The relationship between two weak forms of the axiom of choice

by

Paul E. Howard, (Ipsilanti, Mich.) **Herman Rubin,** and **Jean E. Rubin** (Lafayette, Ind.)

Abstract. Let P and Q be the following propositions:

P: For every set $x \neq \emptyset$, if R is a transitive relation on x such that every R-anti-symmetrically ordered subset has an R-upper bound, then x has an R-maximal element.

Q: There is a choice function on every well-ordered, non-empty set of non-empty sets.

We show first that $P \to Q$. Secondly we show by means of a Fraenkel-Mostowski model that the implication $Q \to P$ does not hold. And finally we derive some consequences of Q. All theorems are in ZF without the axiom of choice (AC).

1. Introduction. Let P and Q be the following propositions:

P: *For every set $x \neq \emptyset$, if R is a transitive relation on x such that every R-anti-symmetrically ordered subset has an R-upper bound, then x has an R-maximal element.*

Q: *There is a choice function on every well-ordered, non-empty set of non-empty sets.*

It is clear that Zorn's Lemma implies P and although it will not be used it is interesting to note that P is equivalent to the following proposition.

P′: *For every set $x \neq \emptyset$, if R is a transitive relation on x such that every R-anti-symmetrically ordered subset has an R-upper bound, then x has an R-upper bound.*

2. The proof that $P \to Q$.

THEOREM 1. $P \to Q$.

Proof. Let x be a non-empty, well-ordered set of non-empty sets and let y be the set of all choice functions on initial segments of x. Define the relation R on y by

$$fRg \quad \text{if and only if} \quad D(f) \subseteq D(g).$$

(For any function h, $D(h)$ is the domain of h.) R is transitive and since x is well-ordered any subset of y which is R-anti-symmetrically ordered is actually well-ordered by R.

Suppose y' is an R-anti-symmetrically ordered subset of y. We choose an upper bound f for y' as follows:

$$D(f) = \bigcup_{g \in y'} D(g)$$

and for each $u \in D(f)$, $f(u) = g(u)$ where g is the R-first function in y' which has u in its domain.

If we assume property P we can conclude that y has an R-maximal element g. And g is clearly a choice function on x. Hence property Q holds.

3. The Fraenkel-Mostowski model. We now construct a Fraenkel-Mostowski model of ZFU (Zermelo-Fraenkel set theory weakened to permit the existence of urelements) in which Q holds and P fails.

Suppose $\mathfrak{m} = \langle V, \epsilon, \emptyset \rangle$ is a model of ZFU in which AC holds and which has a countable set U of urelements. We construct a Fraenkel-Mostowski model as follows (The construction takes place in $\mathfrak{m}$.):

Let $\leqslant$ be a dense linear ordering of U without first or last element and let G be the group of all order preserving permutations of U. For each bounded subset A of U, let $G_A = \{\varphi \in G\colon (\forall a \in A)(\varphi(a) = a)\}$.

Each $\varphi \in G$ can be extended to an automorphism φ^* of $\mathfrak{m}$ by ε-induction as follows:

$$\varphi^*(a) = \varphi(a) \quad \text{for} \quad a \in U,$$
$$\varphi^*(\emptyset) = \emptyset,$$
$$\varphi^*(x) = \{\varphi^*(y)\colon y \in x\},$$

and since $(\varphi^{-1})^* = (\varphi^*)^{-1}$ and $(\varphi\psi)^* = \varphi^*\psi^*$ for all $\varphi, \psi \in G$, we can identify φ and φ^*.

For each x in V define

$$S(x) \leftrightarrow (\exists A \subseteq U)\big(A \text{ bounded and } (\forall \varphi \in G_A)(\varphi(x) = x)\big)$$

and

$$W(x) \leftrightarrow \big(S(x) \wedge (\forall y \in \mathrm{TC}(x))(S(y))\big),$$

where $\mathrm{TC}(x)$ denotes the transitive closure of x. Finally let $V' = \{x \in V\colon W(x)\}$ and let $\mathfrak{m}' = \langle V', \epsilon, \emptyset \rangle$. Let $\mathfrak{F}$ be the filter generated by $\{G_A\colon A$ is a bounded subset of $U\}$ then it is easy to verify that

1. $a \in U \to \{\varphi \in G\colon \varphi(a) = a\} \in \mathfrak{F}$ and
2. $H \in \mathfrak{F}$ and $\varphi \in G \to \varphi H \varphi^{-1} \in \mathfrak{F}$.

Hence the proofs in [2] show that $\mathfrak{m}'$ is a model of ZFU.

THEOREM 2. Q *holds in* $\mathfrak{m}'$.

Proof. Suppose x is a well-ordered set of non-empty sets (in $\mathfrak{m}'$), $\lhd$ is a well-ordering of x in the model and A is a bounded subset of U such that $\varphi \in G_A \to \varphi(\lhd) = \lhd$. One can easily show that for every $y \in x$ and $\varphi \in G_A$, $\varphi(y) = y$.

Let $A' = [v, v'] = \{u \in U\colon v \leqslant u \leqslant v'\}$ be an interval such that $A \subseteq [v, v']$ and there are two urelements w and w' such that $v < w < w' < v'$ and

$$[v, w] \cap A = \emptyset = [w', v'] \cap A.$$

For each $y \in x$ choose $t(y) \in y$. (The function t may not be in the model $\mathfrak{m}'$.) Suppose, for each $y \in x$, that $A_{t(y)}$ is a bounded subset of U such that

$$\varphi \in G_{A_{t(y)}} \to \varphi(t(y)) = t(y).$$

Since $\leqslant$ is a dense linear ordering and $A_{t(y)}$ is bounded, there is a permutation $\psi_{t(y)} \in G_A$ such that $\psi_{t(y)}(A_{t(y)}) \subseteq A'$.

Define $F\colon x \to \bigcup x$ by $F(y) = \psi_{t(y)}(t(y))$. It now remains to show two things: First that F is in the model $\mathfrak{m}'$ and second that F is a choice function on x.

To show that F is in $\mathfrak{m}'$ it suffices to show that every $\varphi \in G_{A'}$, fixes the set of ordered pairs $\{\langle y, \psi_{t(y)}(t(y))\rangle\colon y \in x\} = F$. So suppose $y \in x$ and $\varphi \in G_{A'}$. Since $A \subseteq A'$, $G_{A'} \subseteq G_A$, hence $\varphi(y) = y$. Now we note that $\psi_{t(y)}^{-1}\varphi\psi_{t(y)}(v) = v$ for every $v \in A_{t(y)}$, hence $\psi_{t(y)}^{-1}\varphi\psi_{t(y)}(t(y)) = t(y)$ and so $\varphi\psi_{t(y)}(t(y)) = \psi_{t(y)}(t(y))$. And combining this with the previous result we get

$$\varphi\big(\langle y, \psi_{t(y)}(t(y))\rangle\big) = \langle y, \psi_{t(y)}(t(y))\rangle.$$

Therefore φ fixes F.

F is a choice function because $t(y) \in y$ and hence since $\psi_{t(y)}(y) = y$, $F(y) = \psi_{t(y)}(t(y)) \in y$.

THEOREM 3. P *fails in* $\mathfrak{m}'$.

Proof. Choose $u_0 \in U$ and let x be the set of ordered pairs

$$\{\langle a, b\rangle\colon u_0 < a < b \;\&\; a, b \in U\}.$$

Define the relation R on x by $\langle a, b\rangle R \langle a', b'\rangle$ if $a \leqslant a'$. It is clear that R is transitive, R is in $\mathfrak{m}'$ and x has no maximal element.

Suppose y is an R-anti-symmetrically ordered subset of x in $\mathfrak{m}'$, then

1. $\langle a, b\rangle$ and $\langle a, b'\rangle \in y$ implies $b = b'$ since $\langle a, b\rangle R \langle a, b'\rangle$ and $\langle a, b'\rangle R \langle a, b\rangle$.

2. $\{b\colon (\exists a)(\langle a, b\rangle \in y)\}$ is contained in some interval $[c, d]$ of U. For suppose A is a bounded subset of U such that $\varphi \in G_A \to \varphi(y) = y$. We may assume without loss of generality that A is an interval $[c, d]$ of U. If for some $\langle a, b\rangle \in y$, $b \notin [c, d] = A$, there is a $\psi \in G_A$ such that $\psi(b) \neq b$ and $\psi(a) = a$. Then since $\psi(y) = y$, we have $\langle \psi(a), \psi(b)\rangle = \langle a, \psi(b)\rangle \in y$ contradicting 1.

3. By 2, we get $\{a\colon (\exists b)(\langle a, b\rangle \in y \text{ or } \langle b, a\rangle \in y)\}$ is contained in some bounded interval $[c, d]$ of U. Hence y has an upper bound $\langle a', b'\rangle$ where $d < a' < b'$. Therefore P fails in $\mathfrak{m}'$.

The model $\mathfrak{m}'$ is similar to the model constructed by Mostowski in [3] and used by Halpern in [1] to show the independence of the Boolean prime ideal theorem from the axiom of choice. In $\mathfrak{m}'$ however, the linear ordering theorem (and therefore the Boolean prime ideal theorem) is false. The set of all countably infinite subsets of the urelements is an example of a set which cannot be linearly ordered in $\mathfrak{m}'$.

4. Consequences of Q. Suppose x is an infinite set such that $|x| = m$. Let $\aleph(m) = |\{a: a \precsim x\}|$ (we use lower case greek letters for ordinal number variables.) Then $\aleph(m)$ is Hartogs' aleph and has the property that

(*) $\aleph(m)$ is the smallest aleph which is not $\leqslant m$.

Let Q_1-Q_4 be the following four statements.

Q_1: *For each infinite cardinal m there is an ordinal a such that* $\aleph(m) = \aleph_{a+1}$.

Q_2: *For each infinite cardinal m,* $\aleph_0 \leqslant m$.

Q_3: *The union of a well-ordered set of well-orderable sets is well-orderable.*

Q_4: *The union of a countable set of countable sets is countable.*

We shall show that $Q \to Q_1 \to Q_2$, $Q \to Q_3$, and $Q \to Q_4$. But first, as a preliminary lemma, we shall show there is a form of Q_3 which is provable in ZF without AC.

LEMMA 4. *If for each $a < \beta$ the relation R_a well-orders the set x_a, then $\bigcup_{a<\beta} x_a$ can be well-ordered.*

Proof. Let $y = \bigcup_{a<\beta} x_a$. Define a relation R on y as follows: let $u_1, u_2 \in y$ and for $i = 1, 2$ let $a_i =$ smallest element of $\{a: u_i \in x_a\}$, then

$$u_1 R u_2 \leftrightarrow [a_1 < a_2 \text{ or } (a_1 = a_2 = a \text{ and } u_1 R_a u_2)].$$

It is clear R well-orders y.

THEOREM 5. $Q \to Q_1$.

Proof. Suppose x is an infinite set such that $|x| = m$. Let

$$y = \{a: a \precsim x \text{ and } (a \in \omega \text{ or } a \text{ is an initial ordinal})\}.$$

For each $a \in y$, let z_a be the set of all 1-1 functions mapping a into x.

It follows from Q that there is a choice function F on $\{z_a: a \in y\}$. For each $a \in y$, let $F(z_a) = f_a$ and let $u = \bigcup_{a \in y} f_a'' a$. Since, by hypothesis, x is infinite, u must be infinite, and, it follows from Lemma 4 that u can be well-ordered. Therefore, there is an ordinal a such that $|u| = \aleph_a$. Consequently, the initial ordinal ω_a is the largest element of y. This implies that ω_a is the largest initial ordinal β such that $\beta \precsim x$. Thus, it follows from (*) that $\aleph(m) = \aleph_{a+1}$.

COROLLARY 6. $Q_1 \to Q_2$.

Proof. Theorem 5 and (*).

THEOREM 7. $Q \to Q_3$.

Proof. Let $x = \{y_a: a < \beta\}$, where for each $a < \beta$, y_a can be well-ordered. For each $a < \beta$, let $z_a = \{R: R \text{ well-orders } y_a\}$. Q implies there is a choice function F on $\{z_a: a < \beta\}$. For each $a < \beta$, $F(z_a)$ well-orders y_a. Therefore, it follows from Lemma 4 that $\bigcup x$ can be well-ordered.

THEOREM 8. $Q \to Q_4$.

Proof. The proof is similar to the proof of Theorem 7, but here $x = \{y_a: a < \omega\}$ and each y_a is countable. Let z_a be the set of all relations R such that R well-orders y_a and the order type of $\langle y_a, R\rangle$ is ω. Q implies there is a choice function on $\{z_a: a < \omega\}$. Now use the Cantor diagonal procedure to count the elements of $\bigcup x$, omitting repetitions.

It seems at first that $Q_3 \to Q_4$, but with closer inspection one sees that the proof is not obvious. In fact, we do not see how to prove it. It is also not known to us whether proposition P above imples the axiom of choice.

Added in proof. Ulrich Felgner (Heidelberg) in an unpublished paper, "Abzählbarkeit und Wohlordenbarkeit", has constructed a Cohen model in which Q_3 holds but Q_4 does not, thereby showing that it is not the case that $Q_3 \to Q_4$ in ZF without AC.

References

[1] J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. 55 (1964), pp. 57–66.

[2] A. Mostowski, *The axiom of choice for finite sets*, ibidem 33 (1945), pp. 137–168.

[3] — *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, ibidem 32 (1939), pp. 201–252.

[4] H. Rubin and J. Rubin, *A weakened form of Zorn's lemma and some of its consequences*, Notices Amer. Math. Soc. 19 (1972), p. A-456.

[5] — — *Equivalents of the Axiom of Choice*, Amsterdam 1963.

Reçu par la Rédaction le 2. 8. 1972