Proof. The proposition follows immediately from Propositions 7, 6 and 8 q.e.d.

Remark. Since every space from $\sqrt{MC}$ is a locally compact separable metric space, the theorem follows immediately from Proposition 9.

References


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Limit mappings and projections of inverse systems

by

E. Puzio (Warszawa)

Abstract. Let $S = (X_s, \Pi_s, \Sigma)$ and $S' = (Y_s, \Pi'_s, \Sigma')$ be inverse systems and $(\varphi, f_s)$ be a mapping of $S$ into $S'$. For some classes $K$ of mappings we discuss the problem when $f_s \in K$ implies $\lim (\varphi, f_s) \in K$ and when $\Pi_s \in K$ implies $\Pi_s \in K$.

In this paper we are concerned with limits of inverse systems, their projections and limit mappings induced by mappings of inverse systems. More precisely, we show how the projections depend on bonding mappings and how the limit mapping depends on the mapping of systems inducing it.

To begin with, we recall some definitions and simple facts about inverse systems and give two auxiliary examples. Our terminology and notations are consistent with those used in [3]; except that by a mapping of an inverse system $S = (X_s, \Pi_s, \Sigma)$ into $S' = (Y_s, \Pi'_s, \Sigma')$ we understand a system $(\varphi, f_s)$ satisfying besides the usual commutativity conditions also the condition that $\varphi(\Sigma)$ is cofinal in $\Sigma$.

The diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow s \downarrow a \\
T \xrightarrow{h} Z
\end{array}
\]

(1)

is said to be exact (see [8], p. 19) if it is commutative and the following implication is true:

if $h(y) = k(t)$, then $g^{-1}(t) \cap f^{-1}(y) \neq \emptyset$.

The diagram (1) is exact (see p. 19 of [8]) if and only if

(2) $fg^{-1}(B) = h^{-1}k(B)$ for $B \subset T$

or, equivalently,

(2') $g^{-1}(A) = k^{-1}h(A)$ for $A \subset Y$.

Obviously, the diagram (1) is commutative if and only if

(3) $fg^{-1}(B) \subset h^{-1}k(B)$ for $B \subset T$
or, equivalently,

\[ g^{-1}(A) \subseteq k^{-1}(A) \quad \text{for} \quad A \subseteq Y. \]

A mapping \( \{ \phi_i, f_i \} \) of the system \( S = (X, P, \Sigma) \) into the system \( S' = (Y, P', \Sigma') \) is said to be exact if the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_i} & Y' \\
\downarrow{\pi_j} & & \downarrow{\pi'_j} \\
X' & \xrightarrow{f'_i} & Y'
\end{array}
\]

is exact for each pair \( \alpha' \in \Sigma \) with \( \alpha' < \alpha \).

It is said to be limit-exact if the diagram

\[
\begin{array}{ccc}
\lim S & \xrightarrow{f} & \lim S' \\
\downarrow{\pi_j} & & \downarrow{\pi'_j} \\
\lim X & \xrightarrow{f'_i} & \lim Y'
\end{array}
\]

is exact for every \( \alpha' \in \Sigma' \).

It is easy to see that if \( \Sigma \) and \( \Sigma' \) are the sets of all natural numbers, then an exact mapping of systems is limit-exact.

Recall that the limit of \( S = (X, P, \Sigma) \) is non-empty if the spaces \( X \) are compact for every \( \alpha \in \Sigma \) (see [3], Theorem 3.2.10). In the case of countable \( \Sigma \), the limit of a system of non-empty spaces is non-empty if the bonding mappings \( P \) are onto, but it may be empty if \( P \) are not onto. In the case of non-countable \( \Sigma \) the limit of an inverse system of non-empty spaces with bonding mappings onto may be empty:

**Example 1** ([5]), simplified in [6]. Let \( \Sigma \) be the set of all ordinal numbers less than \( \alpha \), and \( \alpha \) the real line. For every \( \alpha < \alpha \), we take \( W = (y : y < \alpha) \) with the ordinal topology (see [3], Example 3.5.1) and let \( X \) be the set of all homeomorphic embeddings \( f : W \rightarrow R \) with the discrete topology. For \( \beta < \alpha \) we define mappings \( P \) : \( X \rightarrow X \) as restrictions, i.e., we put \( P(f) = f|W \) for \( f \in X \). One shows that the limit of the inverse system \( S = (X, P, \Sigma) \) is empty.

Now we shall describe a similar inverse system of countable spaces:

**Example 2** ([7]). Let \( S = (X, P, \Sigma) \) be the inverse system from Example 1. For every \( \alpha \in \Sigma \) we define by transfinite induction a countable set \( Y \subseteq X \) consisting of strictly increasing embeddings of \( W \) into \( R \), satisfying the following condition:

\[ \text{for all } \beta < \alpha < \alpha, \text{ a positive integer } n \text{ and } f \in Y \text{ there exists an } f_n \in Y \text{ such that } P(f_n) = f_n \text{ and } f_n(\alpha) - f_n(\beta) < 1/2^n. \]

Let \( Y \) be an arbitrary one-point subspace of \( X \). Suppose that the sets \( Y \) satisfying condition (6) are defined for \( \alpha < \alpha \); let us consider two cases: when \( \alpha = \alpha + 1 \) and when \( \alpha \) is a limit ordinal number.

If \( \alpha = \alpha + 1 \), then for every \( f \in Y \) and \( n \in N \) we take an extension \( f_n \) of \( f \) over \( W \) such that \( f_n(\alpha) = f_n(\alpha) + 1/2^n \). Clearly, \( f_n \in X \) and \( f_n(\alpha) - f_n(\beta) < 1/2^n \). Let \( Y' \) be the set of all functions constructed in this way for every \( f \in Y \) and \( n \in N \), taken with the discrete topology. It is easy to see that \( Y' \) has the required properties.

Now, let \( \alpha \) be a limit ordinal and \( \alpha \) an arbitrary ordinal less than \( \alpha \). For \( f \in Y \) and \( n \in N \) choose an increasing sequence of ordinals \( \alpha = \alpha_1 < \alpha_2 < \alpha_3 \ldots \) convergent to \( \alpha \). By the inductive assumption there exists a sequence \( f_n = f_{n_1} \circ f_{n_2} \circ f_{n_3} \ldots \) such that \( f_n \in Y ' \), \( f_n(\alpha) = f_n(\alpha) + 1/2^n \), \( f_n(\alpha) = f_n(\alpha) + 1/2^n \) for every \( m \in N \). Let us consider the combination \( f' : (y : y < \alpha) \rightarrow R \) of \( \{ f_n \} \) and its extension \( f \) over \( W \) such that \( f(\alpha) = \sup f(y) \). The function \( f \) is one-to-one (being strictly increasing) and continuous, it is also closed as a continuous mapping of a compact space into a Hausdorff space. Thus, \( f \in X \) and

\[ f(\alpha) - f(\beta) < \sum_{n=1}^{\infty} \frac{1}{2^n}. \]

Let \( Y' \) be the set of all functions constructed in this way for every \( \alpha < \alpha \), \( f \in Y \) and \( n \in N \), taken with the discrete topology. Since the set \( \{ \alpha \in \Sigma : \alpha < \alpha \} \) and all \( Y \) are countable, the set \( Y' \) is also countable. Moreover, \( Y' \) consists of strictly increasing functions and satisfies the condition (6).

Let us denote \( Y' \), \( Y' \) be the inverse system \( S = (X, P, \Sigma) \) is an inverse system such that \( Y \) is a countable discrete space for each \( \alpha \), \( X \) is onto for \( \beta \neq \alpha \) and \( \Sigma' = \varnothing \).

We shall now examine limit mappings. In section 3 of [4] there is an example showing that \( f = \lim f \) need not be onto, when each \( f \) is onto even if both of the systems are countable with the bonding mappings onto. However, it can easily be verified that if the mapping \( \{ f, f \} \) is limit-exact and each \( f \) is onto, then \( f = \lim f \) is onto.

Hence, if the mapping \( \{ f, f \} \) of countable systems is exact and each \( f \) is onto, then the limit mapping is onto.

Let us consider the projection \( P \) : \( \lim X, P, \Sigma) \rightarrow X \). If the inverse system \( S \) is countable and \( P \) is onto, then the projection \( P \) is onto, but it is not necessarily onto, if \( S \) is uncountable, this follows at once from Example 1.

**Theorem 1.** For every inverse system \( S = (X, P, \Sigma) \) and \( \alpha < \Sigma \) there exist a system \( S' \) and a mapping \( \{ f, f \} \) of \( S \), where \( f, f \) are bonding mappings...
of $S$, and a homeomorphism $h: \lim S' \to X_\infty$ such that $\Pi_{\infty} = h \lim \{\varphi, f_\alpha\}$. Moreover, we can assume that $\varphi(S' \to \Sigma) = \Sigma$.

Proof. Let $S = (Y_\infty, \Pi_{\infty}, \Sigma)$, where $\Sigma = \{\alpha \in \Sigma: \alpha \equiv \alpha_0\}$, $Y_\infty = X_\infty$, for every $\alpha' \in \Sigma'$ and $\Pi_{\infty} = \id_{X_\infty} \to Y_\infty$ for $\alpha' \equiv \alpha_0$. Let $\varphi: \Sigma' \to \Sigma$ be the natural injection and let $f_\alpha = \Pi_{\alpha}: X_\infty \to X_\infty$ for each $\alpha \equiv \alpha_0$. The diagram

$$
\begin{array}{ccc}
X_\infty & \xrightarrow{\varphi} & \Sigma' \\
\downarrow_{\Pi_{\infty}} & & \downarrow_{\varphi} \\
X_\infty & \xrightarrow{\id_{X_\infty}} & Y_\infty
\end{array}
$$

commutes for $\alpha \equiv \alpha_0$, and so $\{\varphi, f_\alpha\}$ is a mapping of $S$ into $S'$. The space $\lim S'$ is the diagonal of the product $\lim \{Y_\alpha, \Pi_{\alpha}, \Sigma\}$.

We state without proof the following well-known theorems:

**Theorem 2.** For an arbitrary subspace $A$ of the limit $X = \lim \{X_\alpha, \Pi_{\alpha}, \Sigma\}$ the family $S_A = (\bar{A}, \Pi_{\bar{A}}, \varphi(\Sigma))$, where $A = \Pi_{\bar{A}}(A)$, is an inverse system and $\lim S_A = \bar{A} \subset X$.

**Corollary.** A closed subspace $A$ of the limit $X = \lim \{X_\alpha, \Pi_{\alpha}, \Sigma\}$ is the limit of the inverse system $S_A = (\bar{A}, \Pi_{\bar{A}}, \varphi(\Sigma))$, where $A = \Pi_{\bar{A}}(A)$, is a closed subspace of $X$.

**Theorem 3.** If the mapping $\{\varphi, f_\alpha\}$ of the system $S = (X_\alpha, \Pi_{\alpha}, \Sigma)$ into $S' = (Y_\alpha', \Pi_{\alpha}', \Sigma')$ satisfies the condition $\varphi(\Sigma) = \Sigma'$, then there exists a homeomorphic embedding $h: \lim S' \to \lim S$, where $Z_\infty = X_{\infty}'$ onto a closed subspace of $P Z_\infty$ such that $\lim \{\varphi, f_\alpha\} = \{P, f_\alpha\} h$.

We now pass to the main subject of this paper, i.e., to the determination when (under what conditions for $\varphi, f_\alpha$ and for the inverse systems) for the given class of mappings $\mathcal{R}$ the limit mapping $h(\varphi, f_\alpha)$ belongs to $\mathcal{R}$, and (which is a special case by Theorem 1) when projections $\Pi_{\alpha}$ belong to $\mathcal{R}$.

We shall consider the following classes of mappings: 1) open, 2) closed and perfect, 3) quotient and hereditarily quotient, 4) monotone.

1. **Open mappings.** In [4], K. R. Gentry has given an example of a mapping $\{\id_{\infty}, f_\lambda\}$ of an inverse system $S = (X_\infty, \Pi_{\infty}, N)$ into a system $S' = (X_\infty', \Pi_{\infty}', N')$ such that $f_\lambda$ are open and onto and the limit mapping $f = \lim \{\id_{\infty}, f_\lambda\}$ is not open and onto. In fact in that example $\Pi_{\infty}'$ is closed and open and $\Pi_{\infty}$ is neither closed nor open. By a small modification of Gentry's example we can obtain similar systems for which $f$ is an onto mapping but is not quotient (1).

For this purpose it is sufficient to add to the space $X_\infty$ for $n = 1, 2, 3, ...$ an isolated point $y(y)$ for each $y = \{y_n\} \in \lim S'/\lim S$, and to define $\Pi_{\infty} m(y) = y(y)$, $f_\lambda m(y) = y_n$ for $m, n \in N$.

In [4] it is shown that for an exact mapping $\{\id, f_\lambda\}$ of a system $S = (X_\infty, \Pi_{\infty}, N)$ into $S' = (Y_\infty', \Pi_{\infty}', N')$ if each $f_\lambda$ is open, then so is $\lim \{\id, f_\lambda\}$. For uncountable systems an analogous theorem is not true. This follows from Example 4 given below.

The following theorem is a generalization of Gentry's result:

**Theorem 4.** Let $S = (X_\alpha, \Pi_{\alpha}, \Sigma)$ and $S' = (Y_\alpha', \Pi_{\alpha}', \Sigma')$ be two inverse systems and let $\{\varphi, f_\alpha\}$ be a limit-exact mapping of $S$ into $S'$ such that each $f_\alpha$ is open. Then the limit mapping is open.

Proof. Let $U = \Pi_{\infty}^{-1}(U')$, where $U'$ is open in $X_{\infty}'$. Let $\{\varphi, f_\alpha\}$ be an element of the base of $\lim S$. As the diagram (5) is exact, we have $f(U) = \Pi_{\infty}^{-1}(U')$. It follows that $f(U)$ is open in $\lim S'$ and that $f$ is open.

**Theorem 5.** Let $S = (X_\alpha, \Pi_{\infty}, N)$ be an inverse system such that each $\Pi_{\infty}$ is open and onto. Then the projection $\Pi_{\infty}$ is open.

Proof. It is easily seen that the mapping $\{\varphi, f_\alpha\}$ of $S$ into $S'$ from Theorem 1 is exact. Thus, it follows from the above theorem that the projection $\Pi_{\infty}$ is open.

All the assumptions of Theorem 5 are essential. The following example shows that Theorem 5 does not hold for uncountable systems.

**Example 3.** Let $S = (X_\alpha, \Pi_{\alpha}, \Sigma)$ be the system from Example 1. For each $\alpha \in \Sigma$ let $Y_\alpha$ be the hedgehog with $m$ pricks (see Example 4.1.3 in [3]), where $m = X_{\infty}$; that is $Y_\alpha = (X_{\infty} \times [0, 1]) \cup T$, where

$$
[(f, t) \in (f', t')] = \begin{cases}
(t - t') & \text{if } f = f', \\
(t + t') & \text{if } f \neq f'.
\end{cases}
$$

with the topology induced by the metric

$$
\rho([[f, t]], [[f', t']]) = \begin{cases}
|t - t'| & \text{if } f = f', \\
|t + t'| & \text{if } f \neq f'.
\end{cases}
$$

Let the mapping $\Pi_{\infty}$ be defined for $\alpha \equiv \beta$ by the formula:

$$
\Pi_{\infty}([f, t]) = \{[[f, f], t]\} \text{ for } ([f, t]) \in Y_\infty.
$$

As $\Pi_{\infty}$ is onto, so is $\Pi_{\infty}'$; moreover, it is easy to see that $\Pi_{\infty}$ is open. At the limit of the inverse system $S' = (Y_\infty', \Pi_{\infty}', \Sigma)$ is a one-point space, the projection $\Pi_{\infty}$ is not open.

**Example 4.** Using Example 3 and Theorem 1 let us consider the mapping $\{\varphi, f_\lambda\}$ such that $\Pi_{\infty}$ is the composition of $\lim \{\varphi, f_\lambda\}$ and of a homeomorphism. Then $\{\varphi, f_\lambda\}$ is exact but $\lim \{\varphi, f_\lambda\}$ is not open.
2. Closed and perfect mappings. The following theorem was proved in [4] in the case of countable systems.

**Theorem 6.** Let \( S = (X, \Pi^0, \Sigma) \) and \( S' = (X', \Pi'^0, \Sigma') \) be two inverse systems, where all \( X, \Sigma \) are \( T^0 \)-spaces, and let \((\varphi, f_{\varphi})\) be a mapping of \( S \) into \( S' \) such that \( f_{\varphi} \) is perfect for each \( \varphi' \in \Sigma' \). Then \( f = \lim_{\varphi}(\varphi, f_{\varphi}) \) is perfect.

The proof is exactly the same as in [4] for countable systems.

The above-mentioned example of K. R. Gentry shows that an analogous theorem for closed mappings is not true, even if both systems are countable. A slight modification of that example shows that the assumption of \((\varphi, f_{\varphi})\) being exact does not help.

The following two theorems are concerned with the projection \( \Pi^0 \).

**Theorem 7.** ([9], Theorem 4.3, [10], Lemma 2.4). Let \( S = (X, \Pi^0, \Sigma) \) be an inverse system of \( T^0 \)-spaces such that the bonding mappings \( \Pi^0 \) are perfect. Then the projection \( \Pi^0 \) is perfect.

**Proof.** This follows at once from Theorem 1 and Theorem 6.

**Theorem 8.** Let \( S = (X, \Pi^0, \Sigma) \) be an inverse system such that the bonding mappings \( \Pi^0 \) are closed. Then the projection \( \Pi^0 \) is closed.

**Proof.** This was proved by P. Zener in [10] under the additional assumption that \( \Pi^0 \) are onto, which is not used in the proof.

Theorem 8 does not hold for unordered systems, even if bonding mappings are onto. This is shown by the following example:

**Example 5.** Let \( S = (X, \Pi^0, \Sigma) \) be the inverse system from Example 2. We take \( X = [0, a) \cap (\bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\})) \), where \( X_{\gamma} = X \times [0, \gamma] \) and the discrete topology. Since the set \( \Gamma = \{\gamma: \gamma < \alpha\} \) and all \( X_{\gamma} \) are countable, \( X_{\gamma} \) are Lindelöf spaces.

For each \( \alpha \in \Sigma \) let \( X_{\alpha} = (X_{\alpha} \times \{\alpha\}) \cap (\bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\})) \), where \( X_{\alpha} = (X_{\alpha} \times [0, \alpha]) \) and the discrete topology. Since the set \( \Gamma = \{\gamma: \gamma < \alpha\} \) and all \( X_{\gamma} \) are countable, all \( X_{\gamma} \) are Lindelöf spaces.

The mapping \( \Pi^0: X \to X_{\alpha} \) is defined for \( \beta < \alpha \) in the following way:

- If \((y, \beta) \in X \times X_{\alpha} \), then \( \Pi^0((y, \beta)) = (y, \beta) \) for \( \beta \leq \alpha \) and \( \Pi^0((y, \beta)) = (y, \alpha) \times \{\gamma\} \) for \( \beta > \alpha \).
- If \((y, \beta) \in X \times (0, \alpha) \), then \( \Pi^0((y, \beta)) = (y, \beta) \times \gamma \).

It is easy to see that \( \Pi^0 \) is a projection onto the set \( \bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\}) \).

The mapping \( \Pi^0 \) is continuous on the discrete set \( \bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\}) \) and \( \Pi^0 \) is also continuous on that set. Since \( X_{\alpha} \) is the union of those open-and-closed disjoint sets, the mapping \( \Pi^0 \) is continuous.

We shall show that \( \Pi^0 \) is closed. Take a closed set \( F \subset X_{\alpha} \). Then \( \Pi^0(F) = \bigcup_{\gamma \in \mathbb{C}} (F \times \{\gamma\}) \cup A \), where \( A \subset X_{\alpha} \) and \( F \times \{\gamma\} \) is empty or is an image of \( \bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\}) \) by the bonding mappings \( \Pi^0 \). We have \( \Pi^0(F) = \bigcup_{\gamma \in \mathbb{C}} (F \times \{\gamma\}) \cup A \), where \( \Pi^0(F) \) is empty or is an image of the bonding mappings \( \Pi^0 \). We see that \( \Pi^0(F) \) is closed. As \( \Pi^0 \) is onto, the mapping \( \Pi^0 \) is also onto.

Now let \( X = (\lim_{\alpha \in \mathbb{C}} X_{\alpha}, \Pi^0, \Sigma) \). We shall prove that for each \( \alpha \in X \) we have

\[
(\{y_{\alpha} \times \gamma\} \cap (\bigcup_{\gamma \in \mathbb{C}} (X_{\gamma} \times \{\gamma\})) \cup (\{y_{\alpha} \times \{\gamma\}\} \times \{\gamma\})) \subset (\Pi^0(F)) \cup A.
\]

For \((y_{\alpha}, \delta) \in (Y_{\alpha} \times (0, \alpha)) \) we consider two cases: \( \delta > \alpha \) and \( \delta \leq \alpha \). Then we choose \( y_{\alpha} \in (\Pi^0)^{-1}(y_{\alpha}) \) and define

\[
z_{\alpha} = \begin{cases} (y_{\alpha}, \delta) \times \{\gamma\} & \text{for } \delta > \alpha, \\ (\Pi^0(y_{\alpha}), \delta) & \text{for } \delta \leq \alpha. \end{cases}
\]

It is easy to see that \( z_{\alpha} \) is a thread such that \( \Pi^0(z_{\alpha}) = (y_{\alpha}, \delta) \) if \( \delta < \alpha \), then we define

\[
z_{\alpha} = \begin{cases} (\Pi^0(y_{\alpha}), \delta) & \text{for } \alpha > a, \\ (y_{\alpha}, \delta) \times \{\gamma\} & \text{for } \alpha < a. \end{cases}
\]

In this case also one can easily see that \( z_{\alpha} \) is a thread such that \( \Pi^0(z_{\alpha}) = (y_{\alpha}, \delta) \). On the other hand, if \( (x_{\alpha}, \delta) \times (\gamma) \) then for the thread \( z_{\alpha} \), where

\[
z_{\alpha} = \begin{cases} (x_{\alpha}, \delta) \times \{\gamma\} & \text{for } \delta > \alpha, \\ (\Pi^0(x_{\alpha}), \delta) \times \{\gamma\} & \text{for } \delta < \alpha, \end{cases}
\]

we have \( \Pi^0(z_{\alpha}) = (x_{\alpha}, \delta) \times \{\gamma\} \).

The subsystem \( S = (Y_{\alpha} \times (0, \alpha), (\Pi^0)^{-1}(y_{\alpha})) \) has the empty limit, because \( \lim_{\alpha \in \mathbb{C}} Y_{\alpha} = \emptyset \). Therefore, \( \Pi^0(X) = X_{\alpha} \times (0, \alpha) \) and, as this set is not closed in \( X_{\alpha} \), the mapping \( \Pi^0 \) is not closed.

Observe that the existence of an inverse system \( S = (X, \Pi^0, \Sigma) \) of \( T^0 \)-spaces with the bonding mappings closed and onto such that the projection \( \Pi^0: \lim_{S} X_{\alpha} \to X_{\alpha} \) is not closed leads to an inverse system of nonempty spaces with the bonding mappings onto and with the empty limit; so we can hardly expect a simple example of such a system.

Indeed, suppose that \( F \) is a closed subset of \( \lim_{S} F \) such that \( \Pi^0(F) \) is not closed in \( X_{\alpha} \). Let \( F_{\alpha} = \Pi^0(F) \) for each \( \alpha \in \Sigma \) and \( z_{\alpha} \in F_{\alpha} \cap \Pi^0(F_{\alpha}) = \emptyset \).
Since the mappings \( \Pi^*_m \) are closed, \( \Pi^*_m(F_x) = F_x \) for all \( x, \xi \); thus for each \( x, \xi \), the set \( (\Pi^*_m)^{-1}(x) \cap F_x \) is non-empty. The family \( S' = \{ A_x, (\Pi^*_m)^{-1}(x) \cap F_x \} \), where \( S' = \{ (\sigma \in \Xi : x, \xi < \sigma) \} \) and \( A_x = (\Pi^*_m)^{-1}(x) \cap F_x \) for \( \sigma \in S' \), is an inverse system of closed and non-empty subspaces of \( X_x \) with the bonding mappings onto. Moreover \( \lim S' = \emptyset \), since \( x_n \notin \Pi^*_m(F_x) \) and \( \lim S' \cap F_x = \emptyset \) by Theorem 2.

3. Quotient and hereditarily quotient mappings. Recall that a mapping \( f : X \rightarrow Y \) is said to be hereditarily quotient if \( f(\lim X) = \emptyset \) and for each \( A \subset Y \) the restriction \( f | f^{-1}(A) : f^{-1}(A) \rightarrow A \) is quotient. Note that the mapping \( f : X \rightarrow Y \) is hereditarily quotient if and only if, for each set \( U \) open in \( X \) and containing \( f^{-1}(y) \), the set \( \text{Int}(U) \) is a neighbourhood of \( y \) in \( Y \). All open mappings onto and closed mappings onto are hereditarily quotient.

A space \( X \) is said to be a Fréchet space if for each \( x \in A \subset X \) there exists a sequence \( \{ x_n \} \) such that \( x = \lim x_n \) and \( x_n \in A \). Every quotient mapping onto a Fréchet \( T_1 \)-space is hereditarily quotient (see [1] and [2]). Since each metric space is a Fréchet space, it follows that each quotient mapping onto a metric space is hereditarily quotient. From the modification of Gentre's example mentioned in § 1 it follows that the mapping \( f = \lim (y, f_x) \), where \( f_x \) is hereditarily quotient and onto, need not be quotient, even being onto.

However, we have the following

**Theorem 9.** Let \( S = \{ X_n, \Pi^*_m, N \} \) be an inverse system such that each \( \Pi^*_m \) is hereditarily quotient. Then the projection \( \Pi_n : \lim S \rightarrow X_n \) is hereditarily quotient for each \( n \in N \).

**Proof.** First we prove that \( \Pi_n \) is quotient. Let \( A \) be a subset of \( X_n \) such that \( \Pi^*_m(A) \) is open in \( X = \lim S \). Suppose that \( A \) is not open, i.e. there exists an \( x_n \in A \) such that \( x_n \notin \text{Fr} A \). Since \( \Pi^*_m(x_n) = X_{n+1} \), it follows that \( (\Pi^*_m)^{-1}(x_n) \subset \Pi^*_m \Pi^*_m \). If we had \( (\Pi^*_m)^{-1}(x_n) \subset \Pi^*_m \Pi^*_m \), then \( \Pi^*_m \) being hereditarily quotient, the set \( \text{Int}(\Pi^*_m(\Pi^*_m \Pi^*_m)) \) would be a neighbourhood of the point \( x_n \) contained in \( A \), which is impossible, because \( x_n \notin \text{Fr} A \). Thus there exists an \( x_n \in \text{Fr}(\Pi^*_m(\Pi^*_m \Pi^*_m)) \). This process may be continued to obtain a thread \( \{ x_n \} \) such that \( x_n \notin \text{Fr}(\Pi^*_m(\Pi^*_m \Pi^*_m)) \) for each \( m \geq n \). Evidently, \( x_m \in \Pi^*_m(\text{Fr}(\Pi^*_m \Pi^*_m \Pi^*_m)) \) for each \( m \in N \); thus since the set \( X = \Pi^*_m(\text{Fr}(\Pi^*_m \Pi^*_m \Pi^*_m)) \), which is in contradiction with \( x_n \in A \). It follows that \( \Pi_n \) is quotient. Now we show that \( \Pi_n \) is hereditarily quotient. For \( y_n \in X_n \) we have \( \Pi^*_m(y_n) = \lim \{ Y_n, \Pi^*_m(Y_n), N \} \), where \( Y_n = \begin{cases} \Pi^*_m(Y_n) & \text{for } m \leq n, \\ (\Pi^*_m)^{-1}(Y_n) & \text{for } m > n. \end{cases} \)

Since each mapping \( \Pi^*_m \mid Y_n \) for \( m \geq n \) is hereditarily quotient, from the first part of our proof it follows that the mapping \( \Pi_n \mid \Pi^*_m(\Pi^*_m(\Pi^*_m(\Pi^*_m(Y_n)))) : \Pi^*_m(N) \rightarrow Y_n \) is quotient.

In order to show that Theorem 9 does not hold for uncountable systems we shall modify the system \( S = \{ X_n, \Pi^*_m, N \} \) described in Example 5, where \( \Pi^*_m \) were closed and onto but \( \Pi^*_m \) was neither closed nor onto, in the following way. We fix a \( x_n \in X \) and add for all \( \xi > x_n \) one isolated point \( \xi(x) \) to the space \( X \) for each \( x \in X \). The projection \( \Pi_n : \lim S \rightarrow X_n \) need not be quotient if the bonding mappings of the system \( S \) are quotient.
where
\[ A_0 = \{0, 1\}, \]
\[ A_k = \bigcup_{n_1 = 1}^{\infty} \bigcup_{n_2 = 1}^{n_1} \cdots \bigcup_{n_k = 1}^{n_{k-1}} \left( \frac{1}{n_1} \times \frac{1}{n_2} \times \cdots \times \frac{1}{n_k} \times (0, 1) \right) \]
for \( k = 1, 2, 3, \ldots \),
\[ A = \left( \bigcup_{n_m \times n_N = n_1} \left( \frac{1}{n_m} \times (0, 1) \right) \right) / R, \]
and the equivalence relation \( R \) is defined by the formula:
\[ \left( \left( \frac{1}{n_m} \times t \right) R \left( \frac{1}{n_m} \times t' \right) \right) \iff \left( t = t' = 1 \right) \text{ or } \left( \frac{1}{n_m} = \frac{1}{n_M} \text{ and } t = t' \right) . \]
The topology in the set \( X_i \) is generated by the following neighbourhood system:

1°) The base at the point \( \frac{1}{n_1} \times \cdots \times \frac{1}{n_k} \epsilon A_{k-1} \) for \( k = 1, 2, 3, \ldots \) consists of all sets of the form
\[ \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_{k-1}} \times \left( \frac{1}{n_k}, \frac{1}{n_k} + \frac{1}{l} \right) \right) \cup B \cup C, \]
where \( \frac{1}{l} < \frac{1}{n_k} - \frac{1}{n_k + 1} \),
\[ (7) \quad B = \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_k} \times (0, \frac{1}{n_k}) \right) \cup \]
\[ \bigcup_{n_{k+1} > n_k} \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_{k+1}} \times \left( 0, \frac{1}{n_{k+1} + 1} \right) \right) \cup \]
\[ \bigcup_{n_{k+2} > n_{k+1}} \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_{k+2}} \times \left( \frac{1}{n_{k+2}}, \frac{1}{n_{k+2}} \right) \right) \times \]
\[ \cdots \times \left( 0, \frac{1}{n_{k+2} + 1} \right) \cup \]
\[ (8) \quad C = \bigcup_{n_m \times n_N = n_1} \left( \frac{1}{n_m} \times (0, \frac{1}{n_m}) \right), \]
\[ T = \left\{ \left( n_m \epsilon \mathbb{N}^*; \frac{1}{n_m} \times \left( \frac{1}{n_m} \right) \epsilon B \text{ for each } \right) \right\}_{i \geq k}, \]
and \( j_{n_1}, l \), \( n_k, n_{k+1}, n_{k+2}, n_{k+3}, n_{k+4}(n_{k+5}), n_{k+5}, n_{k+6}(n_{k+7}), n_{k+7}, \ldots \) are natural numbers.

2°) The base at the point \( \left( \frac{1}{n_1} \right) \times \cdots \times \left( \frac{1}{n_k} \right) \times (t) \epsilon A_k \) for \( \frac{1}{l+1} < t < \frac{1}{l} \), consists of all sets of the form
\[ \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_k} \times \left( \frac{1}{l+1}, t + \frac{1}{l} \right) \right), \]
where \( \frac{1}{l} < \min \left\{ \frac{1}{l+1}, \frac{1}{l}, \frac{1}{l+1} \right\} \) and \( l, 1 \epsilon \mathbb{N} \).

3°) For \( t \epsilon (0, 1) \) the base at the point \( \left( \frac{1}{n_m} \times \{ t \} \right) \epsilon A \) consists of all sets of the form
\[ \left( \frac{1}{n_m} \times \left( \frac{1}{l+1}, t + \frac{1}{l} \right) \right) \cap \left( \frac{1}{n_m} \times (0, 1) \right), \]
where \( l \epsilon \mathbb{N} \).

4°) The base at the point \( \left( \frac{1}{n_m} \times \{ 1 \} \right) \epsilon A \) consists of all sets of the form
\[ \left( \frac{1}{n_m} \times \left( \frac{1}{l+1}, 1 \right) \right) \cap \left( \frac{1}{n_m} \times (0, 1) \right) \cap \left( \frac{1}{n_m} \times (0, 1) \right), \]
where \( l \epsilon \mathbb{N} \).

It can easily be verified that \( X_i \) is a \( T_1 \)-space. The space \( X_i \) is the union of two connected subspaces \( \bigcup_{k=0}^{\infty} A_k \) and \( \bigcup_{k=0}^{\infty} A_k \); since each point \( \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_k} \right) \epsilon A_{k-1} \) belongs to \( A \), the space \( X_i \) is connected.

Take \( k > 1 \) and assume that \( X_k \) are already defined for \( n < k \). Let \( X_{k+1} = X_k \cup \bigcup_{\frac{1}{n_1} \times \cdots \times \frac{1}{n_k} \epsilon A_{k-1}} (0, 1) \), \( \epsilon X_{k+1} \), since each point \( \frac{1}{n_k} \times \cdots \times \frac{1}{n_{k+1}} \epsilon A_{k-1} \) belongs to \( A \), the space \( X_i \) is connected.

The topology in the set \( X_{k+1} \) is generated by the following neighbourhood system:

1°) The base at the point \( \left( \frac{1}{n_{k+1}} \times \cdots \times \frac{1}{n_{k+2}} \right) \epsilon [0, 1]_{n_{k+1}}, \cdots \) consists of all sets of the form \( \left( \frac{1}{l+1}, t \right) \epsilon B \cup C \), where \( B \) and \( C \) are defined by the formula (7) and (8), and \( l \epsilon \mathbb{N} \).

2°) The base at the point \( \left( \frac{1}{n_{k+1}} \times \cdots \times \frac{1}{n_{k+2}} \right) \epsilon [0, 1]_{n_{k+1}}, \cdots \) consists of all sets of the form
\[ \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_{k+1}} \times \left( \frac{1}{l+1}, t + \frac{1}{l} \right) \right) \cup \left( \frac{1}{n_1} \times \cdots \times \frac{1}{n_{k+1}} \times (0, 1) \right), \]
where \( l \epsilon \mathbb{N} \) and \( \frac{1}{l+1} < \frac{1}{l+1} \).
3° The base at the point \( t_{n_0, ..., n_k} \in (0, 1]_{n_0, ..., n_k} \) consists of all intervals
\[
\left( t_{n_0, ..., n_k} - \frac{1}{l}, t_{n_0, ..., n_k} + \frac{1}{l} \right)_{n_0, ..., n_k}, \quad \text{where} \quad l \in \mathbb{N}.
\]

4° For \( k \geq 2 \), the base at the point \( l \in [0, 1]_{n_0, ..., n_{k-1}} \) consists of all sets of the form
\[
\left\{ \frac{1}{l} \right\} \cup \left( \frac{1}{l} \right)_{n_0, ..., n_{k-1}}, \quad \text{where} \quad l \in \mathbb{N}.
\]

5° Bases at all remaining points are the same as in \( X_n \).

The space \( X_{k+1} \) defined in this way is a connected \( T_2 \)-space.

We shall now prove that \( X_n \) is paracompact for each \( n \). Observe that \( X_n = \mathbb{N} \cup (X_n \setminus A) \), where \( A \) is an open, metrizable subspace of \( X_n \) and \( X_n \setminus A \) is a Lindelöf space, because it is a countable union of Lindelöf spaces. Let \( U \) be an arbitrary open covering of \( X_n \). Let \( U^k \) be a countable subfamily of \( U \) such that \( X_n \setminus A \subseteq \bigcup U^k \). It can easily be seen that there exists an open set \( W \) satisfying \( X_n \setminus A \subseteq W \subseteq \bigcup U^k \). Let \( U^k \) be a locally finite open covering of \( A \) which is a refinement of \( U^k = (A \cap U)_{k=1} \). The family \( U^k = (U \cap (X_n \setminus A))^k \cup U \) is a covering of \( X_n \setminus A \); moreover, \( U^k \) is locally finite in \( X_n \). Indeed, if \( x \in X_n \setminus A \), then \( W \) is a neighbourhood of \( x \) disjoint with all elements of \( U^k \). And for any \( x \in A \) there exists, \( U^k \) being locally finite in \( A \), a set \( U \), open in \( A \) and hence also in \( X_n \), which contains \( x \) and meets only a finite number of elements of \( U^k \). Thus \( U^k \cup U_0 \) is a \( \sigma \)-locally finite refinement of \( U \). This proves that \( X_n \) is paracompact (see Theorem 5.1 in [3]).

Now define the mappings \( P_{k+1}^n : X_{k+1} \to X_n \) assuming that
\[
P_{k+1}^n(t_{n_0, ..., n_k}) = \left\{ \frac{1}{n_0} \right\} \times \cdots \times \left\{ \frac{1}{n_k} \right\} \quad \text{for} \quad t_{n_0, ..., n_k} \in [0, 1]_{n_0, ..., n_k}
\]
and
\[
P_{k+1}^n(x) = x \quad \text{if} \quad x \neq t_{n_0, ..., n_k}.
\]
It is easy to verify that the mappings \( P_{k+1}^n \) are quotient, monotone (i.e., \( P_{k+1}^n \) is connected for \( y \in X_n \)) and onto, but (except for \( P_1^n \)) they are not hereditarily quotient.

Obviously \( S = \{X_n, P_n^m, N\} \), where \( P_n^m = P_{n+1}^m P_{n+2}^m \cdots P_{m-1}^m \) for \( m \geq n+1 \), is an inverse system.

The projection \( \lim S \to X_n \) is not quotient, because the set \( \bigcup_{k=0}^{\infty} A_k \) is open in \( \lim S \). Indeed, for each thread \( \{x_k\} \in P_{k+1}^{\infty} (\bigcup_{k=0}^{\infty} A_k) \) there exists a \( k \in N \) such that \( x_m = x_k \) for \( m > k \). If \( x_k = 1_{n_0, ..., n_m} \) for some \( l \in N \), then there exists a neighbourhood \( U_{k+1} \) of the point \( x_{k+1} \) in \( X_{k+1} \) such that \( \pi_{k+1}^m (U_{k+1}) \subset P_{k+1}^m (\bigcup_{k=0}^{\infty} A_k) \).

Otherwise, we can find such a neighbourhood of \( x_{k+1} \) in \( X_{k+1} \).

Finally, note that the limit \( \lim S \) is not connected, because it is the union of two sets \( P_{k+1}^m (\bigcup_{k=0}^{\infty} A_k) \) and \( P_{n+1}^m (A) \), open and disjoint.

4. Monotone mappings. Recall that a mapping \( f : X \to Y \) is monotone if for each \( y \in Y \) the set \( f^{-1}(y) \) is connected.

The following theorem was proved in [4] in the case of countable systems.

**Theorem 10.** Let \( S = (X_n, P_n^m, N) \) and \( S' = (Y_n, P_n^m, N') \) be two inverse systems, where all \( X_n \) are \( T_2 \)-spaces and all \( Y_n \) are compact. Let \( (\psi_n, \phi_n) \) be a mapping of \( S \) into \( S' \) such that \( \phi_n \) is monotone for each \( n \). Then \( f = \lim (\phi_n, \psi_n) \) is monotone.

**Theorem 11.** Let \( S = (X_n, P_n^m, N) \) be an inverse system of compact spaces such that \( P_n^m \) is monotone for each \( m \). Then the projection \( \lim S \to X_n \) is monotone.

**Proof.** The proof follows directly from Theorem 1.

If the spaces \( X_n \) are not compact, then, as the following example shows, the above theorem does not hold even if the limit mapping is closed and onto.

**Example 7.** For \( n = 1, 2, 3, \ldots \) let
\[
X_n = \{ (x, y) : x \in \mathbb{R}, x > 0, 0 < y < \frac{1}{y} \}
\]
with the topology of a subspace of \( \mathbb{R}^2 \) and \( A_n = \{(x, y) : x \in X_n : y = 0 \text{ or } y = 1\} \cup \{(x, y) : y = 0 \text{ or } y = 1\} \cup \{(x, y) : x \in X_n : x > n-1\} \). Denote by \( X_n / R_n \) the quotient space \( X_n / R_n \), where the equivalence relation \( R_n \) in \( X_n \) is defined by the formula:
\[
(x, y) R_n(x', y') \iff [(x, y) = (x', y')] \text{ or } (x, y) = (x', y') \in A_n.
\]
Let \( f_n : X_n \to Y_n \) be the natural quotient mapping, \( P_n^m : X_n \to X_n \) the identity mapping and \( P_n^m ((x, y)) = [(x, y)] \in Y_n \) for \( (x, y) \in X_n \). Then \( S = (X_n, P_n^m, N) \) and \( S' = (Y_n, P_n^m, N') \) are two inverse systems and \( (\text{id}_Y, f_n) \) is a mapping of \( S \) into \( S' \) such that the mappings \( P_n^m : X_n \to X_n \), \( P_n^m : Y_n \to Y_n \), and \( f_n : X_n \to Y_n \) are monotone, closed and onto, and the limit mapping \( f = \lim (\text{id}_Y, f_n) \) is closed and onto but is not monotone.
The corollary to Theorem 10 is not true if the spaces $X_n$ are not compact; this can be seen from Example 6. We give below a simpler example, but the bonding mappings of the given inverse system are not quotient (see the corollary to Theorem 10).

Example 8. Define the inverse system $S = \{X_n, \Pi^n_n, N\}$ by assuming that:

1° $X_n = A_n \cup B_n \cup C_n$, where $A_n = \{(x, y) \in \mathbb{R}^2 : -1 \leq x < 0, 0 \leq y \leq 1\}$, $B_n = \{(x, y) \in \mathbb{R}^2 : x = 0, \frac{1}{n} - 1 \leq y < 1\}$ and $C_n = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq \frac{1}{n}, y = 1 - \frac{1}{n}\}$ for each $n \in \mathbb{N}$ (see Fig. 2).

2° For $m \geq n$ and $(x, y) \in X_m$, 

\[ \Pi^n_m((x, y)) = \begin{cases} 0, \frac{1}{m} - \frac{1}{n} - 1, \frac{1}{n} - 1, \frac{1}{n} & \text{if } (x, y) \text{ satisfies } 0 < x \leq \frac{1}{n} - \frac{1}{m}, \\ \pi_n((x, y)) & \text{if } (x, y) \text{ satisfies } \frac{1}{n} - \frac{1}{m} < x \leq 1 - \frac{1}{m}, \\ (x, y) & \text{otherwise.} \end{cases} \]

The spaces $X_n$ are connected and the bonding mappings $\Pi^n_m$ are monotone and onto. The limit $\lim S$ is not connected, because $\lim S = U_1 \cup U_2$, where

\[ U_1 = \lim \{A_n, \Pi^n_m | A_m, N\} \]

and

\[ U_2 = \lim \{B_n \cup C_n, \Pi^n_m | B_m \cup C_m, N\} \]

are open and disjoint.

Example 9. Let $X_n$ and $\Pi^n_m$ be as in Example 8. Let $X_0 = \{(0, 0)\}$ and $\Pi^n_0((x, y)) = \{(x, y) \in X_0 \text{ and } m \geq 1\}$. The bonding mappings of the inverse system $S = \{X_n, \Pi^n_m, N \cup \{0\}\}$ are monotone and onto, but the projection $\Pi^0_0$ is not monotone.

We shall now prove that the limit of the inverse system of connected spaces is connected under an additional assumption.

Theorem 11. Let $S = \{X_n, \Pi^n_m, N\}$ be an inverse system of connected spaces such that the bonding mappings $\Pi^n_m$ are monotone, hereditarily quotient and onto. Then the limit $\lim S$ is connected.

Proof. Suppose that $\lim S = U_1 \cup U_2$, where $U_1$ and $U_2$ are open, non-empty and disjoint. By Theorem 9 the mapping $\Pi_1^n: \lim S \rightarrow X_n$ is hereditarily quotient. Suppose that $A_n = \Pi_1^n(U_1) \cap \Pi_1^n(U_2) = \emptyset$ for some $n \in \mathbb{N}$. Then $U_i = \Pi_1^n \Pi_1^n(U_i)$ for $i = 1, 2$ and $X_n = \Pi_1^n(U_1) \cup \Pi_1^n(U_2)$.

Since $\Pi_1^n$ is quotient, the sets $\Pi_1^n(U_1)$ and $\Pi_1^n(U_2)$ are open, non-empty and disjoint, but this is impossible by the connectedness of $X_n$; thus all sets $A_n$ are not empty.

Clearly $\Pi^n_m(A_m) \subseteq A_n$ for $m \geq n$. We shall show that $\Pi^n_m(A_m) = A_n$. Take $a_n \in A_n$; since $\Pi^n_m$ is monotone, the set $(\Pi^n_m)^{-1}(a_n)$ is connected.

Let $B_i = (\Pi^n_m)^{-1}(a_n) \cap \Pi^n_m(U_i)$; obviously $B_i \cup B_i = (\Pi^n_m)^{-1}(a_n)$. To see that $B_i \cap B_i = \emptyset$ suppose the contrary. Then $\Pi^n_m(B_i) = U_i \cap (\Pi^n_m)^{-1}(a_n)$ and this set is open in $\Pi^n_m(U_i) \cap (\Pi^n_m)^{-1}(a_n)$. Since the mapping $\Pi^n_m|\Pi^n_m(U_i) \cap (\Pi^n_m)^{-1}(a_n): (\Pi^n_m)^{-1}(a_n) \rightarrow (\Pi^n_m)^{-1}(a_n)$ is quotient, the set $B_i$ is open in $(\Pi^n_m)^{-1}(a_n)$ for $i = 1, 2$, which contradicts the assumption that $(\Pi^n_m)^{-1}(a_n)$ is connected.

The family $S' = \{A_n, \Pi^n_m | A_m, N\}$ is an inverse system of non-empty spaces with the bonding mappings onto. Thus the limit $\lim S'$ is non-empty. Since the sets $U_i$ are closed, by Theorem 2 we have $\lim S' \subseteq U_i \cap U_i'$, which contradicts the assumption that $U_i \cap U_i' = \emptyset$.

Corollary. Let $S = \{X_n, \Pi^n_m, N\}$ be an inverse system such that the bonding mappings are monotone, hereditarily quotient and onto. Then the projection $\Pi_1^n: \lim S \rightarrow X_n$ is monotone.

Proof. Take $a_n \in X_n$; clearly $\Pi_1^n(a_n) = \lim (A_n, \Pi^n_m | A_m, N)$, where

\[ A_n = \begin{cases} (\Pi^n_m)^{-1}(a_n) & \text{for } n > n_0, \\ \Pi^n_m(a_n) & \text{for } n \leq n_0. \end{cases} \]

Since each mapping $\Pi^n_m$ is monotone, hereditarily quotient and onto, each $A_n$ is connected and $\Pi^n_m | A_m: A_m \twoheadrightarrow A_n$ is monotone, hereditarily quotient and onto for each $m, n \in \mathbb{N}$. Thus, by Theorem 11 the limit $\lim (A_n, \Pi^n_m | A_m, N) = (\Pi^n_m)^{-1}(a_n)$ is connected.

One can check that all the assumptions of Theorem 11 are essential.

The author does not know the answer to the following questions:

1. Does there exist an uncountable system of connected spaces with bonding mappings monotone, onto and hereditarily quotient (better: open or closed) and with a disconnected limit?
2. Does there exist an uncountable system $S = (X_\alpha, I_\alpha^p, \Sigma)$ with bonding mappings monotone, onto and hereditarily quotient (better: open or closed) such that the projection $\Pi_\alpha$ is not monotone?

Clearly a positive answer to problem 1 gives a positive answer to problem 2.

The main results of this paper are described in the following tables.

### Table 1

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### Table 2

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