same ordinals. (Note, the proof needs only a finite part of ZF, so the assumption about inner models can be eliminated.)

Some consequences of the lemma are:

1. If $2^{\omega_1} > 2^{\omega_0}$, then there are absolutely measurable sets which are not in the projective hierarchy.

2. Let A be the set of the lemma. We can prove in the theory ZF+ $+\forall a \subset \omega (\omega_1^{L[a]} < \omega_1)$ that A is not Σ_2^1.

3. Let A be the set of the lemma. We can prove in the theory ZF+PD that A is not projective.

To prove 1 use a cardinality argument. To prove 2 and 3 notice that in $\mathrm{ZF}+\forall a \subset \omega (\omega_1^{L[a]} < \omega_1)$ every uncountable Σ_2^1 set contains a perfect subset, Solovay [7], and in ZP+PD every uncountable projective set contains a perfect set, see e.g. [5].

Remark. The lemma does not answer the question about the complexity of absolutely measurable sets in every case. It has been proved consistent by Martin and Solovay [4] that every Σ_2^1 set is absolutely measurable and that every set of cardinality ω_1 is Π_1^1.

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Solution to a problem of Gandy's

by

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Abstract. Consider the hierarchy of ramified analytical sets A_β, where A_0 = finite sets of integers (for simplicity, finite *reals*), $A_{\beta+1}$ = reals definable by analytical predicates with constants from A_β and quantifiers restricted to A_β, $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$, if λ is a limit. One of the authors and Gandy independently confirmed a conjecture of Cohen by proving the existence of a smallest β-model of analysis. Moreover, they identified it to be A_{β_0}, where β_0 is the least place where the hierarchy A_β stops, i.e., the least β such that $A_\beta = A_{\beta+1}$. We prove here that for all $\beta < \beta_0$, $A_{\beta+1}$ = reals definable by analytical predicates *without constants* with quantifiers restricted to A_β. We also show that there is a constant-free predicate which uniformly well-orders the A_β (when its quantifiers are restricted to A_β), and a constant-free predicate which is satisfied by the arithmetically complete sets of order less than β.

Notation. We define the A_β as follows:

(i) $A_\varnothing = \{X \subset N\colon X \text{ is finite}\}$.

(ii) $A_{\beta+1} = \{X \subseteq N\colon X$ is 2-N.T. definable over A_β, using constants to name sets in A_β.

(iii) $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$.

Let $\beta_0 = (\mu\beta)(A_\beta = A_{\beta+1})$. The ramified analytic hierarchy (RAH) is defined to be A_{β_0}.

If $X \in A_{\beta+1} - A_\beta$ we say X is *of order* β. If X has the property, that any Y of order β (a fortiori, any $Y \in A_{\beta+1}$) is arithmetical in X, we say X is *complete of order* β. We shall use the notation "$Y \leqslant_a X$" to express "Y is arithmetical in X". We shall reserve the notation E_β to denote particular complete sets of order β.

Our notation will be otherwise that of [3]. We will assume throughout the results of [1] and [2], and especially the results on equivalences between the RAH and other hierarchies.

Gandy (in lectures in 1967) asked the following question: If we drop the mention of constants from clause (ii) above, do we still have a characterization of the same sets? In other words, if $X \in A_{\beta+1}$, does X have a constant-free definition over A_β? We shall answer Gandy's question in the affirmative. Our theorem is the following:

THEOREM. *Let* $\beta < \beta_0$. *If* $X \in A_{\beta+1}$ *then* X *is definable over* A_β *in* 2-N.T. *without set constants.*

In giving the proof we shall often have to show that a definition is arithmetical, or that a definition can be expressed in 2-N.T. (¹). To facilitate the reading of the argument, we will use English wherever possible, and leave to the reader the essentially mechanical task of verifying that our definition can be given formally in the required style. Thus, if R is a set of integer which defines a well-ordering, we shall write $x \in \mathrm{Field}\,R$ rather than, e.g. $(\mathrm{E}a)(2^a \cdot 3^x \in R \vee 2^x \cdot 3^a \in R)$.

Notice that if X is definable over A without set constants, then so is any Y r.e. in X. To establish this, we use Rogers' characterization of relative recursiveness:

$$W_e^X = \{x\colon (\mathrm{E}a)(\mathrm{E}u)(\mathrm{E}v)(\langle x, a, u, v\rangle \in W_e \,\&\, D_u \subseteq X \,\&\, D_v \subseteq \overline{X})\}\,.$$

If X is 2-N.T. definable over A by $M(x)$ (²), where M is a formula without set constants, then the following will define Y over A:

$$(\mathrm{E}a)(\mathrm{E}u)(\mathrm{E}v)\big(\langle x, a, u, v\rangle \in W_e \,\&\, (z)\big(z \in D_u \Rightarrow M(z) \,\&\, z \in D_v \Rightarrow -M(z)\big)\,.$$

It follows that the class of sets of integers definable over A without set constants is closed under $\leqslant_T$ and closed under jumps, hence is closed under $\leqslant_a$. Consequently, to show that this class is identical with $A_{\beta+1}$, it will be sufficient to show that it contains some E_β, a complete set of order β. We shall prove this fact, and thereby our theorem, by establishing the following:

LEMMA. *There is a two-place predicate of* 2-*N*.T. *without set constants,* $W(X, Y)$, *which defines over each* A_β, $\beta \leqslant \beta_0$, *a well-ordering of the sets in* A_β. *I.e., the relation*

$$X \leqslant Y \equiv A_\beta \models W(X, Y)$$

is a well-ordering of the sets in A_β.

It is easy to see that $A_\beta \models W(X, Y)$ is a well-ordering; e.g. that for any formula $M(X)$,

$$A_\beta \models (\exists X) M(X) \Rightarrow (\exists!\, X)\big(M(X) \,\&\, (Y)\big(M(Y) \Rightarrow W(X, Y)\big)\big)\,.$$

The fact that a single formula works for all $\beta \leqslant \beta_0$ is interesting; it will play no role in the proof.

We now show how to use our lemma to prove the theorem.

(¹) 2-N.T. is the two-sorted language, whose small-type variables range over integers; the capital-letter variables range over 2^N.

(²) By "X is 2-N.T. definable over Ψ by $M(x)$", we mean $X = \{x\colon \Psi \models M(x)\}$.

DEFINITION. Let Ψ be a set of sets of integers. We call a set E a *uniform upper bound* (uub) *on* Ψ if $(\exists z)(X)\big(X \in \Psi \Leftrightarrow (\exists r \in W_z^E)(C_X = \Phi_r^E)\big)$. Thus E is a uub on Ψ if each set in Ψ is $\leqslant_T E$ and further, there is an E-recursive enumeration of the Gödel-numbers in E of the sets in Ψ.

Let $\beta < \beta_0$. We claim that if E is a uub on A_β, definable in 2-N.T. over A_β, then E is a complete set of order β. First, E is of order β. For $E \in A_{\beta+1}$; further, if E were $\in A_\beta$, then $E' \in A_\beta$, so that $E' \leqslant_T E$, which is impossible. Hence $E \in A_{\beta+1} - A_\beta$, i.e. E is of order β. To show E a complete set, let E_1 be a set of order β. Then E_1 is definable over A_β by a formula $M(x)$ in 2-N.T. (perhaps containing names of sets in A_β). Using the fact that E is a uub on A_β, we may paraphrase quantification over sets in A_β by quantification over Gödel-numbers: we may thereby convert the formula $M(x)$ into an E-arithmetical formula which defines E_1 — the set constants will be replaced by numerals. Hence $E_1 \leqslant_a E$. So E is complete of order β.

Consequently, to prove the theorem, it will be sufficient to find a uub on A_β definable over A_β in 2-N.T. without set constants. Now we know there are uubs on A_β definable over A_β in 2-N.T.; let $M_1(x, A_1 \ldots A_n)$ define such a uub, where the A_i name sets in A_β. Then the same uub may be defined by a formula $M(x, A)$, where A names a set in A_β. (A may, for example, be taken as $\{\langle a, i\rangle\colon a \in A_i\}$. We may use M to define a uub on A_β, over A_β, without set constants, as follows: Let B be the W-least set in A_β such that $M(x, B)$ defines a uub on A_β, when the set quantifiers of M are taken to range over A_β. Let E be the uub thus defined by $M(x, B)$. Then E may be defines over A_β in 2-N.T. by the following set-constant-free formula:

$$(\exists X)\big(\{y\colon M(y, X)\} \text{ is a uub on } 2^N \,\&\, (Y)\big(\{y\colon M(y, Y)\} \text{ is a uub on } 2^N \Rightarrow W(X, Y)\big) \,\&\, M(x, X)\big)\,.$$

(We leave to the reader the translation of this formula into 2-N.T.)

This completes the proof of the theorem.

We turn now to proving the lemma. Boolos [1] has found formulas which play the role of W for HYP ordinals β (i.e. for β such that $X \in A_\beta \Rightarrow O^X \in A_\beta$). Further, it is not difficult to produce such formulas for $\beta <$ the least HYP ordinal. Because a limit of HYP ordinals is itself HYP, this leaves only one case: $\beta = \delta + \eta$, where $\beta < \beta_0$ and δ is the greatest HYP ordinal $\leqslant \beta$, $\eta > 0$. Our proof applies indifferently to all these cases.

Let $\beta < \beta_0$. We will construct a parameter-free formula with free set variable, satisfied in A_β only by complete sets. We recall two definitions from [2]:

DEFINITION. If $P \subseteq 2^N$, the recursive union of P $\big(\mathrm{RU}(P)\big)$ is $\{X\colon (\mathrm{E}Y \in P)(X \leqslant_T Y)\}$.

DEFINITION. If $A \subseteq N$, $A_i = \{a\colon \langle a, i\rangle \in A\}$.

We now introduce the notation $\Xi(X, Y)$ to represent the following situation: Y (taken as a set of ordered pairs) defines a well-ordering of integers, and X defines a hierarchy indexed by Field Y in the following sense:

(i) $|i|_Y = 0 \Rightarrow X_i = N$.

(ii) $|i|_Y = |j|_Y + 1 \Rightarrow X_i = X_j^{(\omega)}$.

(iii) $|i|_Y$ a limit $\Rightarrow X_i$ is an arithmetically least uub on $\mathrm{RU}(\{X_j\colon j <_Y i\})$ (i.e., for every uub Z on $\mathrm{RU}(\{X_j\colon j <_Y i\})$, $X_i \leqslant_a Z$).

It is easy to show that an equivalent formulation of condition (iii) is the following:

(iii′) $|i|_Y$ a limit $\Rightarrow X_i$ is an arithmetically least uub on $\{X_j\colon j <_Y i\}$.

If $|Y| = \theta+1$, and if $\Xi(X, Y)$, then the X_i, $i \in \mathrm{Field}\, Y$, constitute the hierarchy of complete sets, up to and including the complete set of order θ. This fact is essentially proved in [2].

For any $\alpha < \beta$ there are X and $Y \in A_\beta$ with $\Xi(X, Y)$ and $|Y| = \alpha+1$. The proof of this fact is implicit in [1] and [2], we sketch here a proof which, though tedious, is perhaps as quick as any:

In [1] it was shown that for $\theta < \beta_0$, $A_\theta = M_\theta \cap 2^N$ (3), where M_θ is the θth level of Gödel's hierarchy of constructible sets. Further, it was shown that for $\theta < \beta_0$ there is a complete set E_θ of order θ which is an arithmetic copy of M_θ in the following sense:

Let Field $E = \{x\colon (\mathrm{E}y)(\langle x, y\rangle \in E \vee \langle y, x\rangle \in E)\}$. Then, taking E_θ as a set of ordered pairs, E_θ defines a binary relation whose doman is Field E_θ. We have $\langle \mathrm{Field}\, E_\theta, E_\theta\rangle \simeq \langle M_\theta, \varepsilon_{M_\theta}\rangle$. For formulas φ in the language of set theory, we shall write

$$E_\theta \models \varphi \quad \text{for} \quad \langle \mathrm{Field}\, E_\theta, E_\theta\rangle \models \varphi.$$

Finally, it was (essentially) shown in [4] that there is a first-order formula in set theory, $\sigma(x)$ such that $M_\gamma \models \sigma(x)$ iff $x = M_\lambda$ for some limit $\lambda < \gamma$. It follows that $E_\theta \models \sigma(x)$ iff x represents in E_θ some M_λ for a limit ordinal λ.

Now given $\alpha < \beta$, choose $E_\alpha \in A_{\alpha+1}$, an arithmetic copy of M_α. We shall use the ordinals of E as an index set Y_1 for a hierarchy of complete sets X_1, with $\Xi(X_1, Y_1)$. Notice, however, that $|Y| = \alpha$. To find an X, Y with $\Xi(X, Y)$ and $|Y| = \alpha+1$, we shall simply add the set E_α to the end of the hierarchy, indexed by a new integer.

Let $Y_1 = \{\langle x, y\rangle \in E_\alpha\colon E_\alpha \models x$ and y are ordinals$\}$.

Let R_1 be the set of $\langle x, i\rangle$ such that i represents a limit ordinal λ and x represents a least complete set of order λ. R_1 may be defined as follows: (arithmetically in E_α):

(3) We let $M_\varnothing = \{x\colon x$ is hereditarily finite$\}$.

$R_1 = \{\langle x, i\rangle\colon (\mathrm{E}z)(E_\alpha \models \sigma(z)$ and i is a limit ordinal and $i =$ the set of ordinals in z and x is a uub on the set of sets of integers in z and v (v is a uub on the sets of integers in $z. \Rightarrow x \leqslant_a v$) and for all x' with this property, $x \leqslant x'$.)

We may now fill in the gaps in R_1: Let

$R_2 = \{\langle x, i\rangle\colon E_\alpha \models i$ is an ordinal and either 1. $E_\alpha \models i$ is a limit ordinal $\&\ \langle x, i\rangle \in R_1$ or 2. $(\mathrm{E}n)(\mathrm{E}j)(E_\alpha \models i = j+n\ \&.\ j$ is a limit $\vee$ $\vee\ j = \bar{0}$ and $(\mathrm{E}y)(\langle y, j\rangle \in R_1\ \&\ E_\alpha \models x$ is the nth ω-jump of $y)\}$ (4).

We may now define $X_1 = \{\langle y, i\rangle\colon (\mathrm{E}x)(\langle x, i\rangle \in R_2\ \&\ E_\alpha \models \bar{y} \in x$.

Finally, let

$$Y = \{\langle 2^i, 2^j\rangle\colon \langle i, j\rangle \in Y_1\} \cup \{\langle 2^i, 3\rangle\colon i \in \mathrm{Field}\, Y_1\},$$

$$X = \{\langle y, 2^i\rangle\colon \langle y, i\rangle \in X_1\} \cup \{\langle y, 3\rangle\colon y \in E_\alpha\}.$$

Using the techniques of [1], it is easy to show that X and Y are both $\leqslant_a E_\alpha$, hence are $\in A_\beta$. Clearly $\Xi(X, Y)$, also $|Y| = \alpha+1$.

We turn now to the problem of expressing the predicate $\Xi(X, Y)$ in A_β. Let $B(X, Y)$ be the obvious 2-N.T. expression of $\Xi(X, Y)$, with the following special features:

1. "Y is a well-ordering" is expressed by "every initial segment of Field Y is either = Field Y or has an l.u.b. in Field Y".

2. The following clause is added to the definition: $i <_Y j \Rightarrow X_i \leqslant_T X_j$.

We claim that for $X, Y \in A_\beta$, $\Xi(X, Y) \Leftrightarrow A_\beta \models B(X, Y)$.

Proof. Notice first that if Ψ is a set of sets of integers closed under $\leqslant_a$, then for $S, T^\omega \in \Psi$, $\Psi \models S \equiv_a T^\omega . \Leftrightarrow . S \equiv_a T^\omega$.

For $S \leqslant_a T^\omega$ is

$$(\mathrm{E}Z)\big(Z = T^\omega\ \&\ (\mathrm{E}R)(\mathrm{E}m)\big(R_1 = Z\ \&\ (\nabla n < m)\big(R_{n+1} = (R_n)'\big)\ \&\ S \leqslant_T R_m\big)\big).$$

Now this predicate $C(S, T)$ is Σ_1^1, so $\Psi \models C(S, T). \Rightarrow S \leqslant_a T^\omega$. Further, if $S \leqslant_a T^\omega$, then the relevant sets T^ω and R are $\in \Psi$, so $\Psi \models C(S, T)$. Similarly, we may show that $T^\omega \leqslant_a S$ is absolute in models closed under $\leqslant_a$. Finally, the relation T_i is a uub on $\mathrm{RU}\{T_j\colon j <_S i\}$, taken as a relation between T, i, and S, is absolute; indeed, it is arithmetical in T join S.

It is now easy to show that for $X, Y \in A_\beta$, $\Xi(X, Y) \Rightarrow A_\beta \models B(X, Y)$. Assume $\Xi(X, Y)$. Because "Y is a well-ordering" is Π_1^1, we then have $A_\beta \models$ "Y is a well-ordering". Clearly, $A_\beta \models |i|_Y = 0 \Rightarrow X_i = N$. By the

(4) The 1-st ω-jump of A is A^ω. The $(n+1)$-th ω-jump of A is the 1-st ω-jump of the nth ω-jump of A.

preceding paragraph, for $|j|_Y = |i|_Y + 1$, $A_\beta \models X_j \equiv_a X_i^\omega$ (5). Also, for $|i|_Y$ a limit, we have $A_\beta \models X_i$ is a uub on $\mathrm{RU}\{X_j\colon j <_Y i\}$. Indeed, for any $Z \in A_\beta$, $A_\beta \models Z$ is a uub on $\mathrm{RU}\{X_j\colon j <_Y i\} \Rightarrow Z$ is a uub on $\mathrm{RU}\{X_j\colon j <_Y i\}$ $\Rightarrow X_i \leqslant_a Z \Rightarrow A_\beta \models X_i \leqslant_a Z$. So $A_\beta \models X_i$ is an arithmetically least uub on $\mathrm{RU}\{X_j\colon j <_Y i\}$. Finally, it is clear that $A_\beta \models i <_Y j \Rightarrow X_i \leqslant_T X_j$.

We now show $A_\beta \models B(X, Y) . \Rightarrow \Xi(X, Y)$ for $X, Y \in A_\beta$. Suppose $A_\beta \models B(X, Y)$. Then Y is a total ordering of integers. Let Head $Y = \{\langle i, j\rangle \in Y\colon Y \upharpoonright \{n\colon n \leqslant_Y j\}$ is a well-ordering. We claim that for $i \in$ Field Head Y, X_i is a complete set of order $|i|_Y$. The claim is easy to establish: Head Y is well-ordered by Y; the induction uses the absoluteness results of the preceding paragraph and the main results of [2].

Observe that if Head $Y \neq Y$, then Field Head Y has no last element. For if Field Head $Y = \{i\colon i \leqslant_Y j\}$ for some $j \in$ Field Head Y then $(\mathrm{E}Z)(Z \in A_\beta \;\&\; Z =$ Field Head $Y)$. So $A \models$ there is a bounded initial segment of Field Y with no l.u.b., which is impossible.

Consequently, either Head $Y = Y$ or (Head $Y \neq Y$ & Head $Y = \lambda$ for some limit ordinal λ.) We will show that in fact Head $Y = Y$. Suppose on the contrary, that Head $Y \neq Y$ and $|\text{Head } Y| = \lambda$. By reasoning similar to that of the last paragraph we may refute this assumption by using it to derive that Head $Y \in A_\beta$, as follows:

We claim $\lambda \ngtr \beta$. For if $\lambda > \beta$, then some $i \in$ Field Head Y is such that $|i|_Y = \beta$. But then X_i is a complete set of order β, so $X_i \notin A_\beta$, which is impossible. We claim now that $\lambda \neq \beta$. For if $\lambda = \beta$, let $i \in$ Field Y— — Field Head Y. Then every complete set of order θ for $\theta < \beta$ is $\leqslant_T X_i$ (6). So every set in A_β is recursive in X_i, whence $X_i \notin A_\beta$, which is impossible. Hence $\lambda < \beta$. Consequently some complete set E_λ of order λ is $\in A_\beta$. Now $i \in$ Field Head $Y \Rightarrow X_i$ is a complete set of some order $< \lambda \Rightarrow X_i \in A_\lambda$. And $i \in$ Field Y—Field Head $Y \Rightarrow$ every complete set of order $< \lambda$ is $\leqslant_T X_i \Rightarrow X_i \notin A_\lambda$. Hence Field Head $Y = \{i \in \text{Field } Y\colon X_i \in A_\lambda\}$. But the predicate $Z \in A_\lambda$ is arithmetic in any complete set of order λ, so Field Head Y is arithmetic in E_λ join Y. So Field Head $Y \in A_\beta$, so Head $Y \in A_\beta$. Hence Head $Y = Y$. It follows that Y is a well-ordering, hence that $\Xi(X, Y)$. Q.E.D.

Consider the predicate

$$D(Z) \equiv (\mathrm{E}X)(\mathrm{E}Y)(\mathrm{E}i)\big(B(X, Y) \;\&\; i \in \text{Field } Y \;\&\; Z \equiv_a X_i\big) .$$

We have shown that this parameter-free 2-N.T. predicate is satisfied

(5) Here we make some obvious assumptions about the absolutness of such relations as $|j|_Y = |i|_Y + 1$. The proofs are trivial.

(6) By the extra condition on the expression $B(X, Y)$, every X_j, for $|j|_Y < \beta$ is $\leqslant_T X_i$. However, every complete set of order $< \beta$ is $\leqslant_a$ some such X_j, and hence — because β is a limit — $\leqslant_T$ some $X_r \leqslant_T X_j$.

in A_β by precisely the complete sets of all orders $< \beta$. We use $D(Z)$ to define a well-ordering as follows:

$W(X, Y)$ if order $X <$ order Y, or (order $X =$ order $Y = \theta$, and some constant-free definition of X over A_θ is shorter than any constant-free definition of Y over A_θ).

The translation of this predicate into 2-N.T. is essentially carried out in [2]. We sketch here some crucial steps:

1. There is a constant-free predicate of 2-N.T. which expresses in A_β the relation E is a complete set & $X \in A_{\text{order } E}$ thus:

$$D(E) \;\&. (\mathrm{E}Y)\Big(Y \leqslant_T E \;\&\; D(Y) \;\&\; E \nleqslant_a Y \;\&\; (Z)\big(Z \leqslant_T E \;\&\; D(Z) \;\&\; E \nleqslant_a Z \Rightarrow Z \leqslant_a Y\big) \;\&\; X \leqslant_a Y\Big) \vee (\mathrm{E}Y)(\mathrm{E}Z)\big(D(Y) \;\&\; D(Z) \;\&\; Y \leqslant_a Z \leqslant_a E \;\&\; Z \nleqslant_a Y \;\&\; E \nleqslant_a Z \;\&\; X \leqslant_a Y\big) .$$

(The first disjunct expresses that order E is a successor, and $X \in a$ complete set of the preceding order; the second deals with the case where order E is a limit.)

2. For any constant-free sentence of 2-N.T., φ, any E_θ a uub on A_θ, the question, does $A_\theta \models \varphi$? may be uniformly translated into an arithmetical question about E_θ, whose length $= \varrho$ (length of φ), for some recursive function ϱ. (Further, for all integers m, n, length $\varphi(\overline{m}) =$ length $\varphi(\overline{n})$). Consequently, T_A^n, the set of all constant-free 2-N.T. sentences of length $\leqslant n$, true in A_θ, is $\in A_{\theta+1}$. Finally, there is a constant-free formula of 2-N.T. $Q(x, E, Z)$, satisfied in A_β by precisely those x, E, Z such that E is a complete set of order θ, some $\theta < \beta$, and $Z = T_A^x$. (This is easy to do in the light of 1. We do not require, of course, that $T_{A\theta}^n \in A_\theta$, to be able to characterize $T_{A\theta}^n$).

3. To say a is the number of a constant-free formula which defines X over $A_{\text{order } E}$ we write $(\mathrm{E}n)$ (n is the length of a & $(s)\big(\delta(a, s) \in T^n_{A\,\text{order } E} \Leftrightarrow s \in X\big)$ where $\delta(t, s)$ is a recursive function such that if t is the Gödel number of $\varphi(x)$ then $\delta(t, s)$ is the Gödel number of $\varphi(\overline{s})$.

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Remarks on countable models (1)

by

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Abstract. Theories with finitely manny countable models are investigated. Several situations are shown in which such theories have at least two universal models. A theorem of Vaught is improved and an example is given which shows that it cannot be generalized.

Introduction. The theme of the major part of this paper is described in the following

CONJECTURE. *If a complete not ω-categorical theory has only finitely many countable models then it has at least two universal models.*

We thought we had a proof of the conjecture; it contained an "undergraduate mistake" but by strengthening the assumption of the conjecture the argument can be saved (see Theorem 3). We prove two other, maybe more interesting results, in the direction of the conjecture. Theorem 1 proves the existence of two universal models from a restriction on interactions between types of the theory. The concept we introduce may be useful in other investigations. Theorem 2 strengthens the hypothesis of the conjecture to: "every complete extension of T by finitely many constants has finitely many countable models". Known examples of theories with finitely many models satisfy either of the last two assumptions and they, of course, agree with the conjecture. On the other hand we are rather ignorant about the power of the assumption in the conjecture. We know that the conclusion boils down to finding an extension of T which omits a certain type (see proof of Theorem 3). But we have no syntactical characterization of theories with, say, 3 countable models. By Ryll-Nardzewski's theorem (see [9]) we know precisely when a theory has one countable model. It is thus surprising that no generalization of Ryll-Nardzewski's theorem has appeared. Other problems which emerged during our work on the conjecture are mentioned later.

Is the conjecture interesting? We, of course, think it is. Firstly, it, as any other conjecture, stimulates research which usually has value

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