Complements of sets of unstable points (*)

by

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To the memory of Tudor Ganea

Abstract. In the present paper the following fact is established: Let X be a locally compact connected finite dimensional space and let Y be a closed subset of X such that each point of Y is "homotopically labil" in the sense of Borsuk and Jaworowski. Then $X \sim Y$ is connected. In order to establish this theorem, the singular homology groups $H_q(X, X \sim Y)$ are investigated by means of the sheaf theory.

§ 0. Introduction. A few weeks before his death, T. Ganea asked the author if a set of unstable points can disconnect a space. If by "unstable" one means "labil" in the sense of Hopf and Pannwitz, then there is an easy example to show that a single "unstable" point can disconnect a space (see 2.1). If, however, "unstable" means "homotopically labil" in the sense of Borsuk and Jaworowski, then one suspects that a set consisting of unstable points cannot disconnect a space provided, of course, the space is connected to start with. In § 2 of the present paper, we justify this suspicion. We show that, if X is locally compact and of finite dimension and if Y is a closed subset of Y consisting of unstable points, then the homology groups of X and $X \sim Y$ are isomorphic. If in addition X is connected, then $X \sim Y$ is connected. These facts are established by a homological method.

The idea of the proof is quite simple. Assume, for the sake of simplicity, that X is a compact, metric and finite dimensional space, and let Y be a closed subspace of X consisting of unstable points. We establish that $H_q(X, X \sim Y) = 0$ for all q by induction on $\dim Y$. If $\dim Y = -1$ this is clear. Assume that $\dim Y = n$. Suppose that there is a non-zero element α in $H_q(X, X \sim Y)$. Write $Y = Y_1 \cup Z_1$ where Y_1 and Z_1 are closed and $\dim(Y_1 \cap Z_1) \leqslant n-1$. By the inductive hypothesis and the relative Mayer-Vietoris sequence of the triad $(X; X \sim Y_1, X \sim Z_1)$, we see that α is mapped to a non-zero element in either $H_q(X, X \sim Y_1)$ or $H_q(X, X \sim Z_1)$. Suppose that the image of α in $H_q(X, X \sim Y_1)$ is non-zero. We may repeat this argument, and obtain a sequence of closed sets

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$Y\supset Y_1\supset Y_2\supset\ldots$ such that $\operatorname{diam}(Y_k)\to 0$ and the image of α in $H_q(X, X\sim Y_k)$ is non-zero for each k. Let $y_0=\bigcap Y_k\in Y$. Then $H_q(X, X\sim\{y_0\})=\operatorname{dir}\lim_k H_q(X, X\sim Y_k)\neq 0$. However, since y_0 is unstable, it is "homologically unstable", i.e. $H_q(X, X\sim\{y_0\})=0$ for all q. This contradiction proves that $H_q(X, X\sim Y)=0$ for all q. In order to obtain the most out of the type of argument just outlined, one usually resorts to sheaf theory as will be done in this paper. In § 1, a sheaf theoretic preparation is carried out. The basic spectral sequence (Theorem 1.7) is a relative version of the usual spectral sequence, and probably is well-known in some quarters. Our chief reference in sheaf theory is Swan [9].

§ 1. Throughout the paper, all topological spaces are assumed to be Hausdorff. Let X be a topological space. For each non-negative integer q, let $\Delta_q(X)$ be the set of all singular q-simplexes of X i.e. the set of all continuous maps $T\colon \Delta_q\to X$ where Δ_q is the standard q-simplex. For each T in $\Delta_q(X)$, let $|T|=T[\Delta_q]$. Since X is assumed to be Hausdorff, $|T|$ is closed. A *locally finite singular p-chain in X* is a formal expression $\sum_{T\in\Delta_q(X)} m(T)\cdot T$, where m is a function on $\Delta_q(X)$ into the group Z of all integers, and where the family $\{|T|\colon m(T)\neq 0\}$ of closed subsets of X is locally finite. For a locally finite singular p-chain $s=\sum_{T\in\Delta_q(X)} m(T)\cdot T$, we define the *support* $|s|$ of s by

$$|s|=\bigcup\{|T|\colon m(T)\neq 0\}.$$

Note that $|s|$ is necessarily a closed subset of X.

A family Φ of subsets of X is called a *family of supports* if it satisfies the following conditions:

1) Each member of Φ is closed.

2) If $A, B\in\Phi$, then $A\cup B\in\Phi$.

3) If $A\in\Phi$ and $B=B^-\subset A$, then $B\in\Phi$.

If a family of supports Φ satisfies the following two additional conditions, then it is called *paracompactifying*:

4) Each member of Φ is paracompact.

5) If $A\in\Phi$, then A is contained in the interior of some member of Φ.

The family of all closed subsets of X is a family of supports. The family c of all compact subsets of X is a family of supports, and c is paracompactifying if X is locally compact.

Let Φ be a family of supports, and let $C_q^\Phi(X)$ be the set of all locally finite singular p-chains s such that $|s|\in\Phi$. We set $C_q^\Phi(X)=0$ for $q<0$. Clearly $C_q^\Phi(X)$ is an abelian group under the obvious operation. Note that $C_q^c(X)$ is the group of usual singular p-chains in X. The boundary map $\partial\colon C_q^c(X)\to C_{q-1}^c(X)$ can easily be generalized to the boundary map $\partial\colon C_q^\Phi(X)\to C_{q-1}^\Phi(X)$. The resulting chain complex $\{C_q^\Phi(X), \partial\}$ is denoted by $C_*^\Phi(X)$. If Φ is the family of all closed subsets of X, then $C_q^\Phi(X)$ will be simply denoted by $C_q(X)$, and this is the group of *all* locally finite singular p-chains in X. Let A be a subspace of X. Then $C_*(A)$ is a subcomplex of $C_*(X)$, and the quotient complex $\{C_q(X)/C_q(A)\}$ is denoted by $\{C_q(X, A)\}$ or $C_*(X, A)$. For each element s of $C_q(X, A)$, we define the support $|s|$ of s as follows: Let s be represented by $\sum_T m(T)\cdot T\in C_q(X)$. Then

$$|s|=\bigcup\{|T|\colon m(T)\neq 0,\ |T|\cap(X\sim A)\neq\emptyset\}.$$

Clearly $|s|$ is well-defined. For a family of supports Φ in X, let

$$C_q^\Phi(X, A)=\{s\colon s\in C_q(X, A),\ |s|\in\Phi\}.$$

Then $C_*^\Phi(X, A)=\{C_q^\Phi(X, A)\}$ is a subcomplex of $C_*(X, A)$. We remark that the complex $C_*^\Phi(X, A)$ is the quotient of the complex $C_*^\Phi(X)$ by a subcomplex $C_*^{\Phi_A}(A)$, where $\Phi_A=\{B\colon B\in\Phi, B\subset A\}$. The qth homology group of the complex $C_*^\Phi(X, A)$ (resp. $C_*^\Phi(X)$) is denoted by $H_q^\Phi(X, A)$ (resp. $H_q^\Phi(X)$).

Let X be a topological space and let A be a subspace of X. For each non-negative integer q, we define a presheaf ("stack" in [9]) $\underline{S}_q$ on X by

$$\underline{S}_q(U)=C_q^c(X)/[C_q^c(A)+C_q^c(X\sim U)].$$

If $V\subset U$, then $C_q^c(A)+C_q^c(X\sim U)\subset C_q^c(A)+C_q^c(X\sim V)$. Therefore the "restriction" map $\underline{S}_q(U)\to\underline{S}_q(V)$ is defined in the obvious way. Let $\mathcal{S}_q$ be the sheaf generated by the presheaf $\underline{S}_q$. The boundary map $C_q^c(X)\to C_{q-1}^c(X)$ induces a presheaf map $\underline{\partial}_q\colon \underline{S}_q\to\underline{S}_{q-1}$ and a sheaf map $\partial_q\colon \mathcal{S}_q\to\mathcal{S}_{q-1}$ such that $\underline{\partial}_{q-1}\underline{\partial}_q=0$ and $\partial_{q-1}\partial_q=0$.

1.1. Lemma. *Let Φ be a family of supports in X. Then the complexes $C_*^\Phi(X, A)$ and $\{\Gamma_\Phi\mathcal{S}_q, \Gamma_\Phi c_q\}$ are isomorphic.*

Proof. We apply [9, pp. 84–87] to an indexed collection of subsets of X given by $\{|T|\colon T\in\Delta_q(X), |T|\cap(X\sim A)\neq\emptyset\}$. The presheaf $\underline{S}'$ in [9, p. 86] is precisely our $\underline{S}_q$. Hence by Proposition 5 [9, p. 86], $\Gamma\mathcal{S}_q$ is isomorphic to the group of all locally finite chains of the family $\{|T|\colon T\in\Delta_q(X), |T|\cap(X\sim A)\neq\emptyset\}$. But the latter is isomorphic to $C_q(X, A)$. Let $\varphi_q\colon C_q(X, A)\to\Gamma\mathcal{S}_q$ be the resulting isomorphism. Then by Lemma 6 [9, p. 87], $|\varphi_q s|=|s|$ for each s in $C_q(X, A)$. Therefore φ_q induces an isomorphism $C_q^\Phi(X, A)\to\Gamma_\Phi\mathcal{S}_q$. Clearly $\{\varphi_q\}$ is a chain map.

In order to make $\mathcal{S}_*$ into a cochain complex of sheaves, we let $\mathcal{S}^q=\mathcal{S}_{-q}$ and $\delta^q=\partial_{-q}\colon \mathcal{S}^q\to\mathcal{S}^{q+1}$. The cochain complex $\{\mathcal{S}^q, \delta^q\}$ is denoted by $\mathcal{S}^*$. The qth homology (resp. cohomology) sheaf of $\mathcal{S}_*$ (resp. $\mathcal{S}^*$) is denoted by $\mathcal{H}_q(\mathcal{S}_*)$ (resp. $\mathcal{H}^q(\mathcal{S}^*)$).

The proof of the following lemma requires only a slight modification of the usual proof (c.f. [9, p. 88]) and is omitted.

1.2. Lemma. *The cochain complex $\mathcal{S}^*$ is homotopically fine.*

1.3. COROLLARY. *Let Φ be a paracompactifying family of supports in X. Then for $p \neq 0$, the following sequence is exact:*

$$\to H^p_\Phi(X, \mathcal{S}^{q-1}) \xrightarrow{\delta^{q-1}_*} H^p_\Phi(X, \mathcal{S}^q) \xrightarrow{\delta^q_*} H^p_\Phi(X, \mathcal{S}^{q+1}) \to \ldots$$

Proof. See [9, p. 76].

Combining Lemma 1.1 and Corollary 1.3, we obtain the following spectral sequence. Recall that Φ-$\dim X \leqslant n$ if and only if $H^p_\Phi(X, \mathcal{F}) = 0$ for all $p > n$ and all sheaves $\mathcal{F}$ on X.

1.4. THEOREM. *Let Φ be a paracompactifying family of supports in X, and let Φ-$\dim X < \infty$. Then there is a convergent spectral sequence such that*

$$E_2^{p,-q} = H^p_\Phi(X, \mathcal{H}_q(\mathcal{S}_*))$$

and $\{E_\infty^{p,-q}\colon q-p = r\}$ is the graded group associated with $H^\Phi_r(X, A)$ suitably filtered

Proof. Apply [9, pp. 116–117] or [5, p. 178] to the cochain complex $\mathcal{S}^*$, and note that $\mathcal{H}^{-q}(\mathcal{S}^*) = \mathcal{H}_q(\mathcal{S}_*)$ and, by Lemma 1.1, $H^\Phi_r(X, A) \cong H_r(\Gamma_\Phi \mathcal{S}_*) = H^{-r}(\Gamma_\Phi \mathcal{S}^*)$.

The following proposition contains some information concerning the sheaves $\mathcal{H}_q(\mathcal{S}_*)$.

1.5. PROPOSITION. *If $x \notin A$, then the stalk $\mathcal{H}_q(\mathcal{S}_*)_x$ is isomorphic to $H^c_q(X, X \sim \{x\})$. If $x \in \operatorname{Int} A$, then $\mathcal{H}_q(\mathcal{S}_*)_x = 0$ for all q.*

Proof. Let $x \in X$ and let $\mathfrak{U}_x$ be the directed family of open neighborhoods of x. Then

$$(\mathcal{S}_q)_x = \operatorname*{dir\,lim}_{U \in \mathfrak{U}_x} \underline{\mathcal{S}}_q(U) = C^c_q(X)/C^c_q(A) + C^c_q(X \sim \{x\}),$$

and the boundary map: $(\mathcal{S}_q)_x \to (\mathcal{S}_{q-1})_x$ is induced by the usual boundary map: $C^c_q(X) \to C^c_{q-1}(X)$. Therefore there is a short exact sequence of chain complexes:

$$0 \to C^c_q(A)/C^c_q(A \sim \{x\}) \xrightarrow{i_\#} C^c_q(X)/C^c_q(X \sim \{x\}) \to (\mathcal{S}_q)_x \to 0$$

where $i_\#$ is induced by the inclusion map $i\colon A \to X$. Consequently, we have the following exact sequence:

$$\ldots \to H^c_q(A, A \sim \{x\}) \xrightarrow{i_*} H^c_q(X, X \sim \{x\}) \to \mathcal{H}_q(\mathcal{S}_*)_x \to H^c_{q-1}(A, A \sim \{x\}) \to \ldots$$

If $x \notin A$, then $H^c_q(A, A \sim \{x\}) = 0$ for all q. Therefore $H^c_q(X, X \sim \{x\}) \xrightarrow{\approx} \mathcal{H}_q(\mathcal{S}_*)_x$ for all q. Assume next that $x \in \operatorname{Int} A$. Then there is a member U of $\mathfrak{U}_x$ which is contained in A. Let j and k be the inclusion maps: $(U, U \sim \{x\}) \to (A, A \sim \{x\})$ and $(U, U \sim \{x\}) \to (X, X \sim \{x\})$ respectively. Then $k = ij$ and k_* and j_* are isomorphisms by the excision theorem. Therefore i_* is an isomorphism for each q. It follows from the long exact sequence that $\mathcal{H}_q(\mathcal{S}_*)_x = 0$ for all q.

From Proposition 1.5, if A is an open subset of X, then

$$\mathcal{H}_q(\mathcal{S}_*)_x \cong \begin{cases} H^c_q(X, X \sim \{x\}) & \text{if } x \notin A, \\ 0 & \text{if } x \in A. \end{cases}$$

Actually there is a slightly better way to identify the sheaf $\mathcal{H}_q(\mathcal{S}_*)$. Let $\mathcal{J}_q$ be the sheaf generated by the presheaf $U \to H^c_q(X, X \sim U)$. For each x in X, the stalk $(\mathcal{J}_q)_x$ is isomorphic to $H^c_q(X, X \sim \{x\})$. Recall that if $\mathcal{F}$ is any sheaf over X and if Y is a locally closed subset of X, then $\mathcal{F}_Y$ is the unique prolongation of $\mathcal{F}|Y$ by zero (see [9, p. 31] or [5, p. 138]). In particular, if A is open, then $\mathcal{F}_A$ is a subsheaf of $\mathcal{F}$ and there is an exact sequence: $0 \to \mathcal{F}_A \to \mathcal{F} \to \mathcal{F}_{X \sim A} \to 0$.

1.6. LEMMA. *Let A be an open subset of X. Then $\mathcal{H}_q(\mathcal{S}_*)$ is isomorphic to $(\mathcal{J}_q)_{X \sim A}$.*

Proof. For any open subset U of X, there is an obvious chain map $C^c_q(X)/C^c_q(X \sim U) \to \underline{\mathcal{S}}_q(U)$. This map induces a sheaf map $\alpha\colon \mathcal{J}_q \to \mathcal{H}_q(\mathcal{S}_*)$ such that, at each point x of X, the maps $\alpha_x\colon (\mathcal{J}_q)_x \cong H^c_q(X, X \sim \{x\}) \to \mathcal{H}_q(\mathcal{S}_*)_x$ of stalks are precisely the maps appearing in the long exact sequence in the proof of Proposition 1.5. Hence, for $x \notin A$, α_x is an isomorphism. If $x \in A$, then $\mathcal{H}_q(\mathcal{S}_*)_x = 0$ as remarked above. Therefore $\mathcal{H}_q(\mathcal{S}_*) \cong (\mathcal{J}_q)_{X \sim A}$.

1.7. THEOREM. *Let Y be a closed subset of a topological space X and let Φ be a paracompactifying family of supports in X such that Φ-$\dim X < \infty$. Then there is a convergent spectral sequence such that*

$$E_2^{p,-q} = H^p_{\Phi_Y}(Y, \mathcal{J}_q|Y)$$

and $\{E_\infty^{p,-q}\colon q-p = r\}$ is the graded group associated with $H^\Phi_r(X, X \sim Y)$ suitably filtered. Here $\Phi_Y = \{B\colon B \in \Phi, B \subset Y\}$.

Proof. In Theorem 1.4, let $A = X \sim Y$, and note that $\mathcal{H}_q(\mathcal{S}_*) \cong (\mathcal{J}_q)_Y$ by Lemma 1.6. Finally by Corollary 1 [9, p. 93] (or by Theorem 4.9.1 [5, p. 187]), $H^p_\Phi(X, (\mathcal{J}_q)_Y) \cong H^p_{\Phi_Y}(Y, \mathcal{J}_q|Y)$.

1.8. Remark. If X is an n-dimensional manifold, then $\mathcal{J}_q = 0$ for $q \neq n$, and $\mathcal{J} = \mathcal{J}_n$ is called the *orientation sheaf of X*. Theorem 1.7 then yields that $H^\Phi_{n-p}(X, X \sim Y) \cong H^p_{\Phi_Y}(Y, \mathcal{J}|Y)$. This is, of course, the well-known Alexander-Lefshetz duality theorem [9, p. 138].

§ 2. In this section, we shall investigate the effect on a topological space of removal of a subset consisting of unstable points.

The following definition of unstability is due to Borsuk and Jaworowski [2], and it is slightly different from the definition given by Hopf and Pannwitz [7].

A point x_0 in a topological space X is *unstable* (in X) if, for each open neighborhood U of x_0, there is a homotopy $h\colon (X, U) \times I \to (X, U)$ such that

i) $h(x,0) = x$ for all x in X,

ii) $h(x,t) = x$ for all x in $X \sim U$ and t in I,

iii) $h(x,1) \neq x_0$ for all x in X.

Here I denotes the unit interval $[0,1]$ as usual. If x_0 is not unstable in X, then x_0 is said to be *stable* (in X). As pointed out by Borsuk and Jaworowski [2, p. 160], the property of being unstable is a local one.

If, in the above definition, condition iii) is replaced by the following weaker condition:

iii)′ $\{h(x,1)\colon x \in X\} \neq X$,

then the point x_0 is called *labil* by Hopf and Pannwitz ([7], also [1, p. 523]).

2.1. Example. Let X be the union of the closed unit disk $\{(x,y)\colon x^2+y^2 \leqslant 1\}$ in the plane and the segment $\{(x,0)\colon 1 \leqslant x \leqslant 2\}$, and let $x_0 = (1,0)$. Then x_0 is labil, and $X \sim \{x_0\}$ is disconnected. On the other hand, the set of all unstable points in X cannot disconnect X. Therefore, for our problem, the difference between conditions iii and iii′ is crucial.

A point x_0 of a topological space X is called *homologically unstable* (in X) if $H_q^c(X, X \sim \{x_0\}) = 0$ for all q. Recall that H^c denotes the usual singular homology theory (c.f. § 1). From the homology sequence of the pair $(X, X \sim \{x_0\})$, it is clear that the point x_0 is homologically unstable if and only if the inclusion map $X \sim \{x_0\} \to X$ induces isomorphisms $H_q^c(X \sim \{x_0\}) \overset{\approx}{\to} H_q^c(X)$ for all q. The point x_0 is *homologically stable* (in X) if it is not homologically unstable.

2.2. Proposition. *If a point x_0 in a topological space X is unstable, then x_0 is homologically unstable.*

Proof. Let $i\colon X \sim \{x_0\} \to X$ be the inclusion map, and let q be a non-negative integer. We must show that $i_*\colon H_q^c(X \sim \{x_0\}) \to H_q^c(X)$ is an isomorphism. Let a be an arbitrary element of $H_q^c(X \sim \{x_0\})$ such that $i_* a = 0$. The element a is represented by a cycle z in $C_q^c(X \sim \{x_0\})$. Let A be the support $|z|$ of z (c.f. § 1); then A is a compact subset of $X \sim \{x_0\}$. Hence $U = X \sim A$ is an open neighborhood of x_0. (Recall that all topological spaces are assumed to be Hausdorff). By the hypothesis x_0 is unstable. Hence there is a homotopy $h\colon (X, U) \times I \to (X, U)$ satisfying i, ii, iii above. Let $h_t(x) = h(x,t)$ and let $g\colon X \to X \sim \{x_0\}$ be the unique map such that $h_1 = ig$. Then $i_* g_* = (h_1)_* = (h_0)_* = \mathrm{id}$. This shows that i_* is onto. For each x in A, $g(x) = h_1(x) = x$. Therefore $a = g_* i_* a = 0$, and hence i_* is one-to-one.

Remark. The converse of Proposition 2.2 is false. Let CX be the (unreduced) cone over a topological space X, and let x_0 be its vertex. Then

$$H_q^c(CX, CX \sim \{x_0\}) \underset{\approx}{\overset{\partial}{\to}} \widetilde{H}_{q-1}^c(CX \sim \{x_0\}) \underset{\approx}{\to} \widetilde{H}_{q-1}^c(X).$$

Therefore the vertex x_0 is homologically unstable if and only if the reduced groups $\widetilde{H}_q^c(X)$ are all 0, i.e. X is *acyclic*. On the other hand, if x_0 is unstable, then X is contractible (see [6, p. 563]). Since there are acyclic spaces that are not contractible, there are homologically unstable points that are stable. The point x_0 in Example 2.1 is labil but homologically stable.

Let Φ be a family of supports in X. Then $\dim_\Phi X = \sup\{\dim A\colon A \in \Phi\}$. By Theorem 1 [9, p. 109], $\Phi\text{-}\dim X \leqslant \dim_\Phi X$ if Φ is a paracompactifying family of supports.

2.4. Theorem. *Let Y be a closed subset of a topological space X such that each point of Y is homologically unstable in X. If Φ is a paracompactifying family of supports in X such that $\dim_\Phi X < \infty$, then $H_q^\Phi(X, X \sim Y) = 0$ for all q, or, equivalently, the inclusion map $i\colon X \sim Y \to X$ induces isomorphisms $i_*\colon H_q^{\Phi X \sim Y}(X \sim Y) \overset{\approx}{\to} H_q^\Phi(X)$ for all q.*

Proof. By the hypotheses, $\Phi\text{-}\dim X \leqslant \dim_\Phi X < \infty$, and $\mathfrak{J}_q | Y = 0$ for all q, where $\mathfrak{J}_q$ is the sheaf as defined in § 1. Hence by Theorem 1.7, $H_q^\Phi(X, X \sim Y) = 0$ for all q. As remarked in § 1, $C_*^\Phi(X, X \sim Y)$ is the quotient of $C_*^\Phi(X)$ by $C_*^{\Phi X \sim Y}(X \sim Y)$. Hence the last conclusion follows from the standard homology sequence argument.

By specializing Theorem 2.4 to the case where Φ is equal to the family c of all compact subsets of X, we obtain the following corollary. Notice that in order to make c paracompactifying, we must require that X be locally compact.

2.5. Corollary. *Let X be a locally compact topological space such that $\dim_c X < \infty$. Let Y be a closed subspace of X such that each point of Y is homologically unstable. Then the inclusion map $i\colon X \sim Y \to X$ induces isomorphisms $i_*\colon H_q^c(X \sim Y) \overset{\approx}{\to} H_q^c(X)$ for all q.*

2.6. Remark. By proceeding more carefully, one can improve Theorem 2.4 and Corollary 2.5. For instance, in Corollary 2.5, assume that Y is a closed subset of X such that $n = \dim_c Y \leqslant \dim_c X < \infty$ and that $H_q^c(X, X \sim \{y\}) = 0$ for $0 \leqslant q \leqslant n+r$ and for each y in Y. Then $H_q^c(X, X \sim Y) = 0$ for $0 \leqslant q \leqslant r$.

A topological space is called *hereditarily normal* if each subspace is normal with respect to the relative topology. Metrizable spaces are hereditarily normal.

2.7. Corollary. *Let X be a pathwise connected, hereditarily normal, locally compact topological space such that $\dim_c X < \infty$. Let L be the set of all homologically unstable points in X. Then $X \sim L$ is connected.*

Proof. Assume that $X \sim Y$ is disconnected. Then there are non-empty subsets A, B of $X \sim Y$ such that $X \sim Y = A \cup B$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Since X is hereditarily normal, there exist open disjoint neighborhoods U and V of A and B respectively (see, for instance, [3, p. 95]). Let $Y = X \sim (U \cup V) \subset L$. Since Y is closed, by Corollary 2.5, $H_0^c(X \sim Y) \cong H_0^c(X) \cong Z$. It follows that $X \sim Y = U \cup V$ is pathwise connected. But this is absurd, because U and V are separated and non-empty.

Next we shall consider spaces that are not necessarily pathwise connected. A *deformation* of a topological space X is a homotopy $h\colon X \times I \to X$ such that $h(x, 0) = x$ for all x in X. A subset A of X is said to be *stable under deformation* if, for any deformation h of X, $h(x, t) \in A$ for all x in A and t in I. If A and B are stable under deformation, then so are $\bar{A}$ and $\bar{A} \cap B$.

2.8. Lemma. *Let A be a subset of a topological space that is stable under deformation. If a point x_0 in A is stable in A, then x_0 is stable in X.*

Proof. Suppose that x_0 is unstable in X. Then we show that x_0 is unstable in A. Let V be an open neighborhood of x_0 in A. Then there is an open subset U of X such that $U \cap A = V$. Since x_0 is assumed to be unstable, there is a homotopy $h\colon (X, U) \times I \to (X, U)$ satisfying i, ii, iii. Since A is stable under deformation, the restriction of h induces a map $h'\colon (A, V) \times I \to (A, V)$ satisfying i, ii, iii relative to A and V. Hence x_0 is unstable in A. This proves the lemma.

We also need the following theorem of T. Ganea [4]. In deference to Hurewicz and Wallman [8], Ganea assumed that the spaces were always separable and metric. However this assumption is superfluous if we use the covering dimension exclusively. We also remark that Ganea's proof, which uses the Čech homology groups with the reals modulo 1 as the coefficients, can easily be converted to a proof that uses more familiar Čech cohomology groups with the coefficient group Z.

2.9. Theorem (Ganea [4]). *Let X be a topological space such that $\dim_c X < \infty$. Then the set of all stable points is dense in X.*

2.10. Theorem. *Let X be a connected locally compact topological space such that $\dim_c X < \infty$. Let Y be a closed subset of X such that each point of Y is unstable. Then $X \sim Y$ is connected.*

Proof. For points x, y in X, we write $x \sim y$ if x and y can be joined by a path in X. By Proposition 2.2 and Corollary 2.5, the inclusion map $X \sim Y \to X$ induces an isomorphism $H_0^c(X \sim Y) \overset{\approx}{\to} H_0^c(X)$. This means:

(a) For each point x in X, $x \sim y$ for some y in $X \sim Y$, and

(b) If $x, y \in X \sim Y$ and $x \sim y$, then x and y can be joined by a path in $X \sim Y$.

Let us suppose that $X \sim Y$ is disconnected. Then $X \sim Y = A_0 \cup B_0$, where $\bar{A}_0 \cap B_0 = A_0 \cap \bar{B}_0 = \emptyset$, $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. If $x \in A_0$ and $y \in B_0$, then it is not true that $x \sim y$. For otherwise, by (b), x and y can be joined by a path in $A_0 \cup B_0$. Let $A = \{x\colon x \sim y$ for some y in $A_0\}$ and $B = \{x\colon x \sim y$ for some y in $B_0\}$. Then, by (a), $X = A \cup B$. Clearly $A_0 \subset A$, $B_0 \subset B$ and $A \cap B = \emptyset$. Since X is connected, either $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. We may assume that $\bar{A} \cap B \neq \emptyset$.

Now each path that begins at a point in A (or B) is contained in A (or B). Therefore A and B are stable under deformation. By Lemma 2.8 each point that is stable in A is also stable in X. Consequently A_0 contains the set of all points that are stable in A. Since $\dim_c A \leqslant \dim_c X < \infty$, it follows from Theorem 2.9 that A_0 is dense in A, i.e. $\bar{A}_0 = \bar{A}$. From $\bar{A} \cap B_0 = \bar{A}_0 \cap B_0 = \emptyset$, we see that $\bar{A} \cap B \subset B \sim B_0 \subset Y$. Since $\dim_c \bar{A} \cap B \leqslant \dim_c X < \infty$ and $\bar{A} \cap B \neq \emptyset$, Theorem 2.9 implies that there is a point x_0 in $\bar{A} \cap B$ that is stable in $\bar{A} \cap B$. Since A and B are stable under deformation, $\bar{A} \cap B$ is also stable under deformation, and consequently the point x_0 is stable in X by Lemma 2.8. This contradicts the hypothesis that Y consists of unstable points. The proof is therefore complete.

We can prove the following corollary in the same way as Corollary 2.7 is proved.

2.11. Corollary. *Let X be a hereditarily normal, connected, locally compact topological space such that $\dim_c X < \infty$, and let L be the set of all unstable points in X. Then $X \sim L$ is connected.*

Final remark. In Theorem 2.10, the condition $\dim_c X < \infty$ is essential, because, in the Hilbert cube, each point is unstable. However we don't know whether or not the locally compactness of X is essential for the validity of Theorem 2.10.

Added in proof. In Corollary 2.11, the condition that X be hereditarily normal can be dropped, because if a subset L without interior of a connected space X disconnects X then some closed (in X) subset of L disconnects X.

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