Forcing with proper classes

by

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Abstract. In the paper a method of forcing with proper classes is presented. Some sufficient conditions for semantic forcing to be definable are formulated. (This part of the paper does not use the power set axiom; also some further proofs depend, in fact, only on satisfying the collection schema in the ground model.)

In the sequel the conditions for a generic set implying the generic extension to be a model for ZF^- (resp. ZFC) are given. As a corollary (using some concrete combinatorial conditions for a notion of forcing) we obtain e.g. the consistency of the theory ZFC^- + "every set is countable" + "the constructible universe is a model for ZF".

0. Introduction. In this paper we shall consider forcing with classes in the theory ZF^- (i.e. the ZF set theory without the power set axiom).

We shall assume the knowledge of the method of unramified forcing due to Shoenfield [10].

In part 1 we give the definition of forcing where the notion of forcing is a proper class and we introduce the definitions of a generic class and also the construction of the model $M[G]$ together with its simplest properties. This construction and lemmas are analogous to those of Shoenfield. Shoenfield [10] defines weak forcing in ZF under the assumption that the notion of forcing is a set. He proves the main three lemmas of Cohen and the usual connection between weak forcing and semantically defined forcing.

In part 2 we show that Shoenfield's result can be proved in ZF^-.

In part 3 we prove that for certain notions of forcing which are proper classes (the semicoherent notions of forcing — see Definition 3.2) the main three Cohen lemmas hold. The proof is divided into a sequence of lemmas. Note that the facts preceding Definition 3.2 are contained in [10], whereas here they are proven in the case of ZF^-.

Part 4 is fundamental to this paper. It was inspired by Chuaqui [1], which is illustrated e.g. by the notion of a sequence of dense sections (with suitable modifications) and the definition of a strong generic set (i.e. property G!!). In the sequel we investigate the properties of $M[G]$ in relation to the properties of the generic set under the assumption that forcing satisfies the three Cohen lemmas.

G!! — Theorem comes from Chuaqui and states that if M is a countable standard model for ZFC, G strongly generic, then $M[G]$ is a c.s.m. for ZFC.

The proof given in the present paper is not based on the definition of forcing formulated in [1]. Since one of the purposes of the paper was to obtain a model for ZF^- collapsing all cardinals onto ω, the generic set used for this purpose had to have properties weaker than strongly generic, yet sufficiently strong to give in the generic extension all the axioms of ZF^- and above all the axiom of comprehension.

The idea was as follows: consider generic sets for which intersecting uniformly defined dense sections is equivalent to intersecting some uniformly defined subset of those sections. This leads to the definition of the set of classes (Definition 4.2) and the main definition of the paper, i.e. the definition of a C-3-generic set (Definition 4.1). This definition is as follows:

G is C-3-generic $\overset{\text{dfn}}{\equiv}$ for every set of classes $\{D_a\}_{a\in b}$ where $b\neq\emptyset$, there exists a function $W\colon b\to V^M$, $W\in M$, such that $(a)_b(W(a)\subseteq D_a)$ and $(a)_b(W(a)\cap G\neq\emptyset)$.

This definition leads to the formulation of our basic theorem, i.e. the 3-Main Theorem: If M is a 3-model ("collection" model), $\langle C,\leqslant\rangle$ is a notion of forcing in M and the three fundamental lemmas about forcing are satisfied, then if G is C-3-generic over M then $M[G]$ is a 3-model.

The proof of this theorem is divided into a sequence of lemmas all based on the main lemma: If M is a c.s.m. for ZF^- and G is 3-generic, then the axiom scheme of comprehension is satisfied in $M[G]$.

Further lemmas, including the proof of the satisfaction of A.C., are proved in the following way: first we prove that some class of names can be restricted to a set of names and then the comprehension is applied. The proof of AC is different from that of [10] since we could not use the fact that K_G is definable in $M[G]$.

In part 5 we show that a coherent notion of forcing (see Chuaqui [1]) is semicoherent and thus we show, by a method different from that of Shoenfield [10], that forcing with a coherent notion of forcing satisfies the Cohen lemmas. It is worth mentioning that Easton [3] and Jensen [5], who forced with classes, used coherent notions of forcing.

We prove a Combinatorial lemma which establishes the fact that a continuous and coherent notion of forcing (see Definition 5.3) with definable well-ordering of the class of conditions satisfies the set — c.c. In this case every generic set is 3-generic.

In part 6 we prove the following theorem:

Consis (ZF) $\leftrightarrow$ Consis ($ZF^-+(ZF)^{\mathfrak{L}}+(x)$ ("x is countable")). This gives a positive solution to a problem formulated by Scott at the end of [9]. The problem was suggested to me by Professor Mostowski and led to

the considerations enclosed in this paper. Using the methods of part 6 one can prove that if M is a c.s.m. of $ZF+V=L$ and $M\models$ "$\varkappa$ is a regular cardinal", then there exists a c.s.m. N for ZF^- such that $M\subseteq N$, $On^M=On^N$, $N\models(ZF)^{\mathfrak{L}}+(x)(|x|\leqslant\varkappa)$; moreover,

$$N\models \mathrm{Card}[\varkappa]\leftrightarrow M\models \mathrm{Card}[\varkappa]\ \&\ \varkappa\leqslant\kappa\,,$$

$$\mathrm{cf}^N(\varkappa)=\mathrm{cf}^M(\varkappa)\ \text{ for }\ \varkappa\leqslant\kappa \quad\text{ and }\quad N\models 2^\varkappa=\varkappa^+\ \text{ for }\ \varkappa<\kappa\,.$$

Part 6 ends with the applications of the results of Chuaqui [1] (with suitable modifications) to a partial solution of the following problem:

Let M be a c.s.m. of ZFC and let $F\colon On^M\to On^M$ be a functional in M such that

$$(\xi_1)(\xi_2)\big(\xi_1<\xi_2\to F(\xi_1)<F(\xi_2)\big)\,.$$

For what F does there exist a c.s.m. N of ZFC such that $M\subseteq N$, $On^M=On^N$ and for all ordinal $\varkappa\in On^M$ we have $N\models\mathrm{Card}[\varkappa]$ iff $M\models(\exists a)(\varkappa=\omega_{F(a)})$?

We believe that forcing with classes will help to prove some problems of higher order arithmetics and set theory. It seems to be very important to know for which partially ordered classes weak forcing can be defined. It is known that if a partially ordered class can be extended to a set-complete Boolean algebra (i.e. complete under the intersections which are sets), then weak forcing can be defined and Cohen's lemmas are satisfied.

1. The construction of the model $M[G]$. In the following paper ZF^- will denote a theory whose language and primitive notions are those of Zermelo-Fraenkel set theory, with the exception of the power set axiom, which is omitted.

In ZF^-, as in ZF, we shall include the regularity axiom in the following form:

$$(x)[x\neq\emptyset\to(\exists y)(y\in x\ \&\ x\cap y=\emptyset)]\,.$$

The remaining axioms of ZF together with the essential explanations are given in an explicit form in [2]; our notation differs from that of [2] only in our use of "( )" instead of "$\forall$" for the general quantifier.

In the remaining part of the paper we shall often refer to some easy theorems of ZF^-; in particular:

$$(x)(\exists t)(\mathrm{Trans}(t)\ \&\ x\in t)$$

$$(t)\Big[\big(\mathrm{Trans}(t)\ \&\ t\neq\emptyset\big)\to(\exists!r)\Big(\mathrm{Func}(r)\ \&\ \mathrm{dom}(r)=t\ \&\ \downarrow$$

$$\downarrow\ (y)\big(y\in\mathrm{rg}(r)\to\mathrm{Ord}(y)\big)\ \&\ (u)\big(u\in t\to r(u)=\mathrm{Sup}\{r(z)\colon z\in u\}\big)\Big)\Big]\,,$$

i.e. for every transitive non-empty set there exists a unique rank function. Hence we deduce that if t_1 and t_2 are non-empty transitive sets, and r_1

and r_2 the respective rank functions, then $r_1 \upharpoonright t_1 \cap t_2 = r_2 \upharpoonright t_1 \cap t_2$, and since for every set x there exists a transitive set t such that $x \in t$, to every set x there corresponds exactly one ordinal number denoted by $\mathrm{rank}(x)$ and such that $\mathrm{rank}(x) = \mathrm{Sup}\{\mathrm{rank}(y)\colon y \in x\}$.

By $\mathfrak{L}_{ZF}$ we shall denote the language of ZF; if M is a set, then $\mathfrak{L}_{ZF}(M)$ denotes the ZF language extended by constants taken from M.

If a is an ordinal, then it is not essential for all sets of rank $< a$ to form a set; if, however, x is a set, then $\{y\colon \mathrm{rank}(y) < a \;\&\; y \in x\}$ is also a set. The latter remarks permit us to transfer the results concerning forcing from ZF to ZF$^-$ in an easy way. In the model-theoretical part of this paper we assume a new axiom, which has a metatheoretical character.

SM AXIOM. *There exists a standard model for* ZF.

FACT 1.1. *There exists a transitive, countable, standard model for* ZF.

Let M be a given transitive, countable, standard model for ZF or for ZF$^-$.

DEFINITION 1.1. A pair $\langle C, \leqslant\rangle$ is a *notion of forcing* if C is a formula with one free variable, and $\leqslant$ a predicate with two free variables, $C, \leqslant \in \mathfrak{L}_{ZF}(M)$, and if the following statements hold in model M:

1° $(x)\big(C(x) \to \leqslant(x, x)\big)$,

2° $(x)(y)\big(C(x) \;\&\; C(y) \;\&\leqslant(x, y) \;\&\leqslant(y, x) \to x = y\big)$,

3° $(x)(y)(z)\big(C(x) \;\&\; C(y) \;\&\; C(z) \;\&\leqslant(x, y) \;\&\leqslant(y, z) \to \leqslant(x, z)\big)$,

4° $(\exists x)\big(C(x) \;\&\; (y)\big(C(y) \to \leqslant(y, x)\big)\big)$.

From 2° & 4° it follows that the element existing by 4° is unique. We shall denoted it by 1_C.

Remark 1.1. If $M \models (\exists x)(y)\big(y \in x \leftrightarrow C(y)\big)$, then the set defined by formula C will also be denoted by C, which — we hope — will not lead to ambiquity. Instead of $\leqslant(x, y)$ we shall write $x \leqslant y$ and instead of $C(x)$ we shall write $x \in C$, even in the case where C does not define a set.

DEFINITION 1.2. A set $X \subseteq M$ is said to be a *class in* M if there exists a formula $\Theta(x_0) \in \mathfrak{L}_{ZF}(M)$ such that $a \in X \equiv M \models \Theta[a]$.

DEFINITION 1.3. D is *dense in* C if D is a class in M such that $(x)(x \in D \to x \in C)$ and $(x)_C(\exists y)_D(y \leqslant x)$.

DEFINITION 1.4. The elements of the class C are said to be *conditions*.

DEFINITION 1.5. Two conditions p and q are *compatible* if there exists an $s \in C$ such that $s \leqslant p$ and $s \leqslant q$. If such an s does not exist, then p and q are incompatible.

Remark 1.2. By V we shall denote the universe of the set theory, i.e. the class of all sets.

DEFINITION 1.6. $G \in V$ is *C-generic over M* if $G \subseteq C$ and

1° $1_C \in G$,

2° $p \in G \;\&\; q \in G \to (\exists s)_G(s \leqslant p \;\&\; s \leqslant q)$,

3° $p \in G \;\&\; q \in C \;\&\; p \leqslant q \to q \in G$,

4° if D is dense in C, then $G \cap D \neq \emptyset$.

LEMMA 1.1. *If $p \in C$, then there exists a G-C-generic over M such that $p \in G$.*

The proof is a direct consequence of the fact that the language $\mathfrak{L}_{ZF}(M)$ is countable; hence there is a countable number of classes dense in C.

Remark 1.3. There is no need to make precise the sense in which the notion of rank is used. Indeed, since M is a transitive, standard model for ZF$^-$, then the rank defined in it and the rank defined for a whole universe coincide with the elements of model M.

DEFINITION 1.7. For $a \in M$ and a G-C-generic over M

$$K_G(a) = \{K_G(b)\colon (\exists p)_G(\langle b, p\rangle \in a)\}.$$

This definition is inductive with respect to $\mathrm{rank}(a)$.

DEFINITION 1.8. $M[G] = \{K_G(a)\colon a \in M\}$.

DEFINITION 1.9. $\check{a} = \{\langle \check{b}, 1_C\rangle\colon b \in a\}$.

Remark 1.4. $a \in M \to \check{a} \in M$ and $K_G(\check{a}) = a$. Hence $M \subseteq M[G]$. Besides, $\langle M[G], \in\rangle$ is a transitive, countable, standard model; accordingly, the axioms of extensionality, of pairs, of the empty set, of regularity, and of infinity ($\omega \in M \subseteq M[G]$) are true in $M[G]$.

The pair axiom holds because if $p, q \in G$ and $a, b \in M$, e.g. $p = q = 1_C$, then $\{\langle a, p\rangle, \langle b, q\rangle\} \in M$ and

$$K_G(\{\langle a, p\rangle, \langle b, q\rangle\}) = \{K_G(a), K_G(b)\}.$$

$M[G]$ is standard, transitive and regular; hence $a \in M[G]$ is an ordinal in $M[G]$ iff it is an ordinal in the universe. Evidently $On^M \subseteq On^{M[G]}$.

LEMMA 1.2. $\mathrm{rank}\big(K_G(a)\big) \leqslant \mathrm{rank}(a)$.

COROLLARY 1.1. $On^M = On^{M[G]}$.

LEMMA 1.3. *If $C \in M$, then $M[G] \models$ "axiom of union", and $G \in M[G]$.*

2. Forcing: Definition and general properties. The main result of this section is the proof of the fact that the three fundamental Cohen lemmas concerning forcing can be proved in ZF$^-$.

Forcing is essential for the discussion of the axiom of replacement in model $M[G]$. Its importance depends on the fact that the properties of $M[G]$ may be examined inside the model M.

In [10] J. R. Shoenfield supplies the definition of weak forcing in a version which I have made use of in this paper.

Further on we shall assume that M is a transitive, countable, standard model for ZF^-, that $\langle C, \leqslant\rangle$ is a notion of forcing in M, and especially for the aims of this section that $C \in M$. If G is C-generic over M, then we shall denote $K_G(a)$ by $\bar{a}$ and we shall say that a is a *name for* $\bar{a}$.

If $\Phi \in \mathcal{L}_{ZF}(M)$ is a sentence, then $\vdash_G \Phi$ will be understood as the fulfilment of sentence Φ in model $M[G]$, where every constant $a \in M$ appearing in Φ is interpreted as $\bar{a}$.

DEFINITION OF FORCING. Let Φ be a sentence of language $\mathcal{L}_{ZF}(M)$ and p a condition. Then we say that $p \Vdash \Phi$ iff (G) (G is C-generic over M and $p \in G \rightarrow \vdash_G \Phi$).

Remark. In the above definition the assumption that $C \in M$ is superfluous. The definition is correct also for notions of forcing which are determined by proper classes in M. In this last case, however, we could not find a general method permitting us to define the relation $p \Vdash \Phi$ in model M. We shall presently show that in case $C \in M$ the relation $p \Vdash \Phi$ is definable in M.

The Main Theorem. If M is a countable, transitive, standard model of ZF^- and $\langle C, \leqslant\rangle$ is a notion of forcing in M with $C \in M$, then we have the following:

DEFINABILITY LEMMA. *For every formula* $\Phi(x_1, \ldots, x_n) \in \mathcal{L}_{ZF}$ *there exists a formula* $\mathrm{Forc}_\Phi(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \in \mathcal{L}_{ZF}$ *such that for every* $p \in C$ *and* $a_1, \ldots, a_n \in M$

$$p \Vdash \Phi(a_1, \ldots, a_n)$$

$$\text{iff} \quad M \Vdash \mathrm{Forc}_\Phi(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2})[p, a_1, \ldots, a_n, C, \leqslant].$$

TRUTH LEMMA. *For every sentence* $\Phi \in \mathcal{L}_{ZF}(M)$ *and every* G-C-*generic over* M

$$\vdash_G \Phi \quad \text{iff} \quad (\exists p)_G (p \Vdash \Phi).$$

EXTENSION LEMMA. *For every sentence* $\Phi \in \mathcal{L}_{ZF}(M)$ *and* $p, q \in C$

$$p \leqslant q \;\&\; q \Vdash \Phi \rightarrow p \Vdash \Phi.$$

In proving these Lemmas we shall follow Shoenfield [10]. He recognizes as primitive the conjunctions $\neg$ and $\vee$ while the remaining ones are defined in the usual way. He also considers the existential quantifier $\exists$ as given and, using it, he defines "( )". Given are the symbols $\in$, $\neq$; $x \notin y$ is defined as $\neg(x \in y)$ and $x = y$ as $\neg(x \neq y)$. All this refers to the language of forcing and is necessary for correctness of an inductive definition of $\Vdash^*$, i.e. weak forcing. In [10] Shoenfield defines $p \Vdash^* \Phi$ for $p \in C$ and $\Phi \in \mathcal{L}_{ZF}(M)$ where Φ is a sentence. Further he shows that $p \Vdash \Phi$ iff $p \Vdash^* \neg\neg\Phi$ and, proving forcing lemmas for $\Vdash^*$, he subsequently transfers the proofs onto $\Vdash$, putting $\neg\neg\Phi$ instead of Φ. Since in model M of ZF^- it is not obligatory that $\{a\colon \mathrm{rank}(a) < \alpha \;\&\; a \in M\}$ be an element of M, we must add certain notes to the definition given by Shoenfield, so that his proof could be applied to ZF^-.

We shall now define $p \Vdash^* \Phi$ in the following five cases:

a) $p \Vdash^* a \neq b$ if $(\exists c)(\exists q)_{\geqslant p}(\langle c, q\rangle \in a \;\&\; p \Vdash^* c \notin b)$
or $(\exists c)(\exists q)_{\geqslant p}(\langle c, q\rangle \in b \;\&\; p \Vdash^* c \notin a)$,

b) $p \Vdash^* a \in b$ if $(\exists c)(\exists q)_{\geqslant p}(\langle c, q\rangle \in b \;\&\; p \Vdash^* a = c)$,

c) $p \Vdash^* \neg\Phi$ if $(q)_{\leqslant p} \neg (q \Vdash^* \Phi)$,

d) $p \Vdash^* \Phi \vee \Psi$ if $p \Vdash^* \Phi$ or $p \Vdash^* \Psi$,

e) $p \Vdash^* (\exists x)\Phi(x)$ if $(\exists b)\big(p \Vdash^* \Phi(b)\big)$.

On the basis of (b), (c), (a) it is clear that $p \Vdash^* a \neq b \equiv (*)$, where $(*)$ is the formula

$$(\exists c)(\exists q)_{\geqslant p}\Big(\langle c, q\rangle \in a \;\&\; (t)_{\leqslant p}(d)(s)_{\geqslant t}\big(\langle d, s\rangle \in b \rightarrow (\exists r)_{\leqslant t}(r \Vdash^* c \neq d)\big)\Big) \vee$$

$$\vee (\exists c)(\exists q)_{\geqslant p}\Big(\langle c, q\rangle \in b \;\&\; (t)_{\leqslant p}(d)(s)_{\geqslant t}\big(\langle d, s\rangle \in a \rightarrow (\exists r)_{\leqslant t}(r \Vdash^* c \neq d)\big)\Big),$$

and it is evident that if T is a transitive set, $a \in T$ and $\langle c, q\rangle \in a$, then $c \in T$ and $\mathrm{rank}(c) < \mathrm{rank}(a)$. Let T be a transitive set, $T \in M$ and $a, b \in T$; then in formula $(*)$ for $(\exists c)$ and (d) we can substitute $(\exists c)_T$, $(d)_T$ respectively.

These remarks suggest the following construction: Let $0 \neq T$ be a transitive set. By the replacement axiom there exists a set $\Pi(T) = \{\mathrm{rank}(x)\colon x \in T\}$. Let f be a function defined on $\Pi(T)$ such that $f(0) = \emptyset$ and

$$f(\xi) = \bigcup_{\eta<\xi} f(\eta) \cup \Big\{\langle p, a, b\rangle\colon p \in C \;\&\; a, b \in T \;\&\; \xi = \max\big(\mathrm{rank}(a), \mathrm{rank}(b)\big) \;\&$$
$$\& \Big(\Big((\exists c)(\exists q)_{\geqslant p}\Big(\langle c, q\rangle \in a \;\&\; (t)_{\leqslant p}(d)(s)_{\geqslant t}\Big(\langle d, s\rangle \in b$$
$$\rightarrow (\exists r)_{\leqslant t}\Big(\langle r, c, d\rangle \in \bigcup_{\eta<\xi} f(\eta)\Big)\Big)\Big)\Big) \vee (\exists c)(\exists q)_{\geqslant p}\Big(\langle c, q\rangle \in b \;\&$$
$$\& (t)_{\leqslant p}(d)(s)_{\geqslant t}\Big(\langle d, s\rangle \in a \rightarrow (\exists r)_{\leqslant t}\Big(\langle r, c, d\rangle \in \bigcup_{\eta<\xi} f(\eta)\Big)\Big)\Big)\Big)\Big\}$$

for $\xi \in \Pi(T)$.

From the replacement axiom and the sum axiom follows the existence and uniqueness of such a function. It will be denoted by f_T. Let $R(T) = \bigcup_{\xi \in \Pi(T)} f_T(\xi)$. It is obvious that if T_1, T_2 are non-empty and transitive, then $R(T_1) \cap R(T_2) = R(T_1 \cap T_2)$; hence $p \Vdash^* a \neq b$ is equivalent to the fact that there exists a set $\emptyset \neq T$, transitive and such that $\langle p, a, b\rangle \in R(T)$. It is easily seen that if $\langle p, a, b\rangle \in R(T)$ for a certain T and T_1 is transitive, then $\langle p, a, b\rangle \in R(T_1)$ for $a, b \in T_1$; hence $p \Vdash^* a \neq b$ is definable in ZF$^-$ and there exists a formula $\mathrm{Forc}^*_{\neq}(x_0, x_1, x_2, x_3, x_4) \in \mathfrak{L}_{ZF}$ such that for every $a, b \in M$ and $p \in C$ $p \Vdash^* a \neq b$ iff

$$M \models \mathrm{Forc}^*_{\neq}(x_0, \ldots, x_4)[p, a, b, C, \leqslant].$$

Hence we have

DEFINABILITY LEMMA. *For every $\Phi(x_1, x_2, \ldots, x_n) \in \mathfrak{L}_{ZF}$ there exists a formula $\mathrm{Forc}^*_{\Phi}(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \in \mathfrak{L}_{ZF}$ such that for every $a_1, \ldots, a_n \in M$ and $p \in C$*

$$p \Vdash^* \Phi(a_1, \ldots, a_n)$$
$$\textit{iff} \quad M \models \mathrm{Forc}^*_{\Phi}(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2})[p, a_1, \ldots, a_n, C, \leqslant].$$

Remark. $C \in M$ is a crucial assumption, since it permits defining by transfinite induction.

EXTENSION LEMMA. *If Φ is a sentence of $\mathfrak{L}_{ZF}(M)$, then*

$$(q)_C(p)_{\leqslant q}(q \Vdash^* \Phi \to p \Vdash^* \Phi).$$

TRUTH LEMMA. $\models_G \Phi$ *iff* $(\exists p)_G(p \Vdash^* \Phi)$.

FORCING LEMMA. $p \Vdash \Phi$ *iff* $p \Vdash^* \neg\neg\Phi$.

The Main Theorem is an easy consequence of the above lemmas, whose simple proofs we shall ask the reader to complete with the aid of [10].

3. Forcing for classes. We shall assume that $C'(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$, $\leqslant(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$

$$M \models (u)(v)\big(C'(u, v) \to \mathrm{Ord}(u)\big) \;\&\; (a)(\exists x)\big[(y)\big(y \in x \leftrightarrow C'(a, x)\big)\big].$$

For every a, C_a will denote a unique set in M such that

$$M \models (y)\big(y \in C_a \leftrightarrow C'(a, y)\big).$$

Let $C(x) \equiv (\exists a) C'(a, x)$. We assume that $\langle C, \leqslant\rangle$ is a notion of forcing in M. On the basis of the axiom of comprehension for every a the set

$$\leqslant_a = \{\langle p, q\rangle\colon p \in C_a \;\&\; q \in C_a \;\&\; \leqslant(p, q)\} \quad \text{belongs to } M.$$

For every a $\langle C_a, \leqslant_a\rangle$ will be a notion of forcing and we shall write $\langle C_a, \leqslant\rangle$ instead of $\langle C_a, \leqslant_a\rangle$. Note that all the notions of forcing $\langle C_a, \leqslant\rangle$ are defined by one formula with a parameter a. Further we shall assume

that $C_a \subseteq C_\beta$ if $a \in \beta$. Obviously M is a standard, transitive and countable model for ZF$^-$. Let $G_a \overset{\mathrm{dfn}}{=} G \cap C_a$ if G is C-generic over M. For $p \in C$ let $\Delta'(p) = \min_a C'(a, p)$. If $a \in M$, then

$$\Delta(a) = \bigcup\Big\{a\colon (\exists b)(\exists p)_C\big(\langle b, p\rangle \in a \;\&\; \big(\Delta(b) = a \vee \Delta'(p) = a\big)\big)\Big\}$$

is definable by transfinite induction.

If $a \in T$ and T is transitive, $\langle b, p\rangle \in a$, then $b \in T$ and $\mathrm{rank}(b) < \mathrm{rank}(a)$. Hence defining Δ by transfinite induction is valid.

For every set S from the universe we can define $K_S(a) \overset{\mathrm{dfn}}{=} \{K_S(b)\colon (\exists p)_S(\langle b, p\rangle \in a)\}$ without any assumptions about S. Such will be the formal sense of K_{G_a}.

FACT 3.1. *If $\Delta(a) \leqslant a$, then $K_{G_a}(a) = K_G(a)$.*

DEFINITION 3.1. $a^{(a)} = \{\langle b^{(a)}, p\rangle\colon \langle b, p\rangle \in a \;\&\; p \in C_a\}$.

FACT 3.2. $\Delta(a^{(a)}) \leqslant a$.

FACT 3.3. $K_G(a^{(a)}) = K_{G_a}(a)$.

FACT 3.4. *If $\Delta(a) \leqslant a, \beta$, then $K_{G_a}(a) = K_{G_\beta}(a)$.*

The simple proofs are left to the reader. From the given facts we infer that

(1) $$M[G] = \bigcup_{a \in On^M} M[G_a],$$

(2) $$\Delta(a) \leqslant a, \beta \to K_{G_a}(a) = K_G(a) = K_{G_\beta}(a),$$

(3) every set $M[G_a]$ is transitive,

(4) $$(a)(G_a \in M[G_a]) \quad \text{and} \quad (a)(G_a \in M[G]),$$

DEFINITION 3.2. $\Delta(a, b, p) = \max\big(\Delta(a), \Delta(b), \Delta'(p)\big)$.

DEFINITION 3.3. Keeping in mind the assumptions mentioned at the beginning of § 3, we shall call C a *semicoherent notion of forcing* if, for every G-C-generic over M and $a \in On^M$, G_a is C_a-generic over M.

Let C be a semicoherent notion of forcing. To every C_a there corresponds a weak forcing in the way described in § 2. We shall denote it by $\Vdash^*_a$. The form of the formulae stating that sentence Φ is forced in the sense of $\Vdash^*_a$ and of that stating that it is forced in the sense of $\Vdash^*_\beta$ is the same. We shall now define forcing $\Vdash^*$ for C. Let $a, b \in M$, $p \in C$; then

$$p \Vdash^* a \neq b \quad \text{if} \quad p \Vdash^*_{\Delta(a,b,p)} a \neq b,$$

$$p \Vdash^* a \in b \quad \text{if} \quad p \Vdash^*_{\Delta(a,b,p)} a \in b,$$

$$p \Vdash^* \neg\Phi \qquad \text{if} \qquad (q)_C\big(q \leqslant p \to \neg(q \Vdash^* \Phi)\big)\,,$$

$$p \Vdash^* \Phi \vee \Psi \qquad \text{if} \qquad p \Vdash^* \Phi \vee p \Vdash^* \Psi\,,$$

$$p \Vdash^* (\exists x)\Phi(x) \qquad \text{if} \qquad (\exists b)(p \Vdash^* \Phi(b)\,.$$

Since $p \in C_{\Delta(a,b,p)}$, it is reasonable to speak about $p \Vdash^*_{\Delta(a,b,p)} a \neq b$ and $p \Vdash^*_{\Delta(a,b,p)} a \in b$; further definitions are inductive with respect to the length of the formula.

LEMMA 3.1. *If* $\Delta(a,b,p) \leqslant \alpha, \beta$, *then*

$$p \Vdash^*_\alpha a = b \leftrightarrow p \Vdash^*_\beta a = b \quad \textit{and} \quad p \Vdash^*_\alpha a \notin b \leftrightarrow p \Vdash^*_\beta a \notin b\,.$$

LEMMA 3.2. *If* $\Delta(a,b,p) \leqslant \alpha, \beta$, *then*

$$p \Vdash^*_\alpha a \neq b \leftrightarrow p \Vdash^*_\beta a \neq b \quad \textit{and} \quad p \Vdash^*_\alpha a \in b \leftrightarrow p \Vdash^*_\beta a \in b\,.$$

We shall omit the simple proofs.

LEMMA 3.3. *If* $p \leqslant q$ *and* $q \Vdash^* a \neq b$, *then* $p \Vdash^* a \neq b$; *if* $q \Vdash^* a \in b$, *then* $p \Vdash^* a \in b$.

Proof. Let $\alpha = \max\big(\Delta(a,b,p), \Delta(a,b,q)\big)$.

$$q \Vdash^* a \neq b \overset{\text{dfn}}{\equiv} q \Vdash^*_{\Delta(a,b,q)} a \neq b \quad \text{and} \quad q \Vdash^* a \neq b \equiv q \Vdash^*_\alpha a \neq b$$

since by Lemma 3.2 $q \Vdash^*_{\Delta(a,b,q)} a \neq b \equiv q \Vdash^*_\alpha a \neq b$. But $p, q \in C_\alpha$ and $p \leqslant q$; hence by the Extension Lemma we have $p \Vdash^*_\alpha a \neq b$ and by Lemma 3.2 we have $p \Vdash^*_{\Delta(a,b,p)} a \neq b$, i.e. $p \Vdash^* a \neq b$.

The proof for $a \in b$ is analogous.

COROLLARY 3.1. *The Extension Lemma is true for* $\Vdash^*$.

The details of the proof are left to the reader.

COROLLARY 3.2. *The Truth Lemma is true for* $\Vdash^*$.

For the proof it suffices to make use of Corollary 3.1 and to prove the Truth Lemma for atomic formulae on the basis of the facts in this section. Thus we have proved the Truth, Definability and Extension Lemmas.

DEFINITION 3.4. $p \Vdash \Phi$ *iff* (G) $(G$-C-*generic over* M *and* $p \in G \to \vdash_G \Phi)$.

LEMMA 3.4. $p \Vdash^* \neg\neg\Phi$ *iff* $p \Vdash \Phi$.

THEOREM 3.1. *For every formula* $\Phi(x_1, \ldots, x_n) \in \mathfrak{L}_{ZF}$ *there exists a formula* $\mathrm{Forc}_\Phi(x_0, x_1, \ldots, x_n) \in \mathfrak{L}_{ZF}(M)$ *such that for every* $a_1, \ldots, a_n \in M$ *and* $p \in C$

$$p \Vdash \Phi(a_1, \ldots, a_n) \quad \textit{iff} \quad M \models \mathrm{Forc}_\Phi(x_0, x_1, \ldots, x_n)[p, a_1, \ldots, a_n]$$

and for every sentence $\Phi \in \mathfrak{L}_{ZF}(M)$ *and* G-C-*generic over* M

$$\vdash_G \Phi \quad \textit{iff} \quad (\exists p)_G(p \Vdash \Phi)\,.$$

If $p \leqslant q$ *and* $q \Vdash \Phi$ *then* $p \Vdash \Phi$.

The proof follows from Lemma 3.4 and the pertinent lemmas for $\Vdash^*$.

All notions of forcing known to the author and such that the formula (G) $(p \in G \to \vdash_G \Phi)$ satisfies the three fundamental lemmas are semi-coherent. Therefore we shall further assume that $C(\cdot), \leqslant(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ and $\langle C, \leqslant\rangle$ is a notion of forcing and the relation $(G)(p \in G \to \vdash_G \Phi)$ satisfies the three fundamental forcing lemmas. These assumption will be taken for granted in the next section.

4. Models $M[G]$**.** In [1] Rolando Chuaqui formulates the following theorem: if M is a countable model for the Morse set theory with the Gödel axiom of choice, and G is C-generic over M such that (G!!): for every $\beta \in On^M$ there exists a $C_1 \in M$ such that $C_1 \subseteq C$ and for every sequence of dense sections $\langle D_\alpha\colon \alpha < \beta\rangle$ (see [1]) there exists a $q \in G$ such that

$$(\alpha)_{<\beta}(\exists p)_{G \cap C_1}(p \wedge q \text{ exists } \& \ p \wedge q \in D_\alpha)\,,$$

then $M[G]$ is a countable model for Morse + Gödel A.C.

Remark. If $p, q \in C$ and $M \models (\exists s)_C\big(s \leqslant p \ \& \ s \leqslant q \ \& \ (t)_C(t \leqslant p \ \& \ \& \ t \leqslant q \to t \leqslant s)\big)$, then $s \overset{\text{dfn}}{=} p \wedge q$.

We shall need some new ideas and also certain modifications of the given definitions, so that we may transfer the results of [1] onto ZF.

DEFINITION 4.1. A class D is a *dense section in* C if D is a class definable in M, $D \subset C$, D is dense in C and $(p)_C(q)_D(p \leqslant q \to p \in D)$.

DEFINITION 4.2. If, for every $\alpha < \beta$, D_α is a dense section in C and there exists a formula $\Psi(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ such that $M \models \Psi(\alpha, p) \equiv p \in D_\alpha$, then we shall call $\langle D_\alpha\colon \alpha < \beta\rangle$ a *sequence of classes*.

If instead of β we have a set $b \in M$ and $\Psi(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ such that $M \models \Psi(a, p) \equiv p \in D_a$ and D_a is a dense section in C for $a \in b$, then we shall call $\{D_a\}_{a \in b}$ a *set of classes*.

DEFINITION 4.3. If G is C-generic over M and fulfils the condition (G!): for every set of classes $\{D_a\}_{a\in b}$, where $b \neq \emptyset$, there exists a function $W\colon b \to V^M$, $W \in M$ and $(a)_b\big(W(a) \subseteq D_a \ \& \ W(a) \in M\big)$ such that $(a)_b\big(W(a) \cap G \neq \emptyset\big)$, then we shall call G C-3-*generic*.

Remark. V^M denotes the universe of M.

The condition (G!!), with such an interpretation of a sequence of classes, can be applied to ZF^- and in $ZF^- + AC$ it is stronger than (G!).

Indeed, since $M \models ZF^- + AC$, it is sufficient to derive from (G!!) the formula (G!), where instead of "for every set of classes $\{D_a\}_{a\in b}$, where $b \neq \emptyset$" we put "for every sequence of classes $\langle D_a\colon a < \beta\rangle$, where $\beta > 0$". Here AC is the formula

$$(x)\big(x \neq \emptyset \to (\exists \alpha)(\exists f)(\mathrm{Func}(f)\ \&\ \mathrm{dom}(f) = \alpha\ \&\ \mathrm{rg}(f) = x)\big).$$

Let (G!!) be fulfilled, and let $\langle D_a\colon a < \beta\rangle$ be a sequence of classes and $q \in G$, $C_1 \subseteq C$ having the required properties. Let

$$W(a) = \{p \wedge q\colon\ p \in C_1\ \&\ p \wedge q \text{ exists } \&\ p \wedge q \in D_a\}.$$

It is clear that $W \in M$, $W(a) \subseteq D_a$ and $W(a) \cap G \neq \emptyset$, since there is a $p \in G \cap C_1$ such that $p \wedge q$ exists and $p \wedge q \in D_a$. Since $p, q \in G$, there is an $s \in G$ such that $s \leqslant p$, $s \leqslant q$; then $s \leqslant p \wedge q$ and hence $p \wedge q \in G$, i.e. $W(a) \cap G \neq \emptyset$.

Fact 4.1. *If C is a set, then every G-C-generic over M is strongly generic, i.e. it satisfies* (G!!) *(we can put $q = 1_C$, $C_1 = C$).*

Main lemma. *If M is a transitive, countable, standard model of ZF^- and G is 3-generic, then the axiom of comprehension is satisfied in $M[G]$.*

Proof. Note that if Θ is a sentence of $\mathfrak{L}_{ZF}(M)$, then $D_\Theta = \{p\colon p \Vdash \Theta \vee \vee p \Vdash \neg\Theta\}$ is a dense section. Let $\Phi(x, a_1, \ldots, a_n) \in \mathfrak{L}_{ZF}(M)$ and $a \in M$. We shall show that $\{\bar b\colon \vdash_G b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\} \in M[G]$. If $\vdash_G b' \in a$, then obviously there is a $b \in \mathrm{Rg}(a)$, where

$$\mathrm{Rg}(a) \overset{\mathrm{dfn}}{=} \{b\colon (\exists p)_C(\langle b, p\rangle \in a)\},$$

such that $\vdash_G b = b'\ \&\ b \in a$. Now let

$$D_b = \big\{p\colon\ \big(p \Vdash b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\big) \vee \big(p \Vdash \neg\big(b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\big)\big)\big\}$$

for $b \in \mathrm{Rg}(a)$. From the definability of forcing and the independence of the shape of the formula Forc_Φ from the parameters appearing in Φ it follows that every class D_b is definable in M, and $\{D_b\}_{b\in \mathrm{Rg}(a)}$ is a set of classes. We assume that $\mathrm{Rg}(a) \neq \emptyset$; otherwise

$$\emptyset = \{\bar b\colon \vdash_G b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\} \in M[G]$$

and this completes the proof.

In virtue of (G!) there is a function $W\colon \mathrm{Rg}(a) \to V^M$, $W \in M$, $W(b) \subset D_b$ for $b \in \mathrm{Rg}(a)$ and $W(b) \cap G \neq \emptyset$. Let

$$a_\Phi = \Big\{\langle b, p\rangle\colon\ b \in \mathrm{Rg}(a)\ \&\ p \in \bigcup_{d\in \mathrm{Rg}(a)} W(d)\ \&\ p \Vdash \big(\Phi(b, a_1, \ldots, a_n)\ \&\ b \in a\big)\Big\}.$$

From the axiom of replacement (in M) and the definability of forcing it follows that $a_\Phi \in M$. If $\vdash_G b' \in a\ \&\ \Phi(b', a_1, \ldots, a_n)$, then we can assume that there exists a $b \in \mathrm{Rg}(a)$ such that $\vdash_G b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)$ and $\vdash_G b = b'$.

From the Truth Lemma we infer that if $p \in G$ then

$$\neg\big(p \Vdash \neg\big(b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\big)\big).$$

Since (G!), we have $p \in W(b) \cap G$ and for this p we have $p \Vdash b \in a\ \&\ \&\ \Phi(b, a_1, \ldots, a_n)$. Hence $\langle b, p\rangle \in a_\Phi$ and $p \in G$, i.e. $\bar b' = \bar b \in \bar a_\Phi$.

If $\bar b' \in \bar a_\Phi$, then there exists a b such that $\bar b = \bar b'$ and for a certain $p \in G$ we get $\langle b, p\rangle \in a_\Phi$, i.e. $p \Vdash \Phi(b, a_1, \ldots, a_n)\ \&\ b \in a$.

We know that $p \in G$ and so, by the Truth Lemma, $\vdash_G \Phi(b, a_1, \ldots, a_n)\ \&\ \&\ b \in a$ and also $\vdash_G \Phi(b', a_1, \ldots, a_n)\ \&\ b' \in a$, i.e.

$$\vdash_G (b)\big(b \in a_\Phi \leftrightarrow b \in a\ \&\ \Phi(b, a_1, \ldots, a_n)\big).$$

This concludes the proof of the axiom of comprehension in $M[G]$.

Axiom of union. *Let $\bar a \in M[G]$, $T \in M$ and T be such a transitive set that $a \in T$. Let, for $d \in T$,*

$$D_d = \big\{p\colon\ \big(p \Vdash (\exists x)(d \in x\ \&\ x \in a)\big) \vee \big(p \Vdash \neg(\exists x)(d \in x\ \&\ x \in a)\big)\big\}.$$

In virtue of (G!) there is a function $W\colon T \to V^M$ which fulfils the known conditions.

Let

$$a_U = \Big\{\langle d, p\rangle\colon\ d \in T\ \&\ p \in \bigcup_{b\in T} W(b)\ \&\ p \Vdash (\exists x)(x \in a\ \&\ d \in x)\Big\}.$$

By the axiom of replacement in M and the definability of forcing, $a_U \in M$ and $\bigcup \bar a \subseteq K_G(a_U)$, which is easy to check. Hence, and also from the axiom of comprehension in $M[G]$, it follows that $\bigcup \bar a \in M[G]$; we apply the axiom of comprehension to the set $\bar a_U$ and the formula

$$(\exists y)(y \in \bar a\ \&\ x \in y).$$

Definition 4.4. Let $\Phi(\cdot,\cdot)$ be a formula with some parameters. Then

$$3_\Phi(x, y) \overset{\mathrm{dfn}}{\equiv} (u)\big(\big(u \in x\ \&\ (\exists t)\Phi(u, t)\big) \to (\exists z)\big(z \in y\ \&\ \Phi(u, z)\big)\big).$$

Definition 4.5. M is a 3-model if M is a standard, countable, transitive model for ZF^- and for every formula $\Phi(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$

$$M \models (x)(\exists y)3_\Phi(x, y).$$

Axiom of replacement. *If M is a 3-model and* (G!) *is satisfied then $M[G] \models$ "axiom of replacement scheme".*

Proof. Let $a \in M$, $\Phi(x, y) \in \mathfrak{L}_{ZF}(M)$ and $\vdash_G (x)(\exists! y)\Phi(x, y)$. We shall attempt to show that

$$\{\bar{d}\colon \vdash_G (\exists b)(b \in a \ \&\ \Phi(b, d))\} \in M[G].$$

Let

$$D_b = \{p\colon (\exists d)(p \Vdash (\exists y)\Phi(b, y) \to \Phi(b, d))\} \quad \text{for} \quad b \in \mathrm{Rg}(a).$$

The reader can easily check that D_b is a dense section for $b \in \mathrm{Rg}(a)$. It is obvious that $\{D_b\}_{b \in \mathrm{Rg}(a)}$ is a set of classes. In virtue of (G!) there is a function $W\colon \mathrm{Rg}(a) \to V^M$ possessing the given properties.

Let

$$\Psi(z, y) \equiv \big(z = \langle b, p\rangle \ \&\ p \Vdash ((\exists y')\Phi(b, y') \to \Phi(b, y))\big).$$

$\Psi \in \mathfrak{L}_{ZF}(M)$ and M is a 3-model. Hence for the set $\mathrm{Rg}(a) \times \bigcup_{b' \in \mathrm{Rg}(a)} W(b')$ there exists a set $\mathfrak{W} \in M$ such that if $z \in \mathrm{Rg}(a) \times \bigcup_{b' \in \mathrm{Rg}(a)} W(b')$ and there exists a y such that $\Psi(z, y)$, then there is a $y_0 \in \mathfrak{W}$ such that $\Psi(z, y_0)$. If $\vdash_G b_1 \in a \ \&\ \Phi(b_1, d_1)$, then we can find a $b \in \mathrm{Rg}(a)$ such that $\vdash_G b \in a \ \&$ $\&\ \Phi(b, d_1)$ and $\vdash_G b = b_1$. It is evident that $W(b) \cap G \neq \emptyset$. Let $p \in W(b) \cap G$; then we have $(\exists y)\Psi(z, y)$, where $z = \langle b, p\rangle$, and there exists a $d \in \mathfrak{W}$ such that $\Psi(z, d)$, i.e. $p \Vdash (\exists y)\Phi(b, y) \to \Phi(b, d)$. Hence $\vdash_G (\exists y)\Phi(b, y) \to \Phi(b, d)$. From $\vdash_G \Phi(b, d_1)$ it follows that $\vdash_G \Phi(b, d)$. Hence we have $\bar{d} = \bar{d}_1$ and thus $\{\bar{d}\colon \vdash_G (\exists b)(b \in a \ \&\ \Phi(b, d))\} \subseteq \overline{\mathfrak{W} \times \{1_C\}}$. $\overline{\mathfrak{W} \times \{1_C\}} \in M[G]$, which together with the axiom of comprehension gives us the axiom of replacement in the model $M[G]$.

In an analogous way we prove that if M is a 3-model and (G!) holds, then for every formula $\Phi(\cdot, \cdot) \in \mathfrak{L}_{ZF}(M)$ the sentence

$$(x)(\exists y)(z)\big(z \in x \to [(\exists u)\Phi(z, u) \to (\exists t)(t \in y \ \&\ \Phi(z, t))]\big)$$

is satisfied in $M[G]$.

Thus we have shown that $M[G]$ is a 3-model if M is a 3-model and G is 3-generic over M.

Axiom of choice. *Let*

$$\mathrm{AC} \equiv (x)\big(x \neq \emptyset \to (\exists f)(\exists \alpha)(\mathrm{Func}(f) \ \&\ f\colon \alpha \xrightarrow{\text{onto}} x)\big).$$

If M is a 3-model such that $M \models \mathrm{AC}$ and (G!) *holds, then $M[G] \models \mathrm{AC}$.*

Proof. Let $a \in M$ and

$$a^* = \{\langle \{\langle \{\langle \{\langle b, p\rangle\}, p\rangle, \langle \{\langle b, p\rangle, \langle \check{f}(b), p\rangle\}, p\rangle\}, p\rangle\colon \langle b, p\rangle \in a\},$$

where $f \in M$ is such a function defined on $\mathrm{Rg}(a)$ that there exists an $\alpha \in M$ such that $f\colon \mathrm{Rg}(a) \xrightarrow[1-1]{\text{onto}} \alpha$. We know that such a function does exist, because $M \models \mathrm{ZF}^- + \mathrm{AC}$. From $\check{d} = \{\langle \check{c}, 1_C\rangle\colon c \in d\}$ and $M \models \mathrm{ZF}^-$ it follows that $a^* \in M$.

Now note that

$$K_G(\{\langle d_1, p\rangle, \langle d_2, p\rangle\}) = \{K_G(d_1), K_G(d_2)\} \quad \text{if} \quad p \in G$$

and

$$K_G(\{\langle d_1, p\rangle, \langle d_2, p\rangle\}) = \emptyset \quad \text{if} \quad p \notin G.$$

Hence we have

$$K_G(a^*) \setminus \{\emptyset\} = \{\langle \bar{b}, f(b)\rangle\colon \langle b, p\rangle \in a \ \&\ p \in G\}.$$

If (G!) holds, then $M[G] \models$ "axiom of comprehension"; hence

$$\bar{a}' \overset{\text{dfn}}{=} K_G(a^*) \setminus \{\emptyset\} \in M[G].$$

Let $h(x) = \min_\beta \{\langle x, \beta\rangle \in \bar{a}'\}$ for $x \in \bar{a}$. Since $On^M = On^{M[G]}$ and $M[G] \models \mathrm{ZF}^-$, we have $h \in M[G]$. It is evident that $h\colon \bar{a} \to On$ and h is 1–1; thus h introduces a well-ordering in $\bar{a}$ and it belongs to $M[G]$. Hence we get $M[G] \models \mathrm{AC}$. Thus we have proved

3-Main Theorem. *If M is a 3-model, $\langle C, \leqslant\rangle$ is a notion of forcing in M and the three fundamental lemmas about forcing are satisfied, then $M[G]$ is a 3-model if G is C-3-generic over M.*

If $M \models \mathrm{AC}$ then $M[G] \models \mathrm{AC}$.

Now we shall formulate the following

(G!!)-Main Theorem. *If the following assumptions are satisfied:*

a) *M is a 3-model,*

b) *$M \models$ the axiom of power set & the axiom of choice,*

c) *$\langle C, \leqslant\rangle$ is a notion of forcing in M,*

d) *the three fundamental forcing lemmas are fulfilled,*

e) *(G!!) holds.*

Then we have $M[G] \models \mathrm{ZF} + \mathrm{AC}$.

Proof. From the 3-Main Theorem it suffices to show that the axiom of power set is satisfied. The proof will be based on the translation of the proof of [1] into the language of the Zermelo-Fraenkel set theory.

Let $a \in M$ and $\bar{a} \neq \emptyset$. There are an ordinal α and a function $f \in M$ such that $f\colon \alpha \xrightarrow{1-1} \mathrm{Rg}(a)$ is a map onto $\mathrm{Rg}(a)$. If $A \in M$ and $\bar{A} \subseteq \bar{a}$, then let

$$D_{\beta, A} = \{p\colon p \Vdash f(\beta) \in A \vee p \Vdash f(\beta) \notin A\} \quad \text{for} \quad \beta < \alpha.$$

$\langle D_{\beta, A}\colon \beta < \alpha\rangle$ is a sequence of classes in M. G is strongly generic; hence there exists a $C_1 \in M$ such that for every $\bar{A} \subseteq \bar{a}$ there exists a $q_A \in G$ fulfilling the condition $(\beta)_{<\alpha}(\exists p)_{G\cap C_1}(p\wedge q_A \in D_{\beta,A})$. Let $\bar{A} \subseteq \bar{a}$. We shall define a set c as follows:

$$c = \{\langle d, p\rangle :\ d \in \mathrm{Rg}(a) \ \&\ p \in C_1 \ \&\ p\wedge q_A \Vdash d \in A\}$$

of course $c\in M$ and $c\subseteq \mathrm{Rg}(a)\times C_1$. We shall prove that $\bar c = \bar A$. Let us assume that $\bar x \in \bar c$. Then there exist a d and a $p\in G$ such that $\langle d,p\rangle \in c$, $\bar x = \bar d$ and we have $p\wedge q_A \Vdash d\in A$, $p\wedge q_A\in G$.

Hence, by the Truth Lemma, $\bar d \in \bar A$, i.e. $\bar x\in\bar A$, which implies $\bar c\subset \bar A$.

If $\bar x\in \bar A$, then there is a $d\in \mathrm{Rg}(a)$ such that $\bar d = \bar x$ and a $\beta<\alpha$ such that $f(\beta)=d$. Hence there exists a $p\in G\cap C_1$ such that $p\wedge q_A\Vdash d\in A$ or $p\wedge q_A \Vdash d\notin A$. Obviously $p\wedge q_A\in G$, and if $p\wedge q_A\Vdash d\notin A$ then $\bar d\notin\bar A$, contrary to $\bar x \in\bar A$ and $\bar x=\bar d$. Therefore we have $p\wedge q_A\Vdash d\in A$, which implies $\langle d,p\rangle\in c$ and $p\in G$. Thus we get $\bar d\in\bar c$ and $\bar x\in \bar c$; hence $\bar c = \bar A$. From this we infer that if $\bar A\subseteq \bar a$ then in $P_M(\mathrm{Rg}(a)\times C_1)$, which is an element of M in virtue of the axiom of power set, there exists a name for $\bar A$. Hence and by the axiom of comprehension in $M[G]$ we have

$$\vdash_G (x)(\exists y)[(t)(t\in y \leftrightarrow t\subseteq x)]$$

and $M[G]\models \mathrm{ZF}+\mathrm{AC}$.

COROLLARY 4.1. *If M is a 3-model, $\langle C,\leqslant\rangle\in M$ is a notion of forcing and G is C-generic over M, then $M[G]$ is a 3-model and $G\in M[G]$. Furthermore, if N is a standard, transitive model of ZF^- and $G\in N$ and $M\subseteq N$, then $M[G]\subseteq N$. $M\models \mathrm{AC}$ implies $M[G]\models \mathrm{AC}$.*

COROLLARY 4.2. *If M is a standard, countable, transitive model for ZF, $\langle C,\leqslant\rangle\in M$ is a notion of forcing and G is C-generic over M, then $M[G]$ is a countable, transitive, standard model for ZF. If $M\models \mathrm{AC}$ then $M[G]\models \mathrm{AC}$. If N is a standard, transitive model for ZF^- and $G\in N$, then $M[G]\subseteq N$, if $M\subseteq N$.*

Remark. If $N\models \mathrm{ZF}^-$, $G\in N$, then the functional

$$K(x) = \{K(y):\ (\exists p)_G(\langle y,p\rangle\in x)\} \quad \text{is definable in } N.$$

The proofs of both corollaries result from the previous theorems and the above remark. As a consequence we also have

COROLLARY 4.3. *If M is a 3-model, $\langle C,\leqslant\rangle$ is a semicoherent notion forcing in M and G is C-generic over M, then, for every α, $M[G_\alpha]$ is a 3-model and $G_\alpha\in M[G]$ and $M[G] = \bigcup_{\alpha\in On^M} M[G_\alpha]$. Hence $M[G]$ is a transitive, countable and standard model, where the axiom of union is satisfied and "rank" is definable. $M\models \mathrm{AC}$ implies $M[G]\models \mathrm{AC}$. All the axioms mentioned in Remark 1.4 are satisfied in $M[G]$.*

5. Continuous and coherent notions of forcing.

Let $C'(\cdot,\cdot)$, $\leqslant(\cdot,\cdot)\in \mathfrak{L}_{\mathrm{ZF}}(M)$, M be a 3-model and let

$$M\models (x)(y)\big(C'(x,y)\to \mathrm{Ord}(x)\big), \qquad M\models (x)(\exists y)\big[(t)\big(t\in y\leftrightarrow C'(x,t)\big)\big].$$

Let $C(x)\equiv(\exists a)C'(a,x)$ and let $\langle C,\leqslant\rangle$ be a notion of forcing in M. Let C_α be a set in M such that

$$M\models (x)\big(x\in C_\alpha\leftrightarrow\big(C'(\alpha,x)\vee x = 1_C\big)\big).$$

DEFINITION 5.1.

$$A(x,y)\equiv(\exists z)\big(C(z)\ \&\ C(x)\ \&\ C(y)\ \&\leqslant(z,x)\ \&\leqslant(z,y)\ \&\ (u)\big(C(u)\ \&\leqslant(u,x)\ \&\leqslant(u,y)\to\ \leqslant(u,z)\big)\big).$$

FACT 5.1. *If $M\models A(x,y)$, then there exists exactly one condition which is the greatest lower bound of x and y. It will be denoted by $x\wedge y$.*

FACT 5.2. *For every α, $\langle C_\alpha,\leqslant_\alpha\rangle$ is a notion of forcing. (The definition of $\leqslant_\alpha$ is a given at the beginning of § 4.) As previously, we shall write $\langle C_\alpha,\leqslant\rangle$ instead of $\langle C_\alpha,\leqslant_\alpha\rangle$. We also assume that $C_\alpha\subseteq C_\beta$, when $\alpha\in\beta$.*

DEFINITION 5.2. In view of the above assumption, $\langle C,\leqslant\rangle$ is said to be a *coherent notion of forcing* if there exists a formula $C''(\cdot,\cdot)\in\mathfrak{L}_{\mathrm{ZF}}(M)$ fulfilling the conditions

1° $M\models(x)(y)\big(C''(x,y)\to\big(\mathrm{Ord}(x)\ \&\ C(y)\big)\big)$,

2° $M\models(\alpha)(x)(y)\big(C'(\alpha,x)\ \&\ C''(\alpha,y)\to A(x,y)\big)$,

3° $M\models(\alpha)(x)(\exists y)(\exists z)(y')(z')\big[C(x)\to\big(\big(C'(\alpha,y)\ \&\ C''(\alpha,z)\ \&\ x = y\wedge z\big)\ \&\ \big(\big(C'(\alpha,y')\ \&\ C''(\alpha,z')\ \&\ x = y'\wedge z'\big)\to(y = y'\ \&\ z = z')\big)\big)\big]$,

4° $M\models(x)(x')(\alpha)(y)(z)(y')(z')\big[\big(C(x)\ \&\ C(x')\ \&\ C'(\alpha,y)\ \&\ C''(\alpha,z)\ \&\ C'(\alpha,y')\ \&\ C''(\alpha,z')\ \&\ x = y\wedge z\ \&\ x' = y'\wedge z'\ \&\leqslant(x,x')\big)\to\big(\leqslant(y,y')\ \&\leqslant(z,z')\big)\big]$.

Definition 5.2 states that for every α A: $C_\alpha\times C^\alpha\to C$ is an order isomorphism of the classes $C_\alpha\times C^\alpha$ and C, where $C^\alpha(\cdot)\equiv C''(\alpha,\cdot)$. If $p\in C$, then by $p_{(\alpha)}$, $p^{(\alpha)}$ we shall understand the unique conditions existing in virtue of 3°, where $p_{(\alpha)}\in C_\alpha$, $p^{(\alpha)}\in C^\alpha$ and $p = p_{(\alpha)}\wedge p^{(\alpha)}$.

FACT 5.3. *If $p,q\in C$, $\alpha\in On^M$, $p\leqslant q$ then $p_{(\alpha)}\leqslant q_{(\alpha)}$, $p^{(\alpha)}\leqslant q^{(\alpha)}$.*

FACT 5.4. $p\in C_\alpha\to p = p_{(\alpha)}$.

FACT 5.5. *If $p\in C_\alpha$, $q\in C$ and p,q are incompatible in C, then p, $q_{(\alpha)}$ are incompatible in C_α.*

DEFINITION 5.3. $\langle C,\leqslant\rangle$ is a *continuous and coherent notion of forcing* if it is a notion of forcing, and there exists a regular cardinal $\varkappa$ and $\gamma\in On^M$ such that, for any limit ordinal λ such that $\mathrm{cf}(\lambda)=\varkappa$ and $\lambda\geqslant\gamma$, $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$.

DEFINITION 5.4. $\Delta'(p) = \min_{a} C'(a, p)$.

FACT 5.6. *If the formula* $\precsim'(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ *is a formula which defines in* M *a well-ordering of class* C, *then the formula*

$$\precsim(x, y) \equiv \big[\big(\Delta'(x) < \Delta'(y)\big) \vee \big(\Delta'(x) = \Delta'(y) \;\&\; \precsim'(x, y)\big)\big]$$

is a well-ordering of type $\leqslant On^M$ in class C.

The idea of the following lemma can be found in Solovay's paper [11].

COMBINATORIAL LEMMA. *Let* $\langle C, \leqslant\rangle$ *be a coherent continuous notion of forcing, let the formula* $\precsim'(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ *define in* M *a well-ordering of class* C, *and let* $\mathcal{F}(\cdot) \in \mathfrak{L}_{ZF}(M)$ *define a class of pairwise incompatible conditions, i.e.* $M \models (x)\big(\mathcal{F}(x) \to C(x)\big)$ *and*

$$M \models (y)(z)\Big(\mathcal{F}(y) \;\&\; \mathcal{F}(z) \;\&\; y \neq z \to \neg (\exists u)\big(C(u) \;\&\leqslant (u, y) \;\&\leqslant (u, z)\big)\Big).$$

Then $\mathcal{F}$ *is a set in* M, *i.e.* $M \models (\exists v)\big[(x)\big(x \in v \leftrightarrow \mathcal{F}(x)\big)\big]$.

Outline of the proof. 1° For every class of pairwise incompatible conditions there exists a maximal class of pairwise incompatible conditions.

2° If $\mathcal{F}$ is a maximal class of pairwise incompatible conditions, then by $h(p)$ we shall denote the earliest element $q \in \mathcal{F}$ in the ordering $\precsim$ which is compatible with $p \in C$.

Let $g(a) = \min_{\beta}\big(h(C_a) \subseteq C_\beta\big)$; g is defined in M, $g\colon On^M \to On^M$, g is increasing and continuous in limit ordinals λ such that $\mathrm{cf}(\lambda) = \varkappa$ and $\lambda \geqslant \gamma$.

Let $a_0 = \gamma$, $a_{\xi+1} = g(a_\xi)$ for $\xi < \varkappa$ and $a_\delta = \bigcup_{\eta<\delta} a_\eta$ for $\delta < \varkappa$ such that δ is a limit ordinal. Then $\beta_0 = \bigcup_{\xi<\varkappa} a_\xi \geqslant g(\beta_0)$ and by Fact 5.5 we get $\mathcal{F} \subseteq C_{\beta_0}$.

3-LEMMA. *If* C *is a continuous coherent notion of forcing in* M *and* $\precsim(\cdot,\cdot)$ *well-orders* C *into the type* $\leqslant On^M$ *and* G *is* C*-generic over* M, *then* G *is* 3*-generic.*

Outline of the proof. Let $\{D_b\}_{b\in a}$ be a set of classes in M, and $\precsim$ a well-ordering of type $\leqslant On^M$ in C. For every $b \in a$ let p_b the be earliest element of class D_b in the ordering $\precsim$. Let $b \in a$. We shall restrict ourselves to the conditions from D_b. Let $\mathcal{F}_b$ be the maximal set of pairwise incompatible conditions belonging to D_b and let $p_b \in \mathcal{F}_b$. If we put $W(b) = \mathcal{F}_b$, then $W \in M$ ($\mathcal{F}_b$ is constructed in a uniform way for all $b \in a$ by means of a formula with a varying parameter b) and $W(b) \cap G \neq \emptyset$.

SEMICOHERENCE LEMMA. *Every coherent notion of forcing is a semicoherence notion of forcing.*

ZF-COMBINATORIAL LEMMA. *Let* M *be a* 3*-model and* $M \models$ *the axiom of power set.*

(1) *Let* $\Phi(\cdot,\cdot) \in \mathfrak{L}_{ZF}(M)$ *and let* $\mathfrak{M}$ *be a class definable in* M. *Let* $\precsim$ *be a definable well-ordering in* M *having a field* $\mathfrak{M}$ *and ordering* $\mathfrak{M}$ *into the type* $\eta(\mathfrak{M}) \leqslant On^M$. *Let* $\mathfrak{N}$ *be a class definable in* M *and*

$$M \models (\mathfrak{m})_{\mathfrak{M}}(\exists x)\Big[(y)\Big(y \in x \leftrightarrow \big(y \in \mathfrak{N} \;\&\; \Phi(\mathfrak{m}, y)\big)\Big)\Big].$$

Let $\leqslant(x, y) \equiv y \subseteq x$.

(2) *Let* $\Psi(\cdot) \in \mathfrak{L}_{ZF}(M)$

$$C(f) \overset{\mathrm{dfn}}{\equiv} \mathrm{Func}(f) \;\&\; f \subseteq \mathfrak{M} \times \mathfrak{N} \;\&\; (\mathfrak{m})(\mathfrak{n})\big(\langle \mathfrak{m}, \mathfrak{n}\rangle \in f \to \Phi(\mathfrak{m}, \mathfrak{n})\big) \;\&\; \Psi(f).$$

Let

$$M \models \Psi(\emptyset) \;\&\; (f)(g)\big(\big(C(f) \;\&\; g \subseteq f\big) \to \Psi(g)\big)$$

and

$$M \models (f)(g)\big(C(f) \;\&\; C(g) \to \big(\mathrm{Func}(f \cup g) \to \Psi(f \cup g)\big)\big).$$

Let

$$a = o(\mathfrak{m}) \overset{\mathrm{dfn}}{\equiv} \mathfrak{m} \textit{ is } a\textit{-th element of class } \mathfrak{M} \textit{ in the ordering } \precsim,$$

$$C'(a, x) \overset{\mathrm{dfn}}{\equiv} C(x) \;\&\; (\mathfrak{m})(\mathfrak{n})\big(\langle \mathfrak{m}, \mathfrak{n}\rangle \in x \to o(\mathfrak{m}) \leqslant a\big),$$

$$C''(a, x) \overset{\mathrm{dfn}}{\equiv} C(x) \;\&\; (\mathfrak{m})(\mathfrak{n})\big(\langle \mathfrak{m}, \mathfrak{n}\rangle \in x \to o(\mathfrak{m}) > a\big).$$

Then $\langle C, \leqslant\rangle$ *is a coherent notion of forcing.*

We omit the elementary proof of this lemma, and also that of the next.

LEMMA 5.1. *If the assumption* (1) *of the* ZF*-combinatorial lemma is satisfied and* $\mathfrak{N}$ *is well-ordered by* $\precsim$ *into the type* $\eta(\mathfrak{n}) \leqslant On^M$, *then in the case of* $\Psi(x) \equiv \mathrm{Fin}(x)$ *the assumption* (2) *of the* ZF*-combinatorial lemma is satisfied,* $\langle C, \leqslant\rangle$ *is a continuous coherent notion of forcing and* C *may be well-ordered into a type* $\leqslant On^M$.

Let a be a fixed ordinal number,

$$\leqslant(x, y) \overset{\mathrm{dfn}}{\equiv} y \subseteq x,$$

$$C'(\xi, f) \equiv \mathrm{Func}(f) \;\&\; \mathrm{dom}(f) \subseteq \xi \times \omega_a \;\&\; \mathrm{rg}(f) \subset On \;\&\; \mathrm{Fin}(f) \;\&$$
$$\&\; (\xi)(\beta)(\eta)\big(\langle\langle \xi, \beta\rangle, \eta\rangle \in f \to \eta < \xi\big).$$

$$C(f) \overset{\mathrm{dfn}}{\equiv} (\exists \xi)\, C'(\xi, f);$$

then by Lemma 5.1 we get

COROLLARY 5.1. $\langle C, \leqslant\rangle$ *is a coherent continuous notion of forcing and* C *may be well-ordered into type* On.

6. Some applications.

THEOREM.

$$\mathrm{Consis}(\mathrm{ZF}) \leftrightarrow \mathrm{Consis}\big(\mathrm{ZF}^- + (\mathrm{ZF})^{\mathfrak{L}} + (x)(\exists f)\big(\mathrm{Func}(f) \,\&\, f\colon \omega \xrightarrow{\text{onto}} x\big)\big)\,.$$

We shall first recall a fact about constructibility. We denote by $\mathfrak{L}(\cdot)$ the formula of constructibility.

FACT 6.1. *Let N_1, N_2 be standard, transitive models for ZF^- having the same ordinals. Then $\{a \in N_1\colon N_1 \models \mathfrak{L}[a]\} = \{a \in N_2\colon N_2 \models \mathfrak{L}[a]\}$.*

All the explanations are given in [4], [8], [2]. We shall add some facts following immediately from the preceding section.

By §§ 3 and 4, the combinatorial lemma, the 3-lemma and the semicoherence lemma we have

THEOREM 6.1. *Let M be a 3-model. Let $\langle C, \leqslant\rangle$ be a coherent continuous notion of forcing in M such that there exists a definable well-ordering of C in M into type $\leqslant On^M$. Then $M[G]$ is a 3-model for every G-C-generic over M. $M \models \mathrm{AC}$ implies $M[G] \models \mathrm{AC}$.*

THEOREM 6.2. *If M is a 3-model and $M \models \mathrm{ZF}$, $a \in M$*

$$C = \{f\colon \mathrm{Func}(f) \,\&\, \mathrm{Fin}(f) \,\&\, \mathrm{dom}(f) \subset On \times \omega_a \,\&\, \mathrm{rg}(f) \subseteq On \,\&\, (\beta)(\xi)(\gamma)(\langle\langle\beta, \xi\rangle, \gamma\rangle \in f \to \gamma < \beta)\}$$

and $\leqslant(x, y) \equiv y \subseteq x$, then $M[G]$ is a 3-model for every G-C-generic over M.

Furthermore, for every ordinal $\xi \in On^M = On^{M[G]}$ there exists a function $F_\xi \in M[G]$ such that $F_\xi\colon \omega_a \xrightarrow{\text{onto}} \xi$. If $M \models \mathrm{AC}$, then

$$M[G] \models (x)\big(x \neq \emptyset \to (\exists f)\big(\mathrm{Func}(f) \,\&\, \mathrm{dom}(f) = \omega_a \,\&\, \mathrm{rg}(f) = x\big)\big)\,.$$

Let $C_\xi = \{f \in C\colon \mathrm{dom}(f) \subseteq \xi \times \omega_a\}$ and $G_\xi = G \cap C_\xi$ and let N be a standard, transitive model for ZF^-.

Let $M \subseteq N$ and $G_\xi \in N$ for $\xi \in On^M$. Then we have $M[G] \subset N$.

Proof. It suffices to prove the existence of function F_ξ. Indeed, $G_{\xi+1} \in M[G]$ and $M[G] \models$ the axiom of union, and so $\bigcup G_{\xi+1} \in M[G]$, and since for $\beta < \omega_a$

$$D_\beta = \{f \in C_{\xi+1}\colon (\exists\eta)(\langle\langle\xi, \beta\rangle, \eta\rangle \in f)\}$$

and for $\eta < \xi+1$

$$D'_\eta = \{f \in C_{\xi+1}\colon (\exists\beta)(\langle\langle\xi, \beta\rangle, \eta\rangle \in f)\}$$

are dense in $C_{\xi+1}$, we have $G_{\xi+1} \cap D_\beta \neq \emptyset$ and $G_{\xi+1} \cap D'_\eta \neq \emptyset$. Since $G_{\xi+1}$ is a set of compatible conditions, we finally conclude that

$$F_\xi \overset{\text{dfn}}{=} \{\langle\beta, \eta\rangle\colon \langle\langle\xi, \beta\rangle, \eta\rangle \in \bigcup G_{\xi+1}\} \quad \text{is a function}\,.$$

By $M[G] \models \mathrm{ZF}^-$ we get $F_\xi\colon \omega_a \xrightarrow{\text{onto}} \xi$ and $F_\xi \in M[G]$.

COROLLARY 6.1. *If $a = 0$, then $M[G]$ is a 3-model and infinite ordinals are countable. If $M \models \mathrm{AC}$, then*

$$M[G] \models \mathrm{ZF}^- + (x)\big(x \neq \emptyset \to (\exists f)\big(\mathrm{Func}(f) \,\&\, f\colon \omega \xrightarrow{\text{onto}} x\big)\big)\,.$$

COROLLARY 6.2. *If there exists a standard model for ZF, then there exists a standard model for the theory*

$$\mathrm{ZF}^- + (\mathrm{ZF})^{\mathfrak{L}} + (x)\big(x \neq \emptyset \to (\exists f)\big(\mathrm{Func}(f) \,\&\, \mathrm{dom}(f) = \omega \,\&\, \mathrm{rg}(f) = x\big)\big)\,,$$

where $\mathfrak{L}(\cdot) \in \mathfrak{L}_{\mathrm{ZF}}$ and $(\mathrm{ZF})^{\mathfrak{L}}$ denotes the axioms of ZF related to $\mathfrak{L}(\cdot)$.

Outline of the proof. Gödel has proved that

$$\mathrm{Cons}(\mathrm{ZF}) \leftrightarrow \mathrm{Cons}\big(\mathrm{ZF} + (x)\mathfrak{L}(x)\big)\,.$$

From the Skolem-Löwenheim theorem and the contraction lemma it follows that if there exists a standard model for ZF, then there exists a countable, transitive, standard model for $\mathrm{ZF} + (x)\mathfrak{L}(x)$. Let M be a such model. By Corollary 6.1 there exists a 3-model N possessing the same ordinals. Obviously $M \models \mathrm{AC}$, and so $N \models \mathrm{AC}$. $N \models \mathfrak{L}[a]$ iff $M \models \mathfrak{L}[a]$ iff $a \in M$; hence

$$N \models \mathrm{ZF}^- + (\mathrm{ZF})^{\mathfrak{L}} + (x)\big(x \neq \emptyset \to (\exists f)\big(\mathrm{Func}(f) \,\&\, \mathrm{dom}(f) = \omega \,\&\, \mathrm{rg}(f) = x\big)\big)\,.$$

Thus we get

$$\mathrm{Consis}(\mathrm{ZF}) \leftrightarrow \mathrm{Consis}\Big(\mathrm{ZF}^- + (\mathrm{ZF})^{\mathfrak{L}} + (x)\big(x \neq \emptyset \to (\exists f)\big(\mathrm{Func}(f) \,\&\, \mathrm{dom}(f) = \omega \,\&\, \mathrm{rg}(f) = x\big)\big)\Big)\,.$$

The notion of forcing used in Corollary 6.1 is homogeneous, and so the last result is correct (see Chuaqui [1], Levy [7]).

The following remark seems to be useful for the reader who is interested in the interpretability of the theory ZF^- in the second order arithmetic and vice versa: (see [12]).

If M is a 3-model and $M \models \mathrm{AC}$, then $M \models$ schema of the axiom of choice.

The following problem remains open:

Let M be a transitive, countable, standard model of $\mathrm{ZF}+\mathrm{AC}$. Let $F\colon On^M \to On^M$ be such a functional in M that

$$(\xi_1)(\xi_2)\big(\xi_1 < \xi_2 \to F(\xi_1) < F(\xi_2)\big)\,.$$

For what F does there exist a countable, standard, transitive model N of $\mathrm{ZF}+\mathrm{AC}$ such that $M \subseteq N$, $On^M = On^N$ and for any ordinal $\varkappa \in On^M = On^N$ we have

$$N \models \mathrm{Card}[\varkappa] \quad \text{iff} \quad M \models (\exists a)(\varkappa = \omega_{F(a)})\ ?$$

We can supply easy proofs of the following facts:

1° $F(0) = 0$ (since ω is absolute).

2° F is continuous, since the supremum of a set of cardinals is a cardinal.

3° $(\alpha)(\omega_{F(\alpha+1)}$ is a cardinal regular in M) since

$$\mathrm{ZF}+\mathrm{AC} \vdash \mathrm{cf}(\omega_{\alpha+1}) = \omega_{\alpha+1} \quad \text{and} \quad \mathrm{cf}^N \leqslant \mathrm{cf}^M .$$

4° If $M \models \mathrm{GCH}$ and there is no inaccesible cardinal in M, then $(\alpha)\big(F(\alpha+1)$ is a non-limit ordinal$\big)$.

We do not know the full answer to the above problem.

The class of functionals for which the problem has a solution fulfils conditions 1°, 2°, the natural condition $(\alpha)\big(F(\alpha+1)$ is a non-limit ordinal$\big)$ and an artificial condition, which, however, is a consequence of hitherto applied methods of forcing by classes, namely:

$$(\alpha)\big(\omega_{F(\alpha)} \text{ is a singular cardinal in } M \to F(\alpha+1) = F(\alpha)+1\big) .$$

All known methods can be reduced to the proof that for a given notion of forcing $\langle C, \leqslant\rangle$ there exist a functional $\mathfrak{m}$: $On \to \mathrm{Card}$ and a functional β: $On \to On$ such that for any ξ, $\langle C, \leqslant\rangle$ satisfies the $\mathfrak{m}_\xi, \beta_\xi$-density condition and the class $\{\mathfrak{m}_\xi\colon \xi \in On\}$ is cofinal with On.

In [1] Rolando Chuaqui defines various combinatorial properties of notions of forcing and proves several model-theoretical and combinatorial lemmas. The proofs and definitions can be transferred onto ZF after a slight modification. Only Lemma 6.11 must undergo a serious change, since from $(\varkappa)\big(\mathrm{cf}^{M[G]}(\varkappa) \leqslant \mathfrak{m} \to \mathrm{cf}^M(\varkappa) \leqslant \mathfrak{m}\big)$ follows $\mathrm{cf}^{M[G]}(\mathfrak{m}^+) = \mathfrak{m}^+$ but not $\mathrm{cf}^{M[G]}(\mathfrak{m}) = \mathfrak{m}$.

Definition 6.1. Let $\mathfrak{m} \in \mathrm{Card}$, $\gamma, \xi \in On$, and let β be an increasing continuous sequence of ordinals with domain γ. Let $\langle C, \leqslant\rangle$ be a coherent notion of forcing.

(a) $\langle C, \leqslant\rangle$ is *$\mathfrak{m}$-closed* if for every linearly ordered subset E of C such that $|E| < \mathfrak{m}$ there is a $p \in C$ such that $p \leqslant q$ for all $q \in E$;

(b) $\langle C, \leqslant\rangle$ *satisfies the $\mathfrak{m}$-chain condition* if, for every subset E of incompatible elements of C, we have $|E| < \mathfrak{m}$;

(c) C *satisfies the $\mathfrak{m}, \xi$-density condition* if, for every sequence $\langle D_\alpha\colon \alpha < \gamma\rangle$ of C-dense sections, where $\gamma < \mathfrak{m}$ and for every $p \in C$, there is a $q \in C$, $q \leqslant p$ and $\Pi \subseteq C_\xi$ such that $|\Pi| < \mathfrak{m}$ and

$$(\alpha)_\gamma (q')_{\leqslant q} (\exists r)_\Pi (r \text{ is compatible with } q' \;\&\; r \wedge q \in D_\alpha) .$$

If, furthermore, $q_{(\xi)} = p_{(\xi)}$, we say that C *satisfies the strong $\mathfrak{m}, \xi$-density condition*.

Lemma 6.1. *Let C_ξ satisfies the $\mathfrak{m}$-chain condition and let C^ξ be $\mathfrak{m}$-closed. Let D be a dense section of C and $p \in C$. Then there is a $q \leqslant p$, $q \in C$ and a set $\Pi \subseteq C_\xi$ such that $|\Pi| < \mathfrak{m}$, $q_{(\xi)} = p_{(\xi)}$ and $(q')_{\leqslant q} (\exists r)_\Pi$ (r is compatible with q' & $r \wedge q \in D$).*

Proof (ZFC). Let $\Phi(f) \equiv \mathrm{Func}(f) \;\&\; \mathrm{Ord}\big(\mathrm{dom}(f)\big) \;\&\; (\alpha)_{\mathrm{dom}(f)}\big(f(\alpha) \leqslant p \;\&\; f(\alpha) \in D\big) \;\&\; (\alpha, \beta)_{\mathrm{dom}(f)} \big[\alpha \neq \beta \to \big(f(\alpha)_{(\xi)}$ is incompatible with $f(\beta)_{(\xi)}$ & $\{f(\alpha)^{(\xi)}, f(\beta)^{(\xi)}\}$ is linearly ordered$\big)\big]$.

(1) $$\Phi(f) \to |\mathrm{dom}(f)| < \mathfrak{m}$$

since C_ξ satisfies $\mathfrak{m}$-c.c.

Let $\Delta_1(f) = \bigcup \{\Delta'(f(\alpha))\colon \alpha \in \mathrm{dom}(f)\}$; then

(2) $$\Phi(f) \;\&\; \Phi(g) \;\&\; f \subseteq g \to \Delta_1(f) \leqslant \Delta_1(g) .$$

Let $L(\gamma, f) \equiv \Phi(f) \;\&\; \mathrm{dom}(f) = \gamma \;\&\; (g)\big(\Phi(g) \;\&\; \mathrm{dom}(g) = \gamma \to \Delta_1(f) \leqslant \Delta_1(g)\big)$.

By (1)

(3) $$X = \{f\colon (\exists \gamma)_{\mathfrak{m}} L(\gamma, f)\} \quad \text{is a set} .$$

Let $E \subseteq X$ be linearly ordered by inclusion; then

(4) $$\Phi(\bigcup E) \;\&\; \Delta_1(\bigcup E) = \bigcup \{\Delta_1(f)\colon f \in E\} .$$

Let g be such that $\Phi(g) \;\&\; \mathrm{dom}(g) = \mathrm{dom}(\bigcup E)$; then $\Delta_1(\bigcup E) \leqslant \Delta_1(g)$. Indeed, if $\Delta_1(g) < \Delta_1(\bigcup E)$, then there exists an $f \in E$ such that $\Delta_1(g) < \Delta_1(f)$. Let $h = g \restriction \mathrm{dom}(f)$. It is easily seen that $\mathrm{dom}(h) = \mathrm{dom}(f)$, $\Phi(h)$ and, in virtue of (2), $\Delta_1(h) \leqslant \Delta_1(g) < \Delta_1(f)$, but this contradicts $f \in X$. Hence

(5) $E \subseteq X$ & E linearly ordered implies $\bigcup E \in X$.

By (5) and the Kuratowski-Zorn lemma there exists a maximal element f. The set $\{f(\alpha)^{(\xi)}\colon \alpha \in \mathrm{dom}(f)\} \subseteq C^\xi$ is linearly ordered, its cardinality is less than $\mathfrak{m}$ and C^ξ is $\mathfrak{m}$-closed; hence there exists an $s \in C^\xi$ such that $s \leqslant f(\alpha)^{(\xi)}$ for $\alpha \in \mathrm{dom}(f)$. It is easy to check that $q = p_{(\xi)} \wedge s$ and $\Pi = \{f(\alpha)_{(\xi)}\colon \alpha \in \mathrm{dom}(f)\}$ have the required properties.

Lemma 6.2. *If $\mathfrak{m}$ is a regular cardinal and C_ξ satisfies the $\mathfrak{m}$-c.c. and C^ξ is $\mathfrak{m}$-closed, then C satisfies the strong $\mathfrak{m}, \xi$-density condition.*

Fundamental lemma 6.3. *Let $\mathfrak{m}$ be an increasing sequence of cardinals in M, cofinal with Card^M. Let β be a sequence of ordinals in M. Suppose that $\langle C, \leqslant\rangle$ is a coherent notion of forcing in M. Let $\langle C, \leqslant\rangle$ satisfy in M the $\mathfrak{m}_\xi, \beta_\xi$-density condition for every $\xi \in On^M$. Then every class G C-generic over M is strongly C-generic over M. Hence $M[G] \models \mathrm{ZF}+\mathrm{AC}$.*

Fundamental lemma 6.4. *Let C satisfy the $\mathfrak{m}, \xi$-density condition in M for some ξ; $\varkappa \in On^M$, and let $\mathfrak{m}$ be a cardinal in M. Then $\mathrm{cf}^M(\varkappa) \geqslant \mathfrak{m}$ iff $\mathrm{cf}^{M[G]}(\varkappa) \geqslant \mathfrak{m}$, where G is C-generic over M. If $\mathfrak{m}$ is a regular cardinal in M, then $\mathfrak{m}$ is a regular cardinal in $M[G]$.*

Proof. Suppose that $\mathfrak{n} = \mathrm{cf}^{M[G]}(\varkappa) < \mathfrak{m}$ and $\mathrm{cf}^M(\varkappa) \geqslant \mathfrak{m}$. There exists a function $\bar{f} \in M[G]$ such that $\bar{f}\colon \mathfrak{n} \to \varkappa$ and the set $\bar{f}(\mathfrak{n})$ is cofinal with $\varkappa$. Let

$$D_\alpha = \big\{p\colon (\exists c)\big(p \Vdash \big((\exists d)(d \in \check{\varkappa} \;\&\; \langle \check{a}, d\rangle \in f) \to \check{c} \in \check{\varkappa} \;\&\; \langle \check{a}, \check{c}\rangle \in f\big)\big)\big\} .$$

$\langle D_\alpha\colon \alpha < \mathfrak{n}\rangle$ is a sequence of sections C-dense in M. In virtue of the $\mathfrak{m}, \xi$-density condition there exist a $q \in G$ and Π such that $|\Pi| < \mathfrak{m}$ in M and

$$(q')_{\leqslant q}(\alpha)_{<\mathfrak{n}}(\exists p)_{\Pi}(p \text{ is compatible with } q' \;\&\; p \wedge q \in D_\alpha)\,.$$

Let $\mathfrak{p} = |\Pi \times \mathfrak{n}|$. Then $\mathfrak{p} < \mathfrak{m}$ and there exists a function $h \in M$ such that $h\colon \mathfrak{p} \xrightarrow{\text{onto}} \Pi \times \mathfrak{n}$. Let g be a function defined on $\Pi \times \mathfrak{n}$ as follows:

$$g(r, \alpha) = \begin{cases} \bigcap\{\delta\colon r \wedge q \Vdash \big((\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\alpha}, d\rangle \in f) \to \check{\delta} \in \check{\varkappa} \;\&\; \langle\check{\alpha}, \check{\delta}\rangle \in f\big)\} \\ \text{if } r \wedge q \text{ exists, and} \\ (\exists\delta)_{<\varkappa} r \wedge q \Vdash \big((\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\alpha}, d\rangle \in f) \to \check{\delta} \in \check{\varkappa} \;\&\; \langle\check{\alpha}, \check{\delta}\rangle \in f\big), \\ 0 \text{ otherwise}, \end{cases}$$

$g \circ h \in M$, $g \circ h\colon \mathfrak{p} \to \varkappa$ and $\{g \circ h(\eta)\colon \eta < \mathfrak{p}\} = \{g(r, \alpha)\colon r \in \Pi \;\&\; \alpha < \mathfrak{n}\}$. Let $\delta_0 = \operatorname{Sup}\{g \circ h(\eta)\colon \eta < \mathfrak{p}\}$. Since $\operatorname{cf}^M(\varkappa) \geqslant \mathfrak{m} > \mathfrak{p}$, we have $\delta_0 < \varkappa$. Suppose that $\bar{f}(\gamma) = \eta$, where $\gamma < \mathfrak{n}$. Then there is an $s \in G$ such that

$$s \Vdash \check{\eta} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{\eta}\rangle \in f \;\&\; \operatorname{Func}(f)\,,$$

and we can assume that $s \leqslant q$. Since $s \leqslant q$, there exists an $r \in \Pi$ compatible with s and $r \wedge q \in D_\gamma$, i.e. there exists a c such that

$$r \wedge q \Vdash (\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\gamma}, d\rangle \in f) \to \check{c} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{c}\rangle \in f\,.$$

Since r is compatible with s, there exists a G' C-generic over M such that $r, s \in G'$; hence $r \wedge q \in G'$ and by the Truth Lemma

$$\vdash_{G'} \check{\eta} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{\eta}\rangle \in f \;\&\; \operatorname{Func}(f)$$

and

$$\vdash_{G'} \big((\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\gamma}, d\rangle \in f) \to \check{c} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{c}\rangle \in f\big)\,,$$

i.e. $\vdash_{G'} \check{\eta} = \check{c}$; in other words, $\eta = c$ and $\eta < \varkappa$. Hence $r \wedge q$ exists and

$$(\exists\delta)_{<\varkappa}\, r \wedge q \Vdash (\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\gamma}, d\rangle \in f) \to \check{\delta} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{\delta}\rangle \in f\,.$$

From the definition of g and δ_0 it follows that there exists a $\eta_0 < \delta_0$ such that

$$r \wedge q \Vdash (\exists d)(d \in \check{\varkappa} \;\&\; \langle\check{\gamma}, d\rangle \in f) \to \check{\eta}_0 \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{\eta}_0\rangle \in f\,.$$

Since $s \Vdash \check{\eta} \in \check{\varkappa} \;\&\; \langle\check{\gamma}, \check{\eta}\rangle \in f \;\&\; \operatorname{Func}(f)$ and since there exists a G' C-generic over M and such that $s, r \wedge q \in G'$, we have $\eta = \eta_0 < \delta_0$. This shows that if $\bar{f}(\gamma) = \eta$ for $\gamma < \mathfrak{n}$, then $\mathfrak{n} < \delta_0 < \varkappa$. This contradicts the assumption that the set $\{\bar{f}(\gamma)\colon \gamma < \mathfrak{n}\}$ is cofinal with $\varkappa$.

In this manner we have proved that for any $\varkappa$

$$\operatorname{cf}^M(\varkappa) \geqslant \mathfrak{m} \quad \text{iff} \quad \operatorname{cf}^{M[G]}(\varkappa) \geqslant \mathfrak{m}\,.$$

If we now assume that $\mathfrak{m}$ is a regular cardinal in M, then $\operatorname{cf}^M(\mathfrak{m}) = \mathfrak{m}$, and hence $\mathfrak{m} \geqslant \operatorname{cf}^{M[G]}(\mathfrak{m}) \geqslant \mathfrak{m}$, which means that $\operatorname{cf}^{M[G]}(\mathfrak{m}) = \mathfrak{m}$, i.e. that $\mathfrak{m}$ is a regular cardinal in $M[G]$.

LEMMA 6.5. *If* $(\varkappa)(\varkappa < \mathfrak{n} \to 2^\varkappa \leqslant \mathfrak{n})$, $\mathfrak{m} < \mathfrak{n}$, $|B| \leqslant \mathfrak{m}$ *and* $\mathfrak{n}$ *is a regular cardinal, then* $H_{\mathfrak{m}}(A, B)$ *fulfils the* $\mathfrak{n}$*-chain condition.*

Proof. Let I be a set of pairwise incompatible conditions. Let $A_0 = \varnothing$, $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ where λ is a limit ordinal. Having defined A_α, we can choose for every $p \in H_{\mathfrak{m}}(A_\alpha, B)$ a $q \in I$ such that $q \restriction A_\alpha = p$ provided that such a q exists. $A_{\alpha+1} = A_\alpha \cup \bigcup_{q\text{-chosen}} \operatorname{dom}(q)$.

We show that if $\alpha \leqslant \mathfrak{m}$, then $|A_\alpha| < \mathfrak{n}$. For $\alpha = 0$ and α-limit the proof is immediate.

If $|A_\alpha| < \mathfrak{n}$, then

$$|H_{\mathfrak{m}}(A_\alpha, B)| \leqslant \sum_{\mathfrak{p}<\mathfrak{m}} (|A_\alpha||B|)^{\mathfrak{p}} < \prod_{\mathfrak{p}<\mathfrak{m}} (|A_\alpha||B|)^{\mathfrak{p}} \leqslant \big((|A_\alpha||B|)^{\mathfrak{m}}\big)^{\mathfrak{m}}$$
$$= (|A_\alpha||B|)^{\mathfrak{m}} \leqslant 2^{|A_\alpha|\,|B|\cdot\mathfrak{m}}\,,$$

but $|A_\alpha| < \mathfrak{n}$, $|B| < \mathfrak{n}$ and $\mathfrak{m} < \mathfrak{n}$; hence $|A_\alpha||B|\mathfrak{m} < \mathfrak{n}$ and we get $2^{|A_\alpha||B|\mathfrak{m}} \leqslant \mathfrak{n}$, i.e. $|H_{\mathfrak{m}}(A_\alpha, B)| < \mathfrak{n}$. Since

$$|\{q\colon q \text{ choosen for } A_\alpha\}| \leqslant |H_{\mathfrak{m}}(A_\alpha, B)| \quad \text{and} \quad |\operatorname{dom}(q) < \mathfrak{m}\,,$$

we get

$$|A_{\alpha+1}| \leqslant |A_\alpha| + |H_{\mathfrak{m}}(A_\alpha, B)|\mathfrak{m} < \mathfrak{n}\,.$$

It suffices to show that $I \subseteq H_{\mathfrak{m}}(A_{\mathfrak{m}}, B)$. Let $p \in I$, Since $|\operatorname{dom}(p)| < \mathfrak{m}$, there exists an $\alpha < \mathfrak{m}$ such that $\operatorname{dom}(p) \cap A_\alpha = \operatorname{dom}(p) \cap A_{\alpha+1}$; $p \restriction A_\alpha \in H_{\mathfrak{m}}(A_\alpha, B)$. There exists a $q' \in I$ such that $q' \restriction A_\alpha = p \restriction A_\alpha$, e.g. $q' = p$. In virtue of the definition $A_{\alpha+1}$ there is a $q \in I$ such that $\operatorname{dom}(q) \subseteq A_{\alpha+1}$ and $q \restriction A_\alpha = p \restriction A_\alpha$.

If $x \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, then $x \in A_{\alpha+1}$; hence $x \in \operatorname{dom}(p) \cap A_{\alpha+1} = \operatorname{dom}(p) \cap A_\alpha$. But $q \restriction A_\alpha = p \restriction A_\alpha$, and hence $p(x) = q(x)$ and $p, q \in I$, i.e. $p = q$. This means that $\operatorname{dom}(p) \subseteq A_{\alpha+1} \subseteq A_{\mathfrak{m}}$; hence $p \in H_{\mathfrak{m}}(A_{\mathfrak{m}}, B)$.

LEMMA 6.6. *Let* F *be a definable, continuous, increasing functional such that* $F\colon On \to On$ *and* $F(0) = 0$. *Let the following conditions be satisfied:*

$(\alpha)\big(F(\alpha+1)$ *is a non-limit ordinal*$\big)$,
$(\lambda)\big(\omega_{F(\lambda)}$ *is a singular cardinal* $\to F(\lambda+1) = F(\lambda)+1\big)$.

Let H *be such a functional that* $(\xi)\big(F(\xi+1) = H(\xi)+1\big)$. *Let*

$$C(f) \equiv \operatorname{Func}(f) \;\&\; \operatorname{dom}(f) \subseteq On \times On \;\&\; \operatorname{rg}(f) \subseteq On \;\&$$
$$\&\; (\xi)(\alpha)(\beta)\big(\langle\langle\xi, \alpha\rangle, \beta\rangle \in f \to H(\xi) \notin \operatorname{rg}(F) \;\&\; \alpha < \omega_{F(\xi)} \;\&$$
$$\&\; \beta < \omega_{H(\xi)}\big) \;\&\; (\xi)\big(H(\xi) \notin \operatorname{rg}(F) \to |\operatorname{dom}(f) \cap \big((\xi+1) \times \omega_{F(\xi)}\big)| < \omega_{F(\xi)}\big)$$
$$\leqslant(f, g) \equiv g \subseteq f.$$

Let

$$C_\xi = \{f\colon\ C(f)\ \&\ \mathrm{l.dom}(f) \subseteq \xi+1\},$$
$$C_\xi = \{f\colon\ C(f)\ \&\ \mathrm{l.dom}(f) \cap (\xi+1) = \emptyset\}$$

where $\mathrm{l.dom} f = \{\eta\colon (\exists\alpha)(\langle\eta, \alpha\rangle \in \mathrm{dom}(f))\}$. *Then C is a coherent notion of forcing. If for every ξ such that $H(\xi) \notin \mathrm{rg}(F)$ and every cardinal $\mathfrak{m} < \omega_{F(\xi+1)}$ we have $2^{\mathfrak{m}} \leqslant \omega_{F(\xi+1)}$, then for every ξ such that $H(\xi) \notin \mathrm{rg} F$ C_ξ fulfils the $\omega_{F(\xi+1)}$-chain condition and C^ξ is $\omega_{F(\xi+1)}$-closed. If $F(\xi+1) = F(\xi)+1$, then C^ξ is $\omega_{F(\xi+1)}$-closed and C_ξ fulfils the $\omega_{F(\xi+1)}$-chain condition under the assumption that $2^{\omega_{F(\xi)}} = \omega_{F(\xi+1)}$.*

COROLLARY.

$$\mathrm{ZFC}+\mathrm{GCH} \vdash (\xi)\big(C \text{ fulfils the } \omega_{F(\xi+1)}, \xi \text{ — density condition}\big).$$

The above corollary and the previous lemmas imply the following

THEOREM. *If M is a standard, transitive and countable model of ZFC, $F\colon On^M \to On^M$ is an increasing, continuous functional in M, $F(0) = 0$ and*

$$(\alpha)\big(F(\alpha+1) \text{ is a non-limit ordinal}\big),$$
$$(\lambda)\big(\lambda \in \mathrm{Lim}\ \&\ \omega_{F(\lambda)} \text{ is a singular cardinal in } M \to F(\lambda+1) = F(\lambda)+1\big),$$

then there exists a standard, transitive, countable model N of ZF+AC such that $On^M = On^N$ and for any ordinal $\varkappa \in On^M = On^N$ $N \models \mathrm{Card}[\varkappa]$ iff $M \models (\exists\alpha)\ (\varkappa = \omega_{F(\alpha)})$.

Remark. Let M be a model of ZF+AC, $\mathfrak{n}$ — a singular cardinal in M, $\langle \mathfrak{m}_\xi\colon \xi < \alpha\rangle$ — a sequence of cardinals, and $\langle \beta_\xi\colon \xi < \alpha\rangle$ — a sequence of ordinals. Let C be such a notion of forcing that

$$(\xi)_\alpha(C \text{ satisfies the } \mathfrak{m}_\xi, \beta_\xi \text{ — density condition}).$$

If G is C-generic over M and $M[G] \models \mathrm{ZF}$, then $\mathfrak{n}^+$ is a regular cardinal in $M[G]$ under the assumption that $\lim\limits_{\xi<\alpha} \mathfrak{m}_\xi = \mathfrak{n}$.

Outline of the proof. $\mathrm{cf}^{M[G]}(\mathfrak{n}) < \mathfrak{n}$. If $\mathrm{cf}^{M[G]}(\mathfrak{n}^+) < \mathfrak{n}^+$, then we would have $\mathrm{cf}^{M[G]}(\mathfrak{n}^+) < \mathfrak{n}$, whence, for a certain $\xi < \alpha$, $\mathrm{cf}^{M[G]}(\mathfrak{n}^+) < \mathfrak{m}_\xi$. In virtue of the $\mathfrak{m}_\xi, \beta_\xi$-density condition, $\mathrm{cf}^M(\mathfrak{n}^+) < \mathfrak{m}_\xi$, and we have a contradiction. This is the reason for introducing a rather artificial condition:

$$(\lambda)\big(\omega_{F(\lambda)} \text{ — singular} \to F(\lambda+1) = F(\lambda)+1\big).$$

References

[1] R. Chuaqui, *Forcing for the impredicative theory of classes*, preprint.
[2] J. P. Cohen, *Set Theory and the Continuum Hypothesis*, 1966.
[3] W. Easton, *Powers of regular cardinals*, Princeton University 1964.

[4] K. Gödel, *The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory*, Ann. Math. Studies 3 (1951).
[5] R. Jensen, *The generalized continuum hypothesis and measurable cardinals*, to appear in Proc. Amer. Math. Soc. Summer Institute on Axiomatic Set Theory, 1967, Univ. of California, Los Angeles, Proc. Symp. Pure Math. 13.
[6] K. Kuratowski, A. Mostowski, *Set Theory*, Warszawa 1967.
[7] A. Levy, *Definability in axiomatic set theory II*, proceedings of an international colloquium Jerusalem 1968, Yehoshua Bar-Hillel 1970, pp. 129–145.
[8] A. Mostowski, *Constructible Sets with Applications*, Warszawa 1969.
[9] D. Scott, *Lectures on Boolean-valued models for set theory*, lecture notes of the U.C.L.A. Summer Institute on Set Theory, 1967.
[10] J Shoenfield, *Unramified forcing*, preprint.
[11] R. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. 92 (1970), pp. 1–24.
[12] P. Zbierski, *Models for higher order arithmetics*, Bull. Acad. Pol. Sci. 19, pp. 557–562.

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