Variations of Keisler's theorem for complete embeddings [1]

by

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Abstract. We improve Keisler's theorem for complete embeddings in two ways, and we prove an analogue which shows when an elementary embedding $j\colon V_a \to M$ can be extended to the whole set theoretic universe so that the range remains a transitive class.

0. Introduction. Throughout the paper we assume $\mathfrak{M} = \langle M, E\rangle$ is a model of a set theory that contains the axioms of extensionality, pairs, unions, comprehension, and the following two principles:

(a) $\forall u \in x\, \exists v \varphi(u, v, y) \to \exists f$ (f is a function with domain

$$x \wedge \forall u \in x \varphi(u, f(u), y)),$$

(b) x is infinite $\to$ there is a function from x onto $x \times x$.

(The first principle together with extensionality and pairs implies that the Cartesian product $x \times x$ exists.)

Since $\mathfrak{M}$ may not be well-founded, we cannot identify E with the standard membership relation ϵ. However, we can (and will) assume without loss of generality that $\{x, y\} = \{x, y\}_{\mathfrak{M}}$, where $\{x, y\}_{\mathfrak{M}}$ is the unique element $z \in M$ such that $\mathfrak{M}$ satisfies that z is the pair of x and y.

We will say X is an $\mathfrak{M}$-*class* if X is an $\mathfrak{M}$-definable subset of M, i.e. if there is an ϵ-formula φ and a parameter $x \in M$ such that $X = \{y \in M\colon \mathfrak{M} \models \varphi(x, y)\}$. If there is some $x \in M$ such that $X = \{y\colon \langle y, x\rangle \in E\}$, then we will call X an $\mathfrak{M}$-*set*. We will call a function an $\mathfrak{M}$-*function* if it is an $\mathfrak{M}$-class.

We say a relational structure $\mathfrak{A} = \langle A, R, \ldots\rangle$ is an $\mathfrak{M}$-*structure* if its universe A and its relations $R, \ldots$ are $\mathfrak{M}$-classes. We do not assume that $\mathfrak{A}$ is in $\mathfrak{M}$ in any other sense. We say an expansion (extension) $\mathfrak{A}'$ of $\mathfrak{A}$ is an $\mathfrak{M}$-*expansion* ($\mathfrak{M}$-*extension*) of $\mathfrak{A}$ if $\mathfrak{A}'$ is an $\mathfrak{M}$-structure.

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[1] The results were originally intended to be contained in a paper titled "Elementary embeddings preserving well-orderings".

Suppose $\mathfrak{B} = \langle B, S, \ldots\rangle$ is an arbitrary structure. We call an embedding $j\colon \mathfrak{A}\to\mathfrak{B}$ an $\mathfrak{M}$-*embedding* if $\mathfrak{A}$ is an $\mathfrak{M}$-structure. (We do not require $\mathfrak{B}$ to be an $\mathfrak{M}$-structure.) If $j\colon \mathfrak{A}\to\mathfrak{B}$ is an elementary embedding, then for a relation R' in $\mathfrak{A}$ we let $\boldsymbol{j}(R')$ be the unique corresponding relation in $\mathfrak{B}$.

We say that an elementary $\mathfrak{M}$-embedding $j\colon \mathfrak{A}\to\mathfrak{B}$ is $\mathfrak{M}$-*complete* if there is an $\mathfrak{M}$-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ containing all relations on A that are $\mathfrak{M}$-sets and an expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that $j\colon \mathfrak{A}'\to\mathfrak{B}'$ remains elementary and for every $x \in B$ there is some $\mathfrak{M}$-set $X\subseteq A$ such that $x \in \boldsymbol{j}(X)$. Clearly we can drop the last clause if A is an $\mathfrak{M}$-set. We also define an $\mathfrak{M}$-embedding $j\colon \mathfrak{A}\to\mathfrak{B}$ to be *weakly* $\mathfrak{M}$-*complete* if there is an $\mathfrak{M}$-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ containing all *binary* relations on A that are $\mathfrak{M}$-sets and an expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that $j\colon \mathfrak{A}'\to\mathfrak{B}'$ is elementary and for all $x \in B$ there is an $\mathfrak{M}$-set $X\subseteq A$ such that $\langle x, x\rangle \in \boldsymbol{j}(X\times X)$. Note that we may assume without loss of generality that $\mathfrak{A}'$ contains all unary relations on A that are $\mathfrak{M}$-sets since we can define $\boldsymbol{j}(X) = \{x \in M\colon \langle x, x\rangle \in \boldsymbol{j}(X\times X)\}$. If $\mathfrak{M}$ is the standard model $V = \langle V, \in\rangle$ of set theory, then (weakly) $\mathfrak{M}$-complete embeddings are just called (weakly) complete embeddings.

H. J. Keisler [1], [3], [4] has shown that the completeness of $j\colon \mathfrak{A}\to\mathfrak{B}$ is a necessary and sufficient condition that there is an elementary embedding $j'\colon V'\to\mathfrak{M}'$ such that $j'\supseteq j$, $\boldsymbol{j}'\supseteq \boldsymbol{j}$ and V' contains all relations that are V-classes. We will prove three variations of Keisler's theorem. In section 1 we will prove that weak completeness is also a sufficient condition, and we will show that Keisler's theorem holds for all $\mathfrak{M}$ in the sense that $j\colon \mathfrak{A}\to\mathfrak{B}$ is $\mathfrak{M}$-complete if and only if there is an elementary $\mathfrak{M}$-embedding $j'\colon \mathfrak{M}'\to\mathfrak{M}''$ such that $j'\supseteq j$, $\boldsymbol{j}'\supseteq \boldsymbol{j}$ and $\mathfrak{M}'$ contains all relations that are $\mathfrak{M}$-classes. In section 2 we will consider the extent to which this is a generalization of Keisler's theorem. For example, we can suppose $\mathfrak{M}$ is a model of the theory of hereditarily finite sets. Finally in section 3 we will define an analogue of completeness which is a necessary and sufficient condition that there is an elementary embedding $j'\colon V'\to\mathfrak{M}'$ such that $\mathfrak{M}'$ is well-founded, $j'\supset j$, $\boldsymbol{j}'\supset \boldsymbol{j}$ and V' contains all relations that are V-classes.

A. Joyal independently proved Keisler's theorem for the case $\mathfrak{M} = V$ using essentially the same proof that we give here. Also most of Theorem 15 was independently proven by H. Gaifman using limit ultrapowers for the case $\mathfrak{M} = V$.

The results of section 1 were essentially proven in the author's thesis [7] for set theories with class variables. The results of the last section were announced in [6].

1. Canonical direct limits of $\mathfrak{M}$- ultrapowers. Let $\mathfrak{A}$ be a fixed $\mathfrak{M}$-structure. In this section we define the category $\mathcal{A}$ of all normal directed systems of $\mathfrak{M}$-ultrafilters concentrating on A, the category $\mathcal{B}$ of all $\mathfrak{M}$-complete embeddings defined on $\mathfrak{A}$, and the category $\mathcal{B}_w$ of all weakly $\mathfrak{M}$-complete embeddings defined on $\mathfrak{A}$. We show that $\mathcal{A}$ and $\mathcal{B}$ are equivalent and that, if A is infinite in $\mathfrak{M}$, then $\mathcal{A}$ and $\mathcal{B}_w$ are equivalent. The two improvements of Keisler's theorem then follow.

An ultrafilter in the field of $\mathfrak{M}$-classes which concentrates on an $\mathfrak{M}$-set we will call an $\mathfrak{M}$-*ultrafilter*. We define $\mathfrak{D} = \langle D, \leqslant, \mathfrak{U}, \mathcal{F}\rangle$ to be a *directed system* (*of* $\mathfrak{M}$-*ultrafilters*) if $\langle D, \leqslant\rangle$ is a directed set and $\mathfrak{U}$ and $\mathcal{F}$ are functions with domains D and $D\times D$, respectively, such that for all $a, b, c \in D$, $F \in \mathcal{F}_{ab}$, $G \in \mathcal{F}_{bc}$, $H \in \mathcal{F}_{ac}$ and $\mathfrak{M}$-classes X, we have

(o) $\Delta \in \mathcal{F}_{aa}$ where $\Delta = \{\langle x, x\rangle\colon x \in M\}$,
(i) $a\leqslant b$ iff $\mathcal{F}_{ab} \neq 0$,
(ii) $\mathcal{F}_{ab}$ is a set of $\mathfrak{M}$-functions with domain M,
(iii) $\mathfrak{U}_a$ is an $\mathfrak{M}$-ultrafilter,
(iv) $X \in \mathfrak{U}_a$ iff $F^{-1}(X) \in \mathfrak{U}_b$,
(v) $\{x \in M\colon F(G(x)) = H(x)\} \in \mathfrak{U}_c$.

Note that $\mathfrak{D}$ is determined by $\mathfrak{U}$ and $\mathcal{F}$ alone. We say $\mathfrak{D}$ *concentrates on* A if each $\mathfrak{U}_a$ concentrates on A, i.e. if $A = \mathfrak{U}_a$. Furthermore, we say $\mathfrak{D}$ is *normal* if we have

$$F\colon M\to M \text{ is an } \mathfrak{M}\text{-function} \wedge \{x \in M\colon F(x) \in A\} \in \mathfrak{U}_b \to \exists a \in D\; F \in \mathcal{F}_{ab}$$

and

$$a \neq b \to \mathcal{F}_{ac} \cap \mathcal{F}_{bc} = 0\,.$$

We define the objects of $\mathcal{A}$ to be normal directed systems of $\mathfrak{M}$-ultrafilters concentrating on A. Morphisms $f\colon \mathfrak{D}\to\mathfrak{D}'$ from $\mathfrak{D}$ to $\mathfrak{D}'$ are defined to be order preserving mappings $f\colon \langle D, \leqslant\rangle\to\langle D', \leqslant'\rangle$ such that $\mathfrak{U}_a = \mathfrak{U}'_{f(a)}$ and $\mathcal{F}_{ab} = \mathcal{F}'_{f(a)f(b)}$. Such mappings must be one-to-one.

We define the objects of $\mathcal{B}_w$ to be the weakly $\mathfrak{M}$-complete embeddings defined on $\mathfrak{A}$. A morphism $k\colon j\to j'$ from $j\colon \mathfrak{A}\to\mathfrak{B}$ to $j'\colon \mathfrak{A}\to\mathfrak{B}'$ is an elementary embedding $k\colon \mathfrak{B}\to\mathfrak{B}'$ such that $j' = k\circ j$. $\mathcal{B}$ is the subcategory whose objects are the $\mathfrak{M}$-complete embeddings defined on $\mathfrak{A}$.

Next we define a functor $\iota\colon \mathcal{A}\to\mathcal{B}$.

We may form an ultrapower of $\mathfrak{A}$ by an $\mathfrak{M}$-ultrafilter $\mathfrak{U}$ almost as usual. For a relation R on A that is an $\mathfrak{M}$-class, we define a relation $R_{\mathfrak{U}}$ between $\mathfrak{M}$-functions $\vec{F}\colon M\to M$ by

$$\langle\vec{F}\rangle \in R_{\mathfrak{U}} \quad\text{iff}\quad \{x \in M\colon \langle\overrightarrow{F(x)}\rangle \in R\} \in \mathfrak{U}\,.$$

Since $=_{\mathfrak{U}}$ is a congruence relation for all relations $R_{\mathfrak{U}}$, we may form the equivalence classes $[F]_{\mathfrak{U}}$ for $\mathfrak{M}$-functions $F\colon M\to M$, and regard the relations $R_{\mathfrak{U}}$ as relations between the equivalence classes. Let $\mathfrak{A}_{\mathfrak{U}} = \langle A_{\mathfrak{U}}, R_{\mathfrak{U}}, \ldots\rangle$ and call $\mathfrak{A}_{\mathfrak{U}}$ the $\mathfrak{M}$-*ultrapower of* $\mathfrak{A}$ *by* $\mathfrak{U}$.

Los' theorem holds in this general setting:

$$\mathfrak{A}_{\mathfrak{U}} \models \varphi(\overrightarrow{[F]_{\mathfrak{U}}}) \quad \text{iff} \quad \{x \in M\colon \mathfrak{A} \models \varphi(\overrightarrow{F(x)})\} \in \mathfrak{U}\,.$$

Let $i_{\mathfrak{U}}\colon \mathfrak{A} \to \mathfrak{A}_{\mathfrak{U}}$ be the embedding defined by $i_{\mathfrak{U}}(x) = [c_x]_{\mathfrak{U}}$, where $c_x\colon M \to M$ is the constant $\mathfrak{M}$-function with value x. By Los' theorem $i_{\mathfrak{U}}\colon \mathfrak{A} \to \mathfrak{A}_{\mathfrak{U}}$ is elementary.

Now suppose $\mathfrak{D}$ is a directed system of $\mathfrak{M}$-ultrafilters. For $a \in D$, let $\mathfrak{A}_a = \mathfrak{A}_{\mathfrak{U}_a}$ and $i_a = i_{\mathfrak{U}_a}\colon \mathfrak{A} \to \mathfrak{A}_a$. For $a \leqslant b$, $\varphi_{ab}\colon \mathfrak{A}_a \to \mathfrak{A}_b$ defined by $\varphi_{ab}([F]_a) = [F \circ G]_b$ (for $G \in \mathcal{F}_{ab}$) is an elementary embedding. And, hence, $\{\mathfrak{A}_a, \varphi_{ab}\}$ is a directed system of structures and $\{i_a\colon \mathfrak{A} \to \mathfrak{A}_a, \varphi_{ab}\}$ is a directed system of elementary embeddings. Let $\mathfrak{A}_{\mathfrak{D}} = \langle A_{\mathfrak{D}}, R_{\mathfrak{D}}, \ldots\rangle$ be the direct limit of $\{\mathfrak{A}_a, \varphi_{ab}\}$, and let $\varphi_a\colon \mathfrak{A}_a \to \mathfrak{A}_{\mathfrak{D}}$ be the natural elementary embedding associated with $a \in D$. We call $\mathfrak{A}_{\mathfrak{D}}$ the *canonical direct limit of $\mathfrak{M}$-ultrapowers of $\mathfrak{A}$ with respect to* $\mathfrak{D}$. Let $i_{\mathfrak{D}}\colon \mathfrak{A} \to \mathfrak{A}_{\mathfrak{D}}$ be the direct limit of $\{i_a\colon \mathfrak{A} \to \mathfrak{A}_a, \varphi_{ab}\}$.

Now if $\mathfrak{D}$ is an object of $\mathcal{A}$, then let $\iota(\mathfrak{D}) = i_{\mathfrak{D}}\colon \mathfrak{A} \to \mathfrak{A}_{\mathfrak{D}}$. If $f\colon \mathfrak{D} \to \mathfrak{D}'$ is a morphism, then define $\iota(f)\colon \iota(\mathfrak{D}) \to \iota(\mathfrak{D}')$ to be the elementary embedding $k\colon \mathfrak{A}_{\mathfrak{D}} \to \mathfrak{A}_{\mathfrak{D}'}$ defined by

$$k\big(\varphi_a([F]_a)\big) = \varphi'_{f(a)}([F]'_{f(a)})\,.$$

It is routine to verify that $\iota\colon \mathcal{A} \to \mathcal{B}$ is a functor.

Before proving that $\iota\colon \mathcal{A} \to \mathcal{B}$ is an equivalence, we state a lemma, whose straight-forward proof we omit.

For a directed system $\mathfrak{D}$ concentrating on A, let $\pi_{\mathfrak{D}}\colon D \to A$ be defined by $\pi_{\mathfrak{D}}(a) = \varphi_a([\Delta]_a)$ where $\Delta = \{\langle x, x\rangle\colon x \in M\}$.

LEMMA. *Suppose $\mathfrak{D}$ is a directed system concentrating on A. Then*

(a) *for $a \in D$ and an $\mathfrak{M}$-class X,*

$$X \in \mathfrak{U}_a \quad \textit{iff} \quad \pi_{\mathfrak{D}}(a) \in X_{\mathfrak{D}}\,,$$

(b) *if $a, b \in D$ and $F\colon M \to M$ is an $\mathfrak{M}$-function, then*

$$F_{\mathfrak{D}}\big(\pi_{\mathfrak{D}}(b)\big) = \pi_{\mathfrak{D}}(a) \quad \textit{iff} \quad \exists G \in \mathcal{F}_{ab}\, [F]_b = [G]_b\,,$$

(c) *if $f\colon \mathfrak{D} \to \mathfrak{D}'$ is a morphism in $\mathcal{A}$, then $\iota(f) \circ \pi_{\mathfrak{D}} = \pi_{\mathfrak{D}'} \circ f$,*

(d) *$\mathfrak{D}$ is normal iff $\pi_{\mathfrak{D}}\colon D \to A_{\mathfrak{D}}$ is a bijection and for all $a, b \in D$ and $\mathfrak{M}$-functions $F\colon M \to M$*

$$F_{\mathfrak{D}}\big(\pi_{\mathfrak{D}}(b)\big) = \pi_{\mathfrak{D}}(a) \to F \in \mathcal{F}_{ab}\,.$$

THEOREM 1. *$\iota\colon \mathcal{A} \to \mathcal{B}$ is an equivalence. If A is infinite in $\mathfrak{M}$, $\iota\colon \mathcal{A} \to \mathcal{B}_w$ is an equivalence.*

Proof. We must show that $\iota\colon \mathcal{A} \to \mathcal{B}$ is a full representative, faithful functor.

For a morphism $k\colon \iota(\mathfrak{D}) \to \iota(\mathfrak{D}')$ in $\mathcal{B}$, let $\varkappa(k) = \pi_{\mathfrak{D}'}^{-1} \circ k \circ \pi_{\mathfrak{D}}$. Then using the lemma we can check that $\varkappa(k)$ is a morphism from $\mathfrak{D}$ to $\mathfrak{D}'$ and $\iota\big(\varkappa(k)\big) = k$, and that $\varkappa\big(\iota(f)\big) = f$ for morphisms $f\colon \mathfrak{D} \to \mathfrak{D}'$. Hence, ι is full and faithful.

To show that ι is representative we must show that for any $\mathfrak{M}$-complete embedding $j\colon \mathfrak{A} \to \mathfrak{B}$ there is a normal directed system $\mathfrak{D}$ concentrating on A such that there is an isomorphism $\pi\colon \mathfrak{B} \cong \mathfrak{A}_{\mathfrak{D}}$ such that $\iota_{\mathfrak{D}} = \pi \circ j$. Since $j\colon \mathfrak{A} \to \mathfrak{B}$ is $\mathfrak{M}$-complete, there is an $\mathfrak{M}$-expansion $\mathfrak{A}'$, of $\mathfrak{A}$ containing all relations on A that are $\mathfrak{M}$-sets and an expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that $j\colon \mathfrak{A}' \to \mathfrak{B}'$ is elementary and for every $x \in B$ there is an $\mathfrak{M}$-set $X \subseteq A$ such that $x \in j(X)$. Let D be the set of finite sequences of elements of B. Let $\mathfrak{U}_a = \{X \subseteq M\colon a \in j(X \restriction A)\}$. Let $a * b$ denote the concatenation of the sequences a and b. For an $\mathfrak{M}$-function F, let $F^* = \{a * b\colon a, b \in D \wedge F(a) = b\}$. Then let $\mathcal{F}_{ab}$ be the set of $\mathfrak{M}$-functions $F\colon M \to M$ such that $b * a \in j(F^*)$. Using the lemma it is routine to check that $\mathfrak{D} = \langle D, \leqslant, \mathfrak{U}, \mathcal{F}\rangle$ is a normal directed system concentrating on A. We only note that $\mathfrak{D}$ is directed since $a \leqslant a * b$ and $b \leqslant a * b$. Finally by the lemma $\pi_{\mathfrak{D}} \restriction B\colon \mathfrak{B} \to \mathfrak{A}_{\mathfrak{D}}$ is an isomorphism such that $i_{\mathfrak{D}} = (\pi_{\mathfrak{D}} \restriction B) \circ j$. Hence, $\iota\colon \mathcal{A} \to \mathcal{B}$ is an equivalence.

Now suppose A is infinite in $\mathfrak{M}$. We have already shown that $\iota\colon \mathcal{A} \to \mathcal{B}_w$ is full and faithful. To show that $\iota\colon \mathcal{A} \to \mathcal{B}_w$ is representative, we alter the construction of $\mathfrak{D}$ slightly. Suppose $j\colon \mathfrak{A} \to \mathfrak{B}$ is weakly $\mathfrak{M}$-complete. Hence, there is an $\mathfrak{M}$-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ containing all binary relations on A that are $\mathfrak{M}$-sets and an expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that $j\colon \mathfrak{A}' \to \mathfrak{B}'$ is elementary and for all $x \in B$ there is an $\mathfrak{M}$-set $X \subseteq A$ such that $x \in j(X)$. Let $\mathfrak{D}$ be defined as follows: Let $D = B$, $\mathfrak{U}_a = \{X \subseteq M\colon a \in j(X \cap A)\}$, and $\mathcal{F}_{ab}$ be the set of all $\mathfrak{M}$-functions $F\colon M \to M$ such that $\langle b, a\rangle \in j(F \cap (A \times A))$.

To show that $\langle D, \leqslant\rangle$ is directed we need the assumption that A is infinite in $\mathfrak{M}$. Suppose $a, b \in D$. Then there is an $\mathfrak{M}$-set $X \subseteq A$ such that $a, b \in j(X)$. And, hence, there is an $\mathfrak{M}$-set $Y \subseteq A$ and an $\mathfrak{M}$-function $F\colon Y \to X \times X$ which is a surjection. For if X is infinite in $\mathfrak{M}$, we can take $Y = X$. In the case that X is finite in $\mathfrak{M}$, we use the fact that A is infinite in $\mathfrak{M}$. Now let $P_i\colon X \times X \to X$ $(i = 1, 2)$ be the projection $\mathfrak{M}$-functions. Let $G = P_1 \circ F$ and $H = P_2 \circ F$. Then there is some $c \in D$ such that $j(G)(c) = a$ and $j(H)(c) = b$. Hence, $\mathfrak{D}$ is directed.

The remainder of the proof is as above.

Note that since $\mathcal{A}$ has direct limits and $\iota\colon \mathcal{A} \to \mathcal{B}$ is an equivalence, $\mathcal{B}$ has direct limits.

Next we give our two improvements of Keisler's theorem.

COROLLARY 2. *$j\colon \mathfrak{A} \to \mathfrak{B}$ is an $\mathfrak{M}$-complete embedding if and only if there is an elementary embedding $j'\colon \mathfrak{M}' \to \mathfrak{M}''$ such that $j' \supseteq j$, $j' \supseteq j$ and $\mathfrak{M}'$ contains all relations that are $\mathfrak{M}$-classes.*

Proof. By the theorem it suffices to show that we can extend $i_{\mathfrak{D}}$: $\mathfrak{A} \to \mathfrak{A}_{\mathfrak{D}}$. But clearly $i_{\mathfrak{D}}$: $\mathfrak{M}' \to (\mathfrak{M}')_{\mathfrak{D}}$ is such an extension for suitable $\mathfrak{M}'$.

We say that $\mathfrak{M}$ is an *ω-model* if its natural numbers are isomorphic to the standard natural numbers.

COROLLARY 3. *If $\mathfrak{M}$ is an ω-model, then an $\mathfrak{M}$-embedding j: $\mathfrak{A} \to \mathfrak{B}$ is weakly $\mathfrak{M}$-complete if and only if there is an elementary embedding j': $\mathfrak{M}' \to \mathfrak{M}''$ such that $j' \supseteq j$, $j' \supseteq j$ and $\mathfrak{M}'$ contains all relations that are $\mathfrak{M}$-classes.*

Proof. The theorem shows that if A is infinite in $\mathfrak{M}$, then j: $\mathfrak{A} \to \mathfrak{B}$ is weakly $\mathfrak{M}$-complete if and only if it is $\mathfrak{M}$-complete. Hence, when A is infinite in $\mathfrak{M}$, Corollary 3 follows from Corollary 2. Since $\mathfrak{M}$ is an ω-model, if A is finite in $\mathfrak{M}$, then it is finite, in which case Corollary 3 is trivial.

2. Existence of $\mathfrak{M}$-complete embeddings. In this section we consider where $\mathfrak{M}$-complete embeddings arise.

We say an extension $\mathfrak{M}'$ of $\mathfrak{M}$ is an *end extension* if $xE'y \in M$ implies $x \in M$. We call an embedding k: $\mathfrak{M} \to \mathfrak{M}'$ an *end embedding* if $\mathfrak{M}'$ is an end extension of the image of k.

First we will show that every elementary embedding j: $\mathfrak{M} \to \mathfrak{M}'$ has a natural factorization as a $\mathfrak{M}$-complete embedding followed by an elementary end embedding. This will be immediate from the next two propositions.

If $\mathfrak{M}' = \langle M', E', \ldots \rangle$ and $\mathfrak{M}'' = \langle M'', E'', \ldots \rangle$ are structures, we say an embedding j: $\mathfrak{M}' \to \mathfrak{M}''$ is *cofinal* if $M'' = \bigcup_{x \in M'} \{y \in M'' : \langle y, j(x) \rangle \in E''\}$.

Then we trivially have

PROPOSITION 4. *An elementary embedding j: $\mathfrak{M} \to \mathfrak{M}'$ is cofinal if and only if it is $\mathfrak{M}$-complete.*

PROPOSITION 5. *Every elementary $\mathfrak{M}$-embedding j: $\mathfrak{M} \to \mathfrak{M}'$ can be factored as a cofinal elementary embedding followed by an elementary end embedding.*

Proof. Let $\mathfrak{M}''$ be the restriction of $\mathfrak{M}'$ to $\bigcup_{x \in M} \{y \in M' : \langle y, j(x) \rangle \in E'\}$. We will show that $\mathfrak{M}''$ is an elementary submodel of $\mathfrak{M}'$. Then we will have that j: $\mathfrak{M} \to \mathfrak{M}''$ is a cofinal elementary embedding and that the identity map id: $\mathfrak{M}'' \to \mathfrak{M}'$ is an elementary end embedding.

Suppose that $\vec{x} \in M''$ and that $\mathfrak{M}' \models \exists y \varphi(\vec{x}, y)$. We will be done if we can show that there is some $y \in M''$ such that $\mathfrak{M}' \models \varphi(\vec{x}, y)$. We can find $u, v \in M$ such that $\vec{x} E' j(u)$ and

$$\mathfrak{M} \models \forall \vec{z} \in u \big(\exists y \varphi(\vec{z}, y) \to \exists y \in v \varphi(\vec{z}, y)\big).$$

Hence,

$$\mathfrak{M}' \models \forall \vec{z} \in j(u) \big(\exists y \varphi(\vec{z}, y) \to \exists y \in j(v) \varphi(\vec{z}, y)\big).$$

Consequently, there is some $y \in M''$ such that $\mathfrak{M}' \models \varphi(\vec{x}, y)$.

In the last section we will consider complete embeddings defined on models of strong set theories. Next we briefly consider models $\mathfrak{M}$ that satisfy (i) all sets are finite, and (ii) all sets are countable.

Every model ω' of Peano arithmetic can be regarded as the finite ordinals of a model $\mathfrak{M}(\omega')$ of the theory of hereditarily finite sets, which is unique up to isomorphism. Furthermore, an elementary embedding j: $\omega' \to \omega''$ between models of Peano arithmetic has a unique extension to an elementary embedding j': $\mathfrak{M}(\omega') \to \mathfrak{M}(\omega'')$. Hence, by Propositions 4 and 5 and Theorem 1, we obtain the following factorization of elementary embeddings between models of Peano arithmetic:

PROPOSITION 6. *If ω' is a model of Peano arithmetic and j: $\omega' \to \omega''$ is an elementary embedding, there is a directed system $\mathfrak{D}$ of $\mathfrak{M}(\omega')$-ultrafilters and an elementary end embedding k: $\omega' \to \omega''$ such that $i_{\mathfrak{D}}$: $\omega' \to \omega'_{\mathfrak{D}}$ is a cofinal elementary embedding and $j = k \circ i_{\mathfrak{D}}$.*

We can obtain a similar result for models of second order arithmetic. If $\mathfrak{A}$ is a model of second order arithmetic, then it determines a model $\mathfrak{M}(\mathfrak{A})$ of $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$ whose sets of natural numbers are just those in $\mathfrak{A}$. (See Zbierski [8], Marek [5]). Furthermore, an elementary embedding j: $\mathfrak{A} \to \mathfrak{B}$ between models of second-order arithmetic extends naturally to an elementary embedding j': $\mathfrak{M}(\mathfrak{A}) \to \mathfrak{M}(\mathfrak{B})$. Thus, we may formulate a result for models of second-order arithmetic similar to the last result for models of Peano arithmetic.

Next we consider $\mathfrak{M}$-complete embeddings that arise from elementary embeddings that are not defined on all of $\mathfrak{M}$.

For the purposes of this paper, let us define a limit ordinal a to be *inaccessible* if the image of each function, whose domain and values have rank less than a, has rank less than a.

For an ordinal a of $\mathfrak{M}$, let $M_a = \{x \in M : \mathfrak{M} \models x \text{ has rank less than } a\}$. Let $\mathfrak{M}_a$ be the restriction of $\mathfrak{M}$ to M_a.

THEOREM 7. *If a is inaccessible in $\mathfrak{M}$, then every cofinal elementary embedding j: $\mathfrak{M}_a \to \mathfrak{M}'$ is $\mathfrak{M}$-complete.*

Proof. By Proposition 4, j: $\mathfrak{M}_a \to \mathfrak{M}'$ is $\mathfrak{M}_a$-complete. Hence, by Theorem 1, there is a directed system of $\mathfrak{M}_a$-ultrafilters and an isomorphism π: $\mathfrak{M}' \cong (\mathfrak{M}_a)_{\mathfrak{D}}$ such that $i_{\mathfrak{D}} = \pi \circ j$. Let $\mathfrak{D}'$ be the directed system of $\mathfrak{M}$-ultrafilters defined as follows: Let $\langle D', \leqslant' \rangle = \langle D, \leqslant \rangle$, $\mathfrak{U}'_a$ be the set of all $\mathfrak{M}$-classes X such that $X \cap M_a \in \mathfrak{U}_a$, and $\mathcal{F}'_{ab}$ be the set of all $\mathfrak{M}$-functions F: $M \to M$ such that $F \cap M_a \in \mathcal{F}_{ab}$. Let k: $(\mathfrak{M}_a)_{\mathfrak{D}} \to (\mathfrak{M}_a)_{\mathfrak{D}'}$, be the embedding defined by $k(\varphi_a([F]_a)) = \varphi'_a([F']_a)$, where F' is any $\mathfrak{M}$-function defined on M that extends F. Clearly $i_{\mathfrak{D}'} = k \circ i_{\mathfrak{D}}$. Since a is inaccessible in $\mathfrak{M}$, for each $a \in D$ and each $\mathfrak{M}$-function F: $M \to M_a$ there is an $\mathfrak{M}_a$-function G: $M_a \to M_a$ such that $\{x \in M_a : F(x) = G(x)\} \in \mathfrak{U}_a$. Hence, k is an isomorphism onto $(\mathfrak{M}_a)_{\mathfrak{D}'}$ and $k \circ \pi$ is an isomorphism from $\mathfrak{M}'$ onto $(\mathfrak{M}_a)_{\mathfrak{D}'}$ such that $i_{\mathfrak{D}'} = (k \circ \pi) \circ j$. Hence $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is $\mathfrak{M}$-complete [1].

The proof that $\iota\colon \mathcal{A} \to \mathcal{B}$ is representative essentially gives the following lemma:

LEMMA. *Suppose $j\colon \mathfrak{A} \to \mathfrak{B}$ is an elementary $\mathfrak{M}$-embedding and $\mathfrak{B}'$ is a substructure of $\mathfrak{B}$ such that there is a function I defined on B' such that for $x \in B'$*

(i) *I_x is an $\mathfrak{M}$-set contained in A,*

(ii) *for $x_1, \ldots, x_n \in B'$, all $\mathfrak{M}$-classes contained in $I_{x_1} \times \ldots \times I_{x_n}$ are n-ary relations in $\mathfrak{A}$,*

(iii) *$x \in \boldsymbol{j}(I_x)$.*

Then there is a directed system $\mathfrak{D}$ of $\mathfrak{M}$-ultrafilters and an isomorphism $\pi\colon \mathfrak{B}' \to \mathfrak{A}_{\mathfrak{D}}$ such that $i_{\mathfrak{D}} = \pi \circ j$.

THEOREM 8. (a) *Suppose in $\mathfrak{M}$, a is a limit ordinal of cofinality $\beta < a$ and $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is a elementary embedding such that j maps the ordinals less than β onto the ordinals less than $j(\beta)$. Then there is some $\mathfrak{M}''$ such that $\mathfrak{M}'$ is an end extension of $\mathfrak{M}''$ and $j\colon \mathfrak{M}_a \to \mathfrak{M}''$ is $\mathfrak{M}$-complete.*

(b) *If $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is a cofinal elementary embedding, then there exists an end extension $\mathfrak{M}''$ of $\mathfrak{M}'$ such that $j\colon \mathfrak{M}_a \to \mathfrak{M}''$ is $\mathfrak{M}$-complete.*

(c) *If a is an infinite ordinal of $\mathfrak{M}$ and $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is an elementary embedding that can be extended to $\mathfrak{M}_{a+1}$, then there is an end extension $\mathfrak{M}''$ of $\mathfrak{M}'$ such that $j\colon \mathfrak{M}_a \to \mathfrak{M}''$ is $\mathfrak{M}$-complete.*

Proof. (a) Let $B' = \bigcup_{x \in M} \{y \in M'\colon \langle y, j(x)\rangle \in E'\}$. Then we can easily expand $\mathfrak{M}_a$ and $\mathfrak{M}'$ to structures $\mathfrak{A}$ and $\mathfrak{B}$ such that the conditions of the preceeding lemma hold. Let $\mathfrak{M}''$ be the restriction of $\mathfrak{M}'$ to B'. Then by the lemma, there is a directed system $\mathfrak{D}$ and an end isomorphism $\pi\colon \mathfrak{M}'' \to (\mathfrak{M}_a)_{\mathfrak{D}}$ such that $i_{\mathfrak{D}} = \pi \circ \boldsymbol{j}$. If we can show that π is a surjection, we will be done. Consider $i_{\mathfrak{D}}$ extended to $\mathfrak{M}$. Since in $\mathfrak{M}$ the cofinality of a is β, there is some $f \in M$ such that in $\mathfrak{M}$ f is an increasing β-sequence of ordinals approaching a. Hence, in $\mathfrak{M}_{\mathfrak{D}}$, $i_{\mathfrak{D}}(a) = \sup_{\gamma < i_{\mathfrak{D}}(\beta)} i_{\mathfrak{D}}(f)(\gamma)$. Then since j maps the ordinals less than β onto the ordinals less than $j(\beta)$, $\pi\colon \mathfrak{M}'' \cong (\mathfrak{M}_a)_{\mathfrak{D}}$ must be an isomorphism onto $(\mathfrak{M}_a)_{\mathfrak{D}}$.

(b) The proof of (b) is similar to (a).

(c) Suppose $j''\colon \mathfrak{M}_{a+1} \to \mathfrak{M}''$ is an extension of $j\colon \mathfrak{M}_a \to \mathfrak{M}'$. Let $\mathfrak{M}^*$ be the restriction $\mathfrak{M}''$ to $M_{j''(a)}$. Then since all relations on M_a that are $\mathfrak{M}$-sets can be coded as elements of M_{a+1}, we have that $j\colon \mathfrak{M}_a \to \mathfrak{M}^*$ is $\mathfrak{M}$-complete.

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[1] We can prove the theorem directly (without use of directed systems) if $\mathfrak{M}_{a+1}$ satisfies the reflection principle. But since $\mathfrak{M}$ may satisfy neither the power set axiom nor the principle of dependent choices, it is not apparent that the reflection principle holds.

Not all cofinal elementary embeddings $j\colon V_a \to \mathfrak{M}'$ are complete. Consider a non-principal ultrafilter $\mathfrak{U}$ on ω and an ordinal a of cofinality ω such that V_a is a model of Zermelo-Fraenkel set theory. By proposition 5, $i_{\mathfrak{U}}\colon V_a \to (V_a)_{\mathfrak{U}}$ can be factored as a cofinal V_a-complete embedding followed by an elementary end embedding. But the cofinal V_a-complete embedding cannot be complete.

3. Nicely complete embeddings. In this section we consider well-founded $\mathfrak{M}$. Thus, we assume that $\mathfrak{M} = \langle M, \epsilon\rangle$ is a standard transitive model. We also assume that ${}^{\omega}M \subseteq M$ and that in $\mathfrak{M}$ every set is well-ordered. Let Ω be the set of ordinals in M.

Since we have considered collections of $\mathfrak{M}$-classes, we have been viewing $\mathfrak{M}$ from the "outside". Thus, unless we make some minor modifications, our results do not literally hold for the standard model $\mathfrak{M} = \mathrm{V}$ of Zermelo Fraenkel set theory. However, it is natural to consider "second order" sets built up from the "first order" sets in V as is done in [2], [7].

We say that a directed system $\mathfrak{D}$ of $\mathfrak{M}$-ultrafilters is *nice* if for any increasing sequence $a = \langle a_n\colon n \in \omega\rangle$ of elements of D and $X \in M \cap \prod_{n \in \omega} \mathfrak{U}_{a_n}$ there exists $s \in \prod_{n \in \omega} X_n$ such that for all $n \in \omega$ and $F \in \mathcal{F}_{a_n a_{n+1}}$, $F\big(s(n+1)\big) = s(n)$. Note that we obtain an equivalent definition if X is not required to be in M. Note also that when $\mathfrak{D}$ is nice, $\mathfrak{U}_a$ must be countably complete for each $a \in D$.

We say an elementary embedding $j\colon \mathfrak{A} \to \mathfrak{B}$ is *nicely $\mathfrak{M}$-complete* if there is an $\mathfrak{M}$-expansion $\mathfrak{A}'$ of $\mathfrak{A}$ whose relations include all binary relations on A that are $\mathfrak{M}$-sets and there is an expansion $\mathfrak{B}'$ of $\mathfrak{B}$ such that (i) $j\colon \mathfrak{A}' \to \mathfrak{B}'$ remains elementary, (ii) for every $x \in B$ there is an $\mathfrak{M}$-set $X \subseteq A$ such that $x \in \boldsymbol{j}(X)$, and (iii) $\boldsymbol{j}(R')$ is a well-ordering on B for every well-ordering R' on A that is an $\mathfrak{M}$-set.

Let $\mathcal{A}_n$ be the subcategory of $\mathcal{A}$ of all nice, normal directed systems of $\mathfrak{M}$-ultrafilters concentrating on A. Let $\mathcal{B}_n$ be the subcategory of $\mathcal{B}_w$ of all nicely $\mathfrak{M}$-complete embeddings defined on $\mathcal{A}$. We will show that $\iota\colon \mathcal{A}_n \to \mathcal{B}_n$ is an equivalence.

We will call a partially ordered set T a *tree* if T is a set of finite sequences with the extension relation $\precsim$ understood. We say T is *well-founded* if T has no infinite chains. A *rank function* for a tree is a function R from T into the ordinals such that for $t \in T$

$$R(t) = \sup\{R(s)+1\colon s \precsim t \wedge s \in T\}.$$

More generally, we call a function from T into the ordinals a *norm for T* if it is order-preserving and one-to-one. If a norm is an $\mathfrak{M}$-function, we all it an *$\mathfrak{M}$-norm*. Every well-founded tree has a rank function, which we denote by R_T. It is the least norm for T in the sense that (i) R_T is a norm for T and (ii) for any norm N for T, $R_T(t) \leqslant N(t)$ for all $t \in T$. Note that if $T \in M$, then $R_T \in M$. Hence, if a tree is in M, it is well-founded if and only if it has an $\mathfrak{M}$-norm. We call $\sup\{R_T(t)+1\colon t \in T\}$ the rank of a well-founded tree T.

Suppose $\mathfrak{D}$ is a directed system of $\mathfrak{M}$-ultrafilters. To each increasing sequence a of elements of D, $X \in M \cap \prod_{n\in\omega} \mathcal{U}_{a_n}$ and $F \in \prod_{n\in\omega} \mathcal{F}_{a_n a_{n+1}}$ we associate an induced tree

$$T = T^{a,X,F} = \bigcup_{n\in\omega} \{t \in \prod_{m\leqslant n} X_m\colon \forall m < n \; F_m\big(t(m+1)\big) = t(m)\}\,.$$

Since we assume $X \in M$, $T \in M$. Clearly T is well-found if and only if there is no $s \in \prod_{n\in\omega} X_n$ such that for all $n \in \omega$, $F_n\big(s(n+1)\big) = s(n)$. Consequently, $\mathfrak{D}$ is nice if and only if every induced tree is not well-founded.

THEOREM 9. *Suppose $\mathfrak{D}$ is a directed system of $\mathfrak{M}$-ultrafilters. Then for $a < \Omega$, $i_{\mathfrak{D}}(a)$ is well-ordered if and only if no induced tree T has an $\mathfrak{M}$-norm $N\colon T \to a$.*

Proof. Suppose that there is some induced tree $T = T^{a,X,F}$ and an $\mathfrak{M}$-norm $N\colon T \to a$ for T. For each $n \in \omega$, let $G_n\colon M \to a$ be the $\mathfrak{M}$-function defined by

$$G_n(x) = \begin{cases} N(t_{\restriction n+1}) & \text{if } t \in T \text{ and } t(n) = x\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then for $n \in \omega$,

$$\{x \in M\colon G_{n+1}(x) < G_n\big(F_n(x)\big)\} \supseteq X_{n+1} \cap \bigcap_{m\leqslant n} (F_m \circ F_{m+1} \circ \ldots \circ F_n)^{-1}(X_m) \in \mathcal{U}_{a_{n+1}}\,.$$

Hence, for $n \in \omega$,

$$\varphi_{a_{n+1}}([G_{n+1}]_{a_{n+1}}) <_{\mathfrak{D}} \varphi_{a_n}([G_n]_{a_n})\,.$$

Consequently $i_{\mathfrak{D}}(a)$ is not well-ordered.

Now suppose that $i_{\mathfrak{D}}(a)$ is not well-ordered. Then since $\langle D, \leqslant\rangle$ is directed, there is an increasing sequence a of elements of D, there are sequences $F \in \prod_{n\in\omega} \mathcal{F}_{a_n a_{n+1}}$ and $\langle G_n\colon M \to a\rangle_{n\in\omega}$ of $\mathfrak{M}$-functions, and there is a sequence $X \in M \cap \prod_{n\in\omega} \mathcal{U}_{a_n}$ such that for $n \in \omega$, $X_{n+1} \subseteq \{x \in M\colon G_{n+1}(x) < G_n\big(F_n(x)\big)\}$. Let $T = T^{a,X,F}$. Let $N\colon T \to a$ be defined by $N(t) = G_n\big(t(n)\big)$, where $n+1$ is the length of $t \in T$. Then clearly N is an $\mathfrak{M}$-norm on T.

COROLLARY 10. *Suppose $\mathfrak{D}$ is a directed system of $\mathfrak{M}$-ultrafilters and $a < \Omega$. Then $i_{\mathfrak{D}}(a)$ is well-ordered if and only if every well-founded induced tree has rank greater than a.*

COROLLARY 11. *$\mathfrak{D}$ is nice if and only if for all $a < \Omega$ $i_{\mathfrak{D}}(a)$ is well-ordered.*

LEMMA 12. *Suppose $\beta \leqslant \Omega$ has cofinality greater than ω. Suppose $\mathfrak{D}$ is a directed system of $\mathfrak{M}$-ultrafilters such that for $a \in D$ $\mathcal{U}_a$ concentrates on an $\mathfrak{M}$-set I_a whose cardinality in $\mathfrak{M}$ is less than β, and for all $a < \beta$ $i_{\mathfrak{D}}(a)$ is well-ordered. Then for all $a < \Omega$ $i_{\mathfrak{D}}(a)$ is well-ordered.*

Proof. Suppose $i_{\mathfrak{D}}(a)$ is not well-ordered for some $a < \Omega$. Then since $\langle D, \leqslant\rangle$ is directed, we can find an increasing sequence $a\colon \omega \to D$ and a sequence $\langle F_n\colon M \to a\rangle_{n\in\omega}$ of $\mathfrak{M}$-functions such that $\langle \varphi_{a_n}([F_n]_{a_n})\rangle_{n\in\omega}$ is decreasing. Since in $\mathfrak{M}$, I_{a_n} has cardinality less than β and β has cofinality greater than ω, the cardinality in $\mathfrak{M}$ of $\bigcup_{n\in\omega} F_n''(I_{a_n})$ is less than β. Hence, for some $\gamma < \beta$ there is an $\mathfrak{M}$-function $G\colon M \to \gamma$ whose restriction to $\bigcup_{n\in\omega} F_n''(I_{a_n})$ is order-preserving and one-to-one. Thus, $\langle \varphi_{a_n}([G \circ F_n]_{a_n})\rangle_{n\in\omega}$ is decreasing so that $i_{\mathfrak{D}}(\gamma)$ is not well-ordered.

THEOREM 13. *$\iota\colon \mathcal{A}_n \to \mathcal{B}_n$ is an equivalence.*

Proof. If $\mathfrak{D}$ is nice, then by Corollary 11, $\iota(\mathfrak{D})$ is nicely $\mathfrak{M}$-complete, and hence, is an object of $\mathcal{B}_n$. The rest of the proof is the same as the second part of the proof of Theorem 1 except that we must check that the directed system $\mathfrak{D}$ that was constructed is nice. But this follows from Lemma 12 and Corollary 11.

COROLLARY 14. *$j\colon \mathfrak{A} \to \mathfrak{B}$ is a nicely $\mathfrak{M}$-complete embedding if and only if there is an elementary embedding $j'\colon \mathfrak{M}' \to \mathfrak{M}''$ such that $j' \supseteq j$, $j' \supseteq j$, $\mathfrak{M}'$ contains all relations that are $\mathfrak{M}$-classes, and for every well-ordering R' that is an $\mathfrak{M}$-set $j'(R')$ is a well-ordering.*

Next we give sufficient conditions for the existence of nicely $\mathfrak{M}$-complete embeddings.

THEOREM 15. *Suppose $a < \Omega$.*

(a) *Suppose a has cofinality greater than ω, and the cardinality in $\mathfrak{M}$ of every element of M_a is less than a. Then if $\mathfrak{M}'$ is well-founded and $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is a cofinal elementary embedding, there is a well-founded end extension $\mathfrak{M}''$ of $\mathfrak{M}'$ such that $j\colon \mathfrak{M}_a \to \mathfrak{M}''$ is nicely $\mathfrak{M}$-complete.*

(b) *If a is inaccessible in $\mathfrak{M}$ and $\mathfrak{M}'$ is well-founded, then every cofinal elementary embedding $j\colon \mathfrak{M}_a \to \mathfrak{M}'$ is nicely $\mathfrak{M}$-complete.*

(c) *If $j\colon V_{a+1} \to V_{\beta+1}$ is an elementary $\mathfrak{M}$-embedding and a is the first ordinal moved, then $j \restriction V_a\colon V_a \to V_\beta$ is nicely $\mathfrak{M}$-complete.*

(d) *If $j\colon V_{a+2} \to \mathfrak{M}'$ is an elementary $\mathfrak{M}$-embedding and $\mathfrak{M}'$ is a standard transitive model such that $M' \supseteq V_{j(a+1)}$, then $j \restriction V_{a+1}\colon V_{a+1} \to V_{j(a+1)}$ is nicely $\mathfrak{M}$-complete.*

Proof. The proofs of (a) and (b) are the same as the proofs of Theorem 8(a) and Theorem 7, respectively, except that we must check that the directed system we obtained is nice. But this follows from Lemma 12 and Corollary 11.

We only prove (c) since the proof of (d) is similar. We are given an elementary embedding $j\colon V_{\alpha+1} \to V_{\beta+1}$ where $\beta = j(\alpha)$ and α is the first ordinal moved. It suffices to show that j preserves well-orderings. Suppose, on the contrary, that R' is a well-ordering of V_α, but that $j(R')$ is not a well-ordering of V_β. Then there is a $j(R')$-descending sequence $s\colon \omega \to V_\beta$. Since $\beta > \omega$ is inaccessible, $s \in V_\beta$. Hence, $V_{\beta+1}$ satisfies

$$\exists s\colon \omega \to V_{j(\alpha)}\ \forall n \in \omega\ \langle s(n+1), s(n)\rangle \in j(R').$$

Then since $j\colon V_{\alpha+1} \to V_{\beta+1}$ is elementary, $V_{\alpha+1}$ satisfies

$$\exists s\colon \omega \to V_\alpha\ \forall n \in \omega\ \langle s(n+1), s(n)\rangle \in R'.$$

But then R' is not a well-ordering.

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Two conjectures regarding the stability of ω-categorical theories

by

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Abstract. It has been conjectured (1) that any ω-categorical first order theory has finite Morley rank, and (2) that any stable ω-categorical theory is totally transcendental. In this paper it is shown that any structure, whose theory is a counterexample to either of these two conjectures, contains a pseudoplane. Here a pseudoplane consists of a universe of "points" and "lines" together with an incidence relation; the axioms are that each line contains infinitely many points, and that two distinct lines meet in at most a finite number of points, together with the duals of these. Thus both conjectures would follow if it could be shown that ω-categorical pseudoplanes do not exist.

The greater part of this paper is motivated by the conjecture:

C1. *If T is stable and ω-categorical then T is totally transcendental.*

In § 1 we prove a conjecture weaker than C1 namely:

C1′. *If T is superstable and ω-categorical then T is totally transcendental.*

The truth of C1′ was first known by Shelah. Allthrough it has not appeared explicity it follows immediately from two lemmas of [6], Lemma 38, p. 106 and Lemma 40, p. 108. The main tool we use namely that of normalizing ranked was also invented by Shelah.

In § 2 we show that if M is a structure which refutes C1 then M contains a pseudoplane. Let "$\bigvee_\omega x$" be read "for at most a finite number of x". A *pseudoplane* is a model for the axioms

$$\bigvee x\big(I(x, y) \vee I(y, x)\big),$$

$$\bigvee x I(x, y) \wedge \bigvee y I(x, y) . \to x \neq y,$$

$$x_0 \neq x_1 \to \bigvee_\omega y\big(I(x_0, y) \wedge I(x_1, y)\big),$$

$$y_0 \neq y_1 \to \bigvee_\omega x\big(I(x, y_0) \vee I(x, y_1)\big),$$

$$\bigvee x I(x, y) \to \neg \bigvee_\omega x I(x, y) . \wedge . \bigvee y I(x, y) \to \neg \bigvee_\omega y I(x, y) .$$