{1,k})$ to S^α_i for each formula $f^\alpha_{1,i,k}$ in S^α_i which is of the form $(\exists x)h^\alpha_{1,i,k}(x)$, and adding $h^\alpha_{1,i,k}(c)$ for all constants c in S^α if $f^\alpha_{1,i,k}$ is of the form $(x)h^\alpha_{1,i,k}(x)$.

Let $g^\alpha_{N+1}(S^\alpha_i)$ be the set formed by arranging $g^\alpha_N(S^\alpha_i)$ as $\{f_{N+1,i,k}: k<\omega_1\}$ and adding $h^\alpha_{N+1,i,k}(c^\alpha_{N+1,k})$ to $g^\alpha_N(S^\alpha_i)$ if $f^\alpha_{N+1,i,k}$ is of the form $(\exists x)h^\alpha_{N+1,i,k}(x)$, and adding $h^\alpha_{N+1,i,k}(c)$ for all constants c in S^α if $f^\alpha_{N+1,i,k}$ is of the form $(x)h^\alpha_{N+1,i,k}(x)$.

Then $S^{\alpha+1}=\bigcup_{i<2^{\omega_0}}\bigcup_{N<\omega_0} g^\alpha_N(S^\alpha_i)$.

If B is a limit ordinal, then let $S^\beta=\bigcup_{\alpha<\beta}S^\alpha$. Thus, our definition of $\{S^\beta:\beta<\omega_1\}$ is complete. Let $G=\bigcup_{\beta<\omega_1}S^\beta$. Note that the set of constants wh ch appear in G has cardinality ω_1. If G is divided into 2^{ω_0} equivalence classes $\{G_j: j<2^{\omega_0}\}$ by the re ation of comparability, then each G_j has the property that $G_j=\bigcup_{\alpha\leqslant\beta<\omega_1}S^\beta_{j\beta}$ for some sequence $\{j\beta: \alpha\leqslant\beta<\omega_1\}$ where $1\leqslant j\beta<2^{\omega_0}$ for $\alpha\leqslant\beta<\omega_1$ so that each set $S^\beta_{j\beta}$ is one of the sets S^β_i mentioned previously, and $S^\alpha_{j\alpha}$ has the property that S^k_i is disjoint from G_j when $k<\alpha$.

It is easy to see that $\bigcup_{\alpha\leqslant\beta<\omega_1}S^\beta_{j\beta}$ has a model with the same universe as the original model U, since if $S^\alpha_{j\alpha}$ has the property that S^k_i is always disjoint from it if $k<\alpha$, then $S^\alpha_{j\alpha}$ contains only substitution instances of Δ, so each constant in $S^\alpha_{j\alpha}$ can be assigned to an element in $|U|$ and each successively introduced constant in $\bigcup_{\alpha\leqslant\beta<\omega_1}S^\beta_{j\beta}$ can obviously be assigned to an element in $|U|$ in such a way as to create a model for $\bigcup_{\alpha\leqslant\beta<\omega_1}S^\beta_{j\beta}$.

We can convert G_j into a set of formulas G'_j such that G'_j has a model

whose universe is $|U|$ and no 2 constants appearing in G'_j are assigned to the same individual.

We define a sequence denoted as $\{(S^\beta_{j\beta})': \alpha\leqslant\beta<\omega_1\}$, together with an assignment of each constant appearing in any set of formulas $S^\beta_{j\beta}$ to an individual in $|U|$. Let $(S^\alpha_{j\alpha})'=S^\alpha_{j\alpha}$. We can assign all constants appearing in $(S^\alpha_{j\alpha})'$ to different individuals in $|U|$, since we assumed that $|U|$ has cardinality equal or greater than ω_1.

Suppose $(S^\beta_{j\beta})'$ has already been chosen and every constant in $(S^\beta_{j\beta})'$ has been assigned to a different element of $|U|$. Then $S^{\beta+1}_{j,\beta+1}$ is chosen as follows, along with an assignment of each constant in $(S^{\beta+1}_{j,\beta+1})'$ to a different element of $|U|$. Arrange and denote $(S^\beta_{j\beta})'$ as $\{g^\beta_{1,j\beta,k}: k<\omega_1\}$ and let $(g^\beta_1)'(S^\beta_{j\beta})'$ be the set formed by successively adding, for $k<\omega_1$,

1) $m^\beta_{1,j\beta,k}(c^\beta_{1,k})$ to $(S^\beta_{j\beta})'$ if $g^\beta_{1,j\beta,k}$ is of the form $(\exists x)m^\beta_{1,j\beta,k}(x)$, and has the property that it is possible to assign $c^\beta_{1,k}$ to an individual a in $|U|$ that no constant in $(S^\beta_{j\beta})'$ and no constant $c^\beta_{1,j}$ where $j<k$ has been assigned to in such a way that the assignment of $c^\beta_{1,k}$ to the element a in $|U|$, together with the assignments of constants in $(S^\beta_{j\beta})'$, gives us a model for $m^\beta_{1,j\beta,k}(c^\beta_{1,k})$. In such a case, we assign $c^\beta_{1,k}$ to some new individual a with the above property.

2) $m^\beta_{i,j\beta,k}(c)$ to $(S^\beta_{j\beta})')$ if $g^\beta_{1,i\beta,k}$ is of the form $(\exists x)m^\beta_{1,j\beta,k}(x)$ and c is a previously introduced constant in $(g^\beta_1)'(S^\beta_{j\beta})'$ and $|U|$ together with the assignment of previously introduced constants gives us a model for $m^\beta_{1,j\beta,k}(c)$.

3) $m^\beta_{1,j\beta,k}(c)$ to $(S^\beta_{i\beta})'$ if $g^\beta_{1,j\beta,k}$ is of the form $(x)m^\beta_{1,j\beta,k}(x)$ and c is a constant in $(S^\beta_{j\beta})'$.

We remark only that $(g^\beta_{N+1})'(S^\beta_{j\beta})'$ is related to $(g^\beta_N)'(S^\beta_{j\beta})'$ in the same way that $g^\beta_{N+1}(S^\beta_{j\beta})$ is related to $g^\beta_N(S^\beta_{j\beta})$ except that the same provision is made as in the definition of $(g^\beta_1)'(S^\beta_{j\beta})'$ to insure that each new constant is assigned to a new individual in $|U|$.

Thus, by induction (and the axiom of choice) we have a model for $\bigcup_{N<\omega_0}(g^\beta_N)'(S^\beta_{j\beta})'$ which has each constant assigned to a different individual in $|U|$.

We define $(S^{\beta+1}_{j,\beta+1})'$ to be $\big(\bigcup_{N<\omega_0}(g^\beta_N)'(S^\beta_{j,\beta})\big)\cup\Delta^{\beta+1}_{j,\beta+1}$ where $\Delta^{\beta+1}_{j,\beta+1}$ is the set of all substitution instances of Δ whose constants are in $S^{\beta+1}$ and which are comparable to the formulas in $\bigcup_{N<\omega_0}(g^\beta_N)'(S^\beta_{j\beta})'$. If we choose any assignment of those constants in $\Delta^{\beta+1}_{j,\beta+1}-\bigcup_{N<\omega_0}(g^\beta_N)'(S^\beta_{j\beta})'$ to individuals in $|U|$, we expand our model for $\bigcup_{N<\omega_0}(g^\beta_N)'(S^\beta_{j\beta})'$ to a model for $(S^{\beta+1}_{j,\beta+1})'$. Thus our induction is complete. We have defined a set $(G_j)'=\bigcup_{\alpha\leqslant\beta<\omega_1}(S^\beta_{j\beta})'$ which has a model with $|U|$ as its universe and no 2 constants assigned to the same individual. This model gives us a valuation V_j for all atomic formulas in $(G_j)'$. For $j < 2^{\omega_0}$, each $(G_j)'$ has such a valuation, and each $(G_j)'$ has exactly the same set C of constants appearing in it, and the valuations do not conflict since $(G_j)'$ and $(G_k)'$ are disjoint if $k \neq j$. Also, any substitution instance of any formula in $\Delta \cup \Gamma$ which is constructed with constants in C appears in some $(G_j)'$. Thus, we have a structure with the set C as the universe, each constant in C assigned to itself, and the predicates determined by the set of valuations V_j. It can easily be shown by induction that every formula in $\bigcup_{j<2^{\omega_1}} (G_j)'$ is satisfied by this structure. So the structure is a model for $\Delta \cup \Gamma$. Q.E.D.

References

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Entscheidungsprobleme der Theorie zweier Äquivalenzrelationen mit beschränkter Zahl von Elementen in den Klassen

von

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Abstract. Let $E_{n,m}$ $(2 \leqslant n \leqslant m < \omega)$ be the class of structures $\langle A, R_0, R_1\rangle$ where R_0, R_1 are equivalence relations such that card $a|R_0 \leqslant n$, card $a|R_1 \leqslant m$ for all $a \in A$. Among other things it is shown that $E_{n,m}$ is recursively decidable iff $n = m = 2$. The same holds for the corresponding classes $E_{n,m}^{\text{fin}}$ of finite structures. Proofs are by model-interpretability.

1. Übersicht. Es sei E die Klasse aller Strukturen $\langle M; R_0, R_1\rangle$, wo R_0, R_1 Äquivalenzrelationen über M sind und überdies die Bedingung

$$(*) \qquad a/R_0 \cap a/R_1 = \{a\} \qquad (a \in M)$$

erfüllt ist. Ferner sei E_n die Klasse derjenigen Strukturen aus E, die darüberhinaus der Bedingung

$$(**) \qquad \text{card}\, a/R_0 \leqslant n < \omega \qquad (a \in M)$$

genügen. In [1] und — unabhängig davon — in [2] wurde die rekursive Unentscheidbarkeit der (elementaren) Theorie $\mathrm{Th}\,E$ (ohne Identität) gezeigt und darüberhinaus, daß E eine Reduktionsklasse im Sinne der Prädikatenlogik ist (d. h. jede Aussage H des Prädikatenkalküls ist effektiv in eine Aussage H' der Sprache von E (ohne Identität) überführbar, so daß H dann und nur dann allgemeingültig ist, wenn $H' \in \mathrm{Th}\,E$. $\mathrm{Th}\,E_1$ kann als Theorie einer Äquivalenzrelation mit Identität aufgefaßt werden; diese Theorie ist bekanntlich entscheidbar, daher erscheint das Entscheidungsproblem für $\mathrm{Th}\,E_n$, insbesondere für $n = 2$ als eine interessante Fragestellung. Die Antwort wird gegeben durch

THEOREM 1. *E_n $(n \geqslant 2)$ ist universell bezüglich Modellinterpretierbarkeit. Damit ist E_n eine Reduktionsklasse für den PK, und daher ist* $\mathrm{Th}\,E_n$ *rekursiv unentscheidbar.*

Weil $E_2 \subset E_n \subset E$ $(n \geqslant 2)$, und weil $\mathrm{Th}\,E_2$ endliche Erweiterung von $\mathrm{Th}\,E_n$ ist, genügt es, den Beweis für $n = 2$ zu führen. $\mathrm{Th}\,E_2$ ist auch endliche Erweiterung von $\mathrm{Th}\,E$, so daß dies Theorem eine Verschärfung des bislang bekannten Resultats ist. Das bezieht sich im besonderen auf die Uni-