Proof. The proposition follows immediately from Propositions 7, 6 and 8. q.e.d.

Remark. Since every space from $\bigvee_{\aleph_0} MC$ is a locally compact separable metric space, the theorem follows immediately from Proposition 9.

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Limit mappings and projections of inverse systems

by

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Abstract. Let $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ and $S' = \{Y_{\sigma'}, \Pi_{\varrho'}^{\sigma'}, \Sigma'\}$ be inverse systems and $\{\varphi, f_{\sigma'}\}$ be a mapping of S into S'. For some classes K of mappings we discuss the problem when $f_{\sigma'} \in K$ implies $\varprojlim \{\varphi, f_{\sigma'}\} \in K$ and when $\Pi_\varrho^\sigma \in K$ implies $\Pi_\sigma \in K$.

In this paper we are concerned with limits of inverse systems, their projections and limit mappings induced by mappings of inverse systems. More precisely, we show how the projections depend on bonding mappings and how the limit mapping depends on the mapping of systems inducing it.

To begin with, we recall some definitions and simple facts about inverse systems and give two auxiliary examples. Our terminology and notations are consistent with those used in [3]; except that by a mapping of an inverse system $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ into $S' = (Y_{\sigma'}, \Pi_{\varrho'}^{\sigma'}, \Sigma'\}$ we understand a system $\{\varphi, f_{\sigma'}\}$ satisfying besides the usual commutativity conditions also the condition that $\varphi(\Sigma')$ is cofinal in Σ.

The diagram

(1) $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow{k} & Z \end{array}$$

is said to be *exact* (see [8], p. 19) if it is commutative and the following implication is true:

$$\text{if} \quad h(y) = k(t), \quad \text{then} \quad g^{-1}(t) \cap f^{-1}(y) \neq \emptyset .$$

The diagram (1) is exact (see p. 19 of [8]) if and only if

(2) $$fg^{-1}(B) = h^{-1}k(B) \quad \text{for} \quad B \subset T$$

or, equivalently,

(2′) $$gf^{-1}(A) = k^{-1}h(A) \quad \text{for} \quad A \subset Y.$$

Obviously, the diagram (1) is commutative if and only if

(3) $$fg^{-1}(B) \subset h^{-1}k(B) \quad \text{for} \quad B \subset T$$

or, equivalently,

(3′) $$gf^{-1}(A) \subset k^{-1}h(A) \quad \text{for} \quad A \subset Y.$$

A mapping $\{\varphi, f_{\sigma'}\}$ of the system $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ into the system $S' = \{Y_{\sigma'}, \Pi^{\sigma'}_{\varrho'}, \Sigma'\}$ is said to be *exact* if the diagram

(4) $$\begin{array}{ccc} X_{\varphi(\sigma')} & \xrightarrow{f_{\sigma'}} & Y_{\sigma'} \\ \downarrow{\scriptstyle \Pi^{\varphi(\sigma')}_{\varphi(\varrho')}} & & \downarrow{\scriptstyle \Pi^{\sigma'}_{\varrho'}} \\ X_{\varphi(\varrho')} & \xrightarrow{f_{\varrho'}} & Y_{\varrho'} \end{array}$$

is exact for each pair $\sigma', \varrho' \in \Sigma'$ with $\varrho' \leqslant \sigma'$.

It is said to be *limit-exact* if the diagram

(5) $$\begin{array}{ccc} \underleftarrow{\lim} S & \xrightarrow{f} & \underleftarrow{\lim} S' \\ \downarrow{\scriptstyle \Pi_{\varphi(\sigma')}} & & \downarrow{\scriptstyle \Pi_{\sigma'}} \\ X_{\varphi(\sigma')} & \xrightarrow{f_{\sigma'}} & Y_{\sigma'} \end{array}$$

is exact for every $\sigma' \in \Sigma'$.

It is easy to see that if Σ and Σ' are the sets of all natural numbers, then an exact mapping of systems is limit-exact.

Recall that the limit of $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ is non-empty if the spaces X_σ are compact for every $\sigma \in \Sigma$ (see [3], Theorem 3.2.10). In the case of countable Σ, the limit of a system of non-empty spaces is non-empty if the bonding mappings Π^σ_ϱ are onto, but it may be empty if Π^σ_ϱ are not onto. In the case of non-countable Σ the limit of an inverse system of non-empty spaces with bonding mappings onto may be empty:

EXAMPLE 1 ([5], simplified in [6]). Let Σ be the set of all ordinal numbers less than ω_1 and R the real line. For every $\alpha < \omega_1$ we take $W_\alpha = \{\gamma\colon \gamma < \alpha\}$ with the ordinal topology (see [3], Example 3.5.1) and let X_α be the set of all homeomorphic embeddings $f\colon W_\alpha \to R$ with the discrete topology. For $\beta \leqslant \alpha$ we define mappings $\Pi^\alpha_\beta\colon X_\alpha \to X_\beta$ as restrictions, i.e. we put $\Pi^\alpha_\beta(f) = f | W_\beta$ for $f \in X_\alpha$. One shows that the limit of the inverse system $S = \{X_\alpha, \Pi^\alpha_\beta, \Sigma\}$ is empty.

Now we shall describe a similar inverse system of countable spaces.

EXAMPLE 2 ([7]). Let $S = \{X_\alpha, \Pi^\alpha_\beta, \Sigma\}$ be the inverse system from Example 1. For every $\alpha \in \Sigma$ we shall define by transfinite induction a countable set $Y_\alpha \subset X_\alpha$ consisting of strictly increasing embeddings of W_α into R, satisfying the following condition:

(6) *for all $\beta < \alpha < \omega_1$, a positive integer n and $f_\beta \in Y_\beta$ there exists an $f_\alpha \in Y_\alpha$ such that $\Pi^\alpha_\beta(f_\alpha) = f_\beta$ and $f_\alpha(\alpha) - f_\beta(\beta) \leqslant 1/2^n$.*

Let Y_1 be an arbitrary one-point subspace of X_1. Suppose that the sets Y_α satisfying condition (6) are defined for $\alpha < \alpha'$; let us consider two cases: when $\alpha' = \alpha''+1$ and when α' is a limit ordinal number.

If $\alpha' = \alpha''+1$, then for every $f_{\alpha''} \in Y_{\alpha''}$ and $n \in N$ we take an extension $f_{\alpha'}$ of $f_{\alpha''}$ over $W_{\alpha'}$ such that $f_{\alpha'}(\alpha') = f_{\alpha''}(\alpha'')+1/2^n$. Clearly, $f_{\alpha'} \in X_{\alpha'}$ and $f_{\alpha'}(\alpha') - f_{\alpha''}(\alpha'') \leqslant 1/2^n$. Let $Y_{\alpha'}$ be the set of all functions constructed in this way for every $f_{\alpha''} \in Y_{\alpha''}$ and $n \in N$, taken with the discrete topology. It is easy to see that $Y_{\alpha'}$ has the required properties.

Now, let α' be a limit ordinal and α an arbitrary ordinal less than α'. For $f_\alpha \in Y_\alpha$ and $n \in N$ choose an increasing sequence of ordinals $\alpha = \alpha_1 < \alpha_2 < \alpha_3 < \ldots$ convergent to α'. By the inductive assumption there exists a sequence $f_\alpha = f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_3}, \ldots$ such that $f_{\alpha_m} \in Y_{\alpha_m}$, $\Pi^{\alpha_{m+1}}_{\alpha_m}(f_{\alpha_{m+1}}) = f_{\alpha_m}$ and $f_{\alpha_{m+1}}(\alpha_{m+1}) - f_{\alpha_m}(\alpha_m) \leqslant 1/2^{n+m}$ for every $m \in N$. Let us consider the combination $f'\colon \{\gamma\colon \gamma < \alpha'\} \to R$ of $\{f_{\alpha_m}\}_{m=1}^\infty$ and its extension f over $W_{\alpha'}$ such that $f(\alpha') = \sup_{\gamma<\alpha'} f'(\gamma)$. The function f is one-to-one (being strictly increasing) and continuous, it is also closed as a continuous mapping of a compact space into a Hausdorff space. Thus, $f \in X_{\alpha'}$ and

$$f(\alpha') - f_\alpha(\alpha) \leqslant \sum_{m=1}^{\infty} \frac{1}{2^{n+m}} = \frac{1}{2^n}.$$

Let $Y_{\alpha'}$ be the set of all functions constructed in this way for every $\alpha < \alpha'$, $f_\alpha \in Y_\alpha$ and $n \in N$, taken with the discrete topology. Since the set $\{\alpha \in \Sigma\colon \alpha < \alpha'\}$ and all Y_α are countable, the set $Y_{\alpha'}$ is also countable. Moreover, $Y_{\alpha'}$ consists of strictly increasing functions and satisfies the condition (6).

Let us denote $\Pi^\alpha_\beta | Y_\alpha$ by $\tilde{\Pi}^\alpha_\beta$. The family $S' = \{Y_\alpha, \tilde{\Pi}^\alpha_\beta, \Sigma\}$ is an inverse system such that Y_α is a countable discrete space for each α, $\tilde{\Pi}^\alpha_\beta$ is onto for $\beta \leqslant \alpha$ and $\underleftarrow{\lim} S' = \emptyset$.

We shall now examine limit mappings. In section 3 of [4] there is an example showing that $f = \underleftarrow{\lim}\{\varphi, f_{\sigma'}\}$ need not be onto, when each $f_{\sigma'}$ is onto even if both of the systems are countable with the bonding mappings onto. However, it can easily be verified that if the mapping $\{\varphi, f_{\sigma'}\}$ is limit-exact and each $f_{\sigma'}$ is onto, then $f = \underleftarrow{\lim}\{\varphi, f_{\sigma'}\}$ is onto. Hence, if the mapping $\{\varphi, f_n\}$ of countable systems is exact and each f_n is onto, then the limit mapping is onto.

Let us consider the projection $\Pi_\sigma\colon \underleftarrow{\lim}\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\} \to X_\sigma$. If the inverse system S is countable and Π^σ_ϱ are onto, then the projection Π_σ is onto, but it is not necessarily onto, if Σ is uncountable, this follows at once from Example 1.

THEOREM 1. *For every inverse system $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ and $\sigma_0 \in \Sigma$ there exist a system S' and a mapping $\{\varphi, f_{\sigma'}\}$ of S, where $f_{\sigma'}$ are bonding mappings* *of S, and a homeomorphism $h\colon \varprojlim S' - X_{\sigma_0}$ such that $\Pi_{\sigma_0} = h\varprojlim\{\varphi, f_{\sigma'}\}$. Moreover, we can assume that $\varphi(\Sigma') = \Sigma$.*

Proof. Let $S' = \{Y_{\sigma'}, \Pi^{\sigma'}_{\varrho'}, \Sigma'\}$, where $\Sigma' = \{\sigma \in \Sigma\colon \sigma \geqslant \sigma_0\}$, $Y_{\sigma'} = X_{\sigma_0}$ for every $\sigma' \in \Sigma'$ and $\Pi^{\sigma'}_{\varrho'} = \mathrm{id}_{X_{\sigma_0}}\colon Y_{\sigma'} \to Y_{\varrho'}$ for $\sigma' \geqslant \varrho'$. Let $\varphi\colon \Sigma' \to \Sigma$ be the natural injection and let $f_\sigma = \Pi^\sigma_{\sigma_0}\colon X_\sigma \to X_{\sigma_0}$ for each $\sigma \geqslant \sigma_0$. The diagram

$$\begin{array}{ccc} X_\sigma & \xrightarrow{\ \Pi^\sigma_{\sigma_0}=f_\sigma\ } & X_{\sigma_0} = Y_\sigma \\ {\scriptstyle \Pi^\sigma_\varrho}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{X_{\sigma_0}} = \Pi'^{\sigma}_{\varrho}} \\ X_\varrho & \xrightarrow{\ \Pi^\varrho_{\sigma_0}=f_\varrho\ } & X_{\sigma_0} = Y_\varrho \end{array}$$

commutes for $\sigma \geqslant \varrho \geqslant \sigma_0$, and so $\{\varphi, f_{\sigma'}\}$ is a mapping of S into S'. The space $\varprojlim S'$ is the diagonal of the product $\underset{\sigma' \in \Sigma'}{P} Y_{\sigma'}$. Let h be the natural homeomorphism of this diagonal onto X_{σ_0}. Then $\Pi_{\sigma_0} = h\varprojlim\{\varphi, f_{\sigma'}\}$ and $\varphi(\Sigma')$ is cofinal in Σ. If we take the system $\{X_\sigma, \Pi^\sigma_\varrho, \varphi(\Sigma')\}$ instead of the system S, then $\varphi(\Sigma') = \Sigma$.

We state without proof the following well-known theorems:

THEOREM 2. *For an arbitrary subspace A of the limit $X = \varprojlim\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ the family $S_A = \{\bar{A}_\sigma, \Pi^\sigma_\varrho | \bar{A}_\sigma, \Sigma\}$, where $A_\sigma = \Pi_\sigma(A)$, is an inverse system and $\varprojlim S_A = \bar{A} \subset X$.*

COROLLARY. *A closed subspace A of the limit $X = \varprojlim\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ is the limit of the inverse system $S_A = \{\overline{\Pi_\sigma(A)}, \Pi^\sigma_\varrho | \overline{\Pi_\sigma(A)}, \Sigma\}$ of closed subspaces of X_σ.*

THEOREM 3. *If the mapping $\{\varphi, f_{\sigma'}\}$ of the system $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ of T_2-spaces into $S' = \{Y_{\sigma'}, \Pi^{\sigma'}_{\varrho'}, \Sigma'\}$ satisfies the condition $\varphi(\Sigma') = \Sigma$, then there exists a homeomorphic embedding $h\colon \varprojlim S \to \underset{\sigma' \in \Sigma'}{P} Z_{\sigma'}$, where $Z_{\sigma'} = X_{\varphi(\sigma')}$ onto a closed subspace of $\underset{\sigma' \in \Sigma'}{P} Z_{\sigma'}$ such that $\varprojlim\{\varphi, f_{\sigma'}\} = (\underset{\sigma' \in \Sigma'}{P} f_{\sigma'})h$.*

Now we pass to the main subject of this paper, i.e. to the determination when (under what conditions for $\varphi, f_{\sigma'}$ and for the inverse systems) for the given class of mappings $\mathfrak{K}$ the limit mapping $\varprojlim\{\varphi, f_{\sigma'}\}$ belongs to $\mathfrak{K}$, and (which is a special case by Theorem 1) when projections Π_σ belong to $\mathfrak{K}$.

We shall consider the following classes of mappings: 1) open, 2) closed and perfect, 3) quotient and hereditarily quotient, 4) monotone.

1. Open mappings. In [4], K. R. Gentry has given an example of a mapping $\{\mathrm{id}_N, f_n\}$ of an inverse system $S = \{X_n, \Pi^m_n, N\}$ into a system $S' = \{Y_n, \tilde{\Pi}^m_n, N\}$ such that f_n are open and onto and the limit mapping $f = \varprojlim\{\mathrm{id}_N, f_n\}$ is not open and is into. In fact in that example Π^m_n, $\tilde{\Pi}^m_n$ and f_n are closed-and-open and f is neither closed nor open. By a small modification of Gentry's example we can obtain similar systems for which f is an onto mapping but is not quotient ([1]). For this purpose it is sufficient to add to the space X_n for $n = 1, 2, 3, \ldots$ an isolated point $n(y)$ for each $y = \{y_n\} \in \varprojlim S' \setminus f(\varprojlim S)$, and to define $\Pi^m_n(m(y)) = n(y)$, $f_n(n(y)) = y_n$ for $m, n \in N$.

In [4] it is shown that for an exact mapping $\{\mathrm{id}_N, f_n\}$ of a system $S = \{X_n, \Pi^m_n, N\}$ into $S' = \{Y_n, \Pi^m_n, N\}$ if each f_n is open, then so is $\varprojlim\{\mathrm{id}_N, f_n\}$. For uncountable systems an analogous theorem is not true. This follows from Example 4 given below.

The following theorem is a generalization of Gentry's result:

THEOREM 4. *Let $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ and $S' = \{Y_{\sigma'}, \Pi^{\sigma'}_{\varrho'}, \Sigma\}$ be two inverse systems and let $\{\varphi, f_{\sigma'}\}$ be a limit-exact mapping of S into S' such that each $f_{\sigma'}$ is open. Then the limit mapping is open.*

Proof. Let $U = \Pi^{-1}_{\varphi(\sigma')}(U_{\sigma'})$, where $U_{\sigma'}$ is open in $X_{\varphi(\sigma')}$, be an element of the base of $\varprojlim S$. As the diagram (5) is exact, we have $f(U) = \Pi^{-1}_{\sigma'} f_{\sigma'}(U_{\sigma'})$. It follows that $f(U)$ is open in $\varprojlim S'$ and that f is open.

THEOREM 5. *Let $S = \{X_n, \Pi^m_n, N\}$ be an inverse system such that each Π^m_n is open and onto. Then the projection Π_n is open.*

Proof. It is easily seen that the mapping $\{\varphi, f_{\sigma'}\}$ of S into S' from Theorem 1 is exact. Thus, it follows from the above theorem that the projection Π_n is open.

All the assumption of Theorem 5 are essential. The following example shows that Theorem 5 does not hold for uncountable systems.

EXAMPLE 3. Let $S = \{X_\alpha, \Pi^\alpha_\beta, \Sigma\}$ be the system from Example 1. For each $\alpha \in \Sigma$ let Y_α be the hedgehog with m prickles (see Example 4.1.3 in [3]), where $m = \overline{\overline{X}}_\alpha$; that is $Y_\alpha = (X_\alpha \times [0, 1])/R$, where

$$[(f, t)R(f', t')] \Leftrightarrow [(f = f' \text{ and } t = t') \text{ or } (t = t' = 1)],$$

with the topology induced by the metric

$$\varrho([(f, t)], [(f', t')]) = \begin{cases} |t - t'| & \text{if} \quad f = f', \\ t + t' & \text{if} \quad f \neq f'. \end{cases}$$

Let the mapping $\tilde{\Pi}^\alpha_\beta\colon Y_\alpha \to Y_\beta$ be defined for $\alpha \geqslant \beta$ by the formula:

$$\tilde{\Pi}^\alpha_\beta([(f, t)]) = [(\Pi^\alpha_\beta(f), t)] \quad \text{for} \quad [(f, t)] \in Y_\alpha.$$

As Π^α_β is onto, so is $\tilde{\Pi}^\alpha_\beta$; moreover, it is easy to see that $\tilde{\Pi}^\alpha_\beta$ is open. As the limit of the inverse system $S' = \{Y_\alpha, \tilde{\Pi}^\alpha_\beta, \Sigma\}$ is a one-point space, the projection $\tilde{\Pi}_\alpha\colon \varprojlim S' \to Y_\alpha$ is not open.

EXAMPLE 4. Using Example 3 and Theorem 1 let us consider the mapping $\{\varphi, f_\alpha\}$ such that $\tilde{\Pi}_\alpha$ is the composition of $\varprojlim\{\varphi, f_\alpha\}$ and of a homeomorphism. Then $\{\varphi, f_\alpha\}$ is exact but $\varprojlim\{\varphi, f_\alpha\}$ is not open.

([1]) $f\colon X \to Y$ onto Y is *quotient* if $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X.

2. Closed and perfect mappings. The following theorem was proved in [4] in the case of countable systems.

THEOREM 6. *Let $S=\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ and $S'=\{Y_{\sigma'}, \Pi^{\sigma'}_{\varrho'}, \Sigma'\}$ be two inverse systems, where all X_σ are T_2-spaces, and let $\{\varphi, f_{\sigma'}\}$ be a mapping of S into S' such that $f_{\sigma'}$ is perfect for each $\sigma'\in\Sigma'$. Then $f=\varprojlim\{\varphi, f_{\sigma'}\}$ is perfect.*

The proof is exactly the same as in [4] for countable systems.

The above-mentioned example of K. R. Gentry shows that an analogous theorem for closed mappings is not true, even if both systems are countable. A slight modification of that example shows that the assumption of $\{\varphi, f_n\}$ being exact does not help.

The following two theorems are concerned with the projection Π_σ.

THEOREM 7 ([9], Theorem 4.2, [10], Lemma 2.4). *Let $S=\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ be an inverse system of T_2-spaces such that the bonding mappings Π^σ_ϱ are perfect. Then the projection Π_σ is perfect.*

Proof. This follows at once from Theorem 1 and Theorem 6.

THEOREM 8. *Let $S=\{X_n, \Pi^m_n, N\}$ be an inverse system such that the bonding mappings Π^m_n are closed. Then the projection Π_n is closed.*

Proof. This was proved by P. Zenor in [10] under the additional assumption that Π^m_n are onto, which is not used in the proof.

Theorem 8 does not hold for uncountable systems, even if bonding mappings are onto. This is shown by the following example:

EXAMPLE 5. Let $S'=\{Y_\alpha, \tilde{\Pi}^\alpha_\beta, \Sigma\}$ be the inverse system from Example 2. We take $Y=\{\alpha\colon \alpha\leqslant\omega_1\}$ with the topology defined by assuming that a subset $F\subset Y$ is closed if and only if $\overline{\overline{F}}\leqslant\aleph_0$ or $F\ni\omega_1$. With this topology Y is a Lindelöf space.

For each $\alpha\in\Sigma$ let $X_\alpha=(Y_\alpha\times Y)\oplus\bigoplus_{\gamma<\alpha}(X'_\gamma\times\{\gamma\})$, where $X'_\gamma=X_\gamma\times\times[0,\gamma]$ has the discrete topology. Since the set $\{\gamma\colon \gamma<\alpha\}$ and all Y_α are countable, all X_α are Lindelöf spaces.

The mapping $\Pi^\alpha_\beta\colon X_\alpha\to X_\beta$ is defined for $\beta\leqslant\alpha$ in the following way: if $(y_\alpha, y)\in Y_\alpha\times Y$, then

$$\Pi^\alpha_\beta\big((y_\alpha, y)\big)=\big(\tilde{\Pi}^\alpha_\beta(y_\alpha), y\big)=(\tilde{\Pi}^\alpha_\beta\times\mathrm{id}_Y)(y_\alpha, y),$$

if $(y_\gamma, \delta)\times\{\gamma\}\in(Y_\gamma\times[0,\gamma])\times\{\gamma\}$, then

$$\Pi^\alpha_\beta\big((y_\gamma, \delta)\times\{\gamma\}\big)=\begin{cases}(y_\gamma, \delta)\times\{\gamma\} & \text{for } \beta>\gamma,\\ \big(\tilde{\Pi}^\gamma_\beta(y_\gamma), \delta\big) & \text{for } \beta\leqslant\gamma.\end{cases}$$

It is easy to see that $\Pi^\beta_\varepsilon\Pi^\alpha_\beta=\Pi^\alpha_\varepsilon$ for $\alpha\geqslant\beta\geqslant\varepsilon$ and $\Pi^\alpha_\alpha=\mathrm{id}_{X_\alpha}$ for each α. The mapping Π^α_β is continuous on the discrete set $\bigoplus_{\gamma<\alpha}(X'_\gamma\times\{\gamma\})$, and $\Pi^\alpha_\beta=\tilde{\Pi}^\alpha_\beta\times\mathrm{id}_Y$ on $Y_\alpha\times Y$ and so Π^α_β is also continuous on that set. Since X_α is the union of those open-and-closed disjoint sets, the mapping Π^α_β is continuous.

We shall show that Π^α_β is closed. Take a closed set $F\subset X_\alpha$. Then $F=\bigcup_{y_\alpha\in Y_\alpha}F(y_\alpha)\cup A$, where $\overline{\overline{A}}\leqslant\aleph_0$ and $F(y_\alpha)$ is empty or is an uncountable subset of $\{\tilde{\Pi}^\alpha_\beta(y_\alpha)\}\times Y$ containing the point $\{y_\alpha\}\times\{\omega_1\}$. We have $\Pi^\alpha_\beta(F)=\bigcup_{y_\alpha\in Y_\alpha}\Pi^\alpha_\beta\big(F(y_\alpha)\big)\cup\Pi^\alpha_\beta(A)$, where $\Pi^\alpha_\beta\big(F(y_\alpha)\big)$ is empty or is an uncountable subset of $\{\tilde{\Pi}^\alpha_\beta(y_\alpha)\}\times Y$ containing the point $\{\tilde{\Pi}^\alpha_\beta(y_\alpha)\}\times\{\omega_1\}$, as $\overline{\overline{\Pi^\alpha_\beta(A)}}\leqslant\aleph_0$, we see that $\Pi^\alpha_\beta(F)$ is closed. As $\tilde{\Pi}^\alpha_\beta$ is onto, the mapping Π^α_β is also onto.

Now let $X=\{\varprojlim X_\alpha, \Pi^\alpha_\beta, \Sigma\}$. We shall prove that for each $\alpha_0\in\Sigma$ we have

$$[(Y_{\alpha_0}\times Y)\setminus(Y_{\alpha_0}\times\{\omega_1\})]\cup\bigotimes_{\gamma<\alpha_0}(X'_\gamma\times\{\gamma\})\subset\Pi_{\alpha_0}(X).$$

For $(y_{\alpha_0}, \delta)\in(Y_{\alpha_0}\times Y)\setminus(Y_{\alpha_0}\times\{\omega_1\})$ we consider two cases: $\delta>\alpha_0$ and $\delta\leqslant\alpha_0$. If $\delta>\alpha_0$, then we choose $y'_{\alpha_0}\in(\Pi^\delta_\alpha)^{-1}(y_{\alpha_0})\neq\emptyset$ and define

$$z_\alpha=\begin{cases}(y'_{\alpha_0}, \delta)\times\{\delta\} & \text{for } \alpha>\delta,\\ \big(\tilde{\Pi}^\delta_\alpha(y'_{\alpha_0}), \delta\big) & \text{for } \alpha\leqslant\delta.\end{cases}$$

It is easy to see that $\{z_\alpha\}$ is a thread such that $\Pi_{\alpha_0}(\{z_\alpha\})=(y_{\alpha_0}, \delta)$. If $\delta\leqslant\alpha_0$, then we define

$$z_\alpha=\begin{cases}\big(\Pi^{\alpha_0}_\alpha(y_{\alpha_0}), \delta\big) & \text{for } \alpha\leqslant\alpha_0,\\ (y_{\alpha_0}, \delta)\times\{\alpha_0\} & \text{for } \alpha>\alpha_0.\end{cases}$$

In this case also one can easily see that $\{z_\alpha\}$ is a thread such that $\Pi_{\alpha_0}(\{z_\alpha\})=(y_{\alpha_0}, \delta)$. On the other hand, if $(x_\gamma, \delta)\times\{\gamma\}\in\bigoplus_{\gamma<\alpha_0}(X'_\gamma\times\{\gamma\})$ then for the thread $\{z_\alpha\}$, where

$$z_\alpha=\begin{cases}(x_\gamma, \delta)\times\{\gamma\} & \text{for } \alpha\geqslant\alpha_0,\\ \Pi^{\alpha_0}_\alpha(x_\gamma, \delta)\times\{\gamma\}) & \text{for } \alpha<\alpha_0,\end{cases}$$

we have $\Pi_{\alpha_0}(\{z_\alpha\})=(x_\gamma, \delta)\times\{\gamma\}$.

The subsystem $S'=\{Y_\alpha\times\{\omega_1\}, \Pi^\alpha_\beta|(Y_\alpha\times\{\omega_1\}), \Sigma\}$ has the empty limit, because $\varprojlim\{Y_\alpha, \tilde{\Pi}^\alpha_\beta, \Sigma\}=\emptyset$. Therefore, $\Pi_{\alpha_0}(X)=X_{\alpha_0}\setminus(Y_{\alpha_0}\times\{\omega_1\})$ and, as this set is not closed in X_{α_0}, the mapping Π_{α_0} is not closed.

Observe that the existence of an inverse system $S=\{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ of T_1-spaces with the bonding mappings closed and onto such that the projection $\Pi_{\sigma_0}\colon \varprojlim S\to X_{\sigma_0}$ is not closed leads to an inverse system of nonempty spaces with the bonding mappings onto and with the empty limit; so we can hardly expect a simple example of such a system.

Indeed, suppose that F is a closed subset of $\varprojlim S$ such that $\Pi_{\sigma_0}(F)$ is not closed in X_{σ_0}. Let $F_\sigma=\overline{\Pi_\sigma(F)}$ for each $\sigma\in\Sigma$ and $x_{\sigma_0}\in F_{\sigma_0}\setminus\Pi_{\sigma_0}(F)$.

Since the mappings Π_ϱ^σ are closed, $\Pi_\varrho^\sigma(F_\sigma) = F_\varrho$ for all σ, ϱ; thus for each $\sigma \geqslant \sigma_0$ the set $(\Pi_{\sigma_0}^\sigma)^{-1}(x_{\sigma_0}) \cap F_\sigma$ is non-empty. The family $S' = \{A_\sigma, \Pi_\varrho^\sigma|A_\sigma, \Sigma\}$, where $\Sigma' = \{\sigma \in \Sigma: \sigma_0 \leqslant \sigma\}$ and $A_\sigma = (\Pi_{\sigma_0}^\sigma)^{-1}(x_{\sigma_0}) \cap F_\sigma$ for $\sigma \in \Sigma'$, is an inverse system of closed and non-empty subspaces of X_σ with the bonding mappings onto. Moreover $\varprojlim S' = \emptyset$, since $x_{\sigma_0} \notin \Pi_{\sigma_0}(F)$, and $\varprojlim S' \subset F$ by Theorem 2.

3. Quotient and hereditarily quotient mappings. Recall that a mapping $f: X \to Y$ is said to be *hereditarily quotient* if $f(X) = Y$ and for each $A \subset Y$ the restriction $f|f^{-1}(A): f^{-1}(A) \to A$ is quotient. Note that the mapping $f: X \to Y$ is hereditarily quotient if and only if, for each set U open in X and containing $f^{-1}(y)$, the set $\operatorname{Int} f(U)$ is a neighbourhood of y in Y. All open mappings onto and closed mappings onto are hereditarily quotient.

A space X is said to be a *Fréchet space* if for each $x \in \bar{A} \subset X$ there exists a sequence $\{x_n\}$ such that $x \in \lim x_n$ and $x_n \in A$. Every quotient mapping onto a Fréchet T_2-space is hereditarily quotient (see [1] and [2]). Since each metric space is a Fréchet space, it follows that each quotient mapping onto a metric space is hereditarily quotient. From the modification of Gentry's example mentioned in § 1 it follows that the mapping $f = \varprojlim\{\varphi, f_n\}$, where f_n is hereditarily quotient and onto, need not be quotient, even being onto.

However, we have the following

THEOREM 9. *Let $S = \{X_n, \Pi_n^m, N\}$ be an inverse system such that each Π_n^m is hereditarily quotient. Then the projection $\Pi_n: \varprojlim S \to X_n$ is hereditarily quotient for each $n \in N$.*

Proof. First we prove that Π_n is quotient. Let A be a subset of X_n such that $\Pi_n^{-1}(A)$ is open in $X = \varprojlim S$. Suppose that A is not open, i. e. that there exists an $x_n \in A$ such that $x_n \in \operatorname{Fr} A$. Since $\Pi_{n+1}(X) = X_{n+1}$, it follows that $(\Pi_n^{n+1})^{-1}(x_n) \subset \Pi_{n+1}\Pi_n^{-1}(A)$. If we had $(\Pi_n^{n+1})^{-1}(x_n) \subset \operatorname{Int}\Pi_{n+1}\Pi_n^{-1}(A)$, then, Π_n^{n+1} being hereditarily quotient, the set $\operatorname{Int}\Pi_n^{n+1}(\operatorname{Int}\Pi_{n+1}\Pi_n^{-1}(A))$ would be a neighbourhood of the point x_n contained in A, which is impossible, because $x_n \in \operatorname{Fr} A$. Thus there exists an $x_{n+1} \in \operatorname{Fr}\Pi_{n+1}\Pi_n^{-1}(A) \cap (\Pi_n^{n+1})^{-1}(x_{n+1})$. This process may be continued to obtain a thread $\{x_m\}$ such that $x_m \in \operatorname{Fr}\Pi_m\Pi_n^{-1}(A) \cap (\Pi_n^m)^{-1}(x_n)$ for each $m \geqslant n$. Evidently, $x_m \in \overline{\Pi_m(X \setminus \Pi_n^{-1}(A))}$ for each $m \in N$; thus since the set $X \setminus \Pi_n^{-1}(A)$ is closed, by Theorem 2 we have $\{x_m\} \in X \setminus \Pi_n^{-1}(A)$, which is in contradiction with $x_n \in A$. It follows that Π_n is quotient.

Now we show that Π_n is hereditarily quotient. For $Y_n \subset X_n$ we have $\Pi_n^{-1}(Y_n) = \varprojlim\{Y_m, \Pi_n^m|Y_m, N\}$, where

$$Y_m = \begin{cases} \Pi_m^n(Y_n) & \text{for } m \leqslant n, \\ (\Pi_n^m)^{-1}(Y_n) & \text{for } n < m. \end{cases}$$

Since each mapping $\Pi_n^m|Y_m$ for $m \geqslant n$ is hereditarily quotient, from the first part of our proof it follows that the mapping $\Pi_n|\Pi_n^{-1}(Y_n): \Pi_n^{-1}(Y_n) \to Y_n$ is quotient.

In order to show that Theorem 9 does not hold for uncountable systems we shall modify the system $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ described in Example 5, where Π_ϱ^σ were closed and onto but Π_σ was neither closed nor onto, in the following way. We fix a $\sigma_0 \in \Sigma$ and add for all $\varrho > \sigma_0$ one isolated point $\varrho(x)$ to the space X_ϱ for each $x \in X_\sigma \setminus \Pi_\sigma(\varprojlim S)$; we define $\Pi_\tau^\varrho(\varrho(x)) = \tau(x)$ for $\varrho \geqslant \tau > \sigma_0$.

The projection $\Pi_n: \varprojlim S \to X_n$ need not be quotient if the bonding mappings of the system S are quotient.

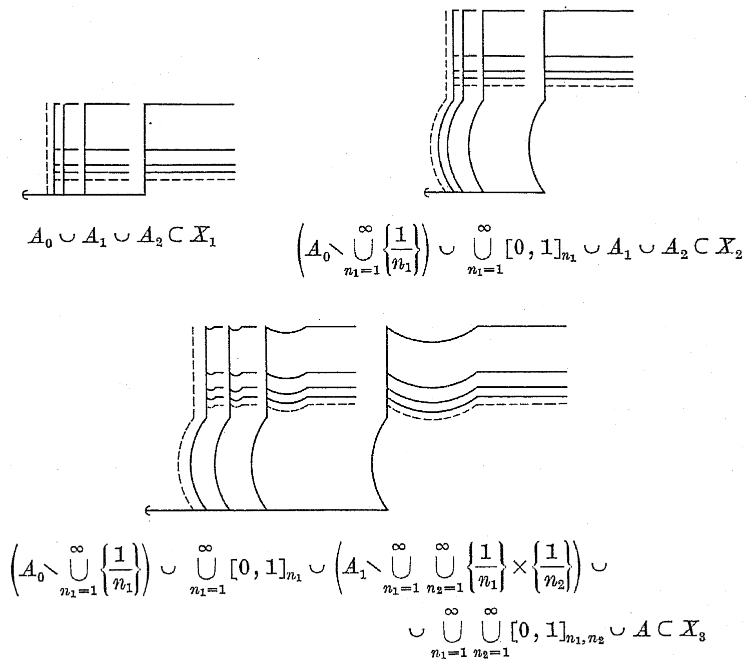

Fig. 1

EXAMPLE 6. We shall define recursively $X_1, X_2, X_3, \ldots$ Let (see Fig. 1)

$$X_1 = \bigcup_{k=0}^{\infty} A_k \cup A,$$

where

$$A_0 = (0, 1],$$

$$A_k = \bigcup_{n_1=1}^{\infty} \bigcup_{n_2=1}^{\infty} \ldots \bigcup_{n_k=1}^{\infty} \left(\left\{\frac{1}{n_1}\right\} \times \left\{\frac{1}{n_2}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times (0, 1]\right) \quad \text{for} \quad k = 1, 2, 3, \ldots,$$

$$A = \left(\bigcup_{\{n_m\} \in N^{\aleph_0}} \left\{\frac{1}{n_m}\right\} \times (0, 1]\right) \Big/ R,$$

and the equivalence relation R is defined by the formula:

$$\left[\left(\left\{\frac{1}{n_m}\right\} \times t\right) R \left(\left\{\frac{1}{n'_m}\right\} \times t'\right)\right] \Leftrightarrow \left[(t = t' = 1) \text{ or } \left(\left\{\frac{1}{n_m}\right\} = \left\{\frac{1}{n'_m}\right\} \text{ and } t = t'\right)\right].$$

The topology in the set X_1 is generated by the following neighbourhood system:

1° The base at the point $\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \in A_{k-1}$ for $k = 1, 2, 3, \ldots$ consists of all sets of the form

$$\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_{k-1}}\right\} \times \left(\frac{1}{n_k} - \frac{1}{l}, \frac{1}{n_k} + \frac{1}{l}\right) \cup B \cup C,$$

$$\text{where} \quad \frac{1}{l} < \frac{1}{n_k} - \frac{1}{n_k+1},$$

$$B = \left[\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times \left(0, \frac{1}{n_{k+1}}\right)\right] \cup \tag{7}$$

$$\cup \left[\bigcup_{n'_{k+1} > n_{k+1}} \left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times \left\{\frac{1}{n'_{k+1}}\right\} \times \left(0, \frac{1}{n_{k+2}(n'_{k+1})}\right)\right] \cup$$

$$\cup \left[\bigcup_{n'_{k+1} > n_{k+1}} \left(\bigcup_{n'_{k+2} > n_{k+2}(n_{k+1})} \left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times \left\{\frac{1}{n'_{k+1}}\right\} \times \left\{\frac{1}{n'_{k+2}}\right\} \times\right.\right.$$

$$\left.\left. \times 0, \left(\frac{1}{n_{k-3}(n'_{k+1}, n'_{k+2})}\right)\right)\right] \cup$$

$$\cup \ldots,$$

$$C = \bigcup_{\{n_m\} \in T} \left(\left\{\frac{1}{n_m}\right\} \times \left(0, \frac{1}{j_{\{n_m\}}}\right)\right), \tag{8}$$

$$T = \left\{\{n_m\} \in N^{\aleph_0}\colon \left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_i}\right\} \in B \text{ for each } i \geqslant k\right\}$$

and $j_{\{n_m\}}, l, n_k, n_{k+1}, n'_{k+1}, n_{k+2}(n'_{k+1}), n'_{k+2}, n_{k+3}(n'_{k+1}, n'_{k+2}), \ldots$ are natural numbers.

2° The base at the point $\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times \{t\} \in A_k$ for $\frac{1}{l+1} < t < \frac{1}{l}$, consists of all sets of the form

$$\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \times \left(t - \frac{1}{l_1}, t + \frac{1}{l_1}\right),$$

$$\text{where} \quad \frac{1}{l_1} < \min\left(t - \frac{1}{l+1}, \frac{1}{l} - t\right) \quad \text{and } l, l_1 \in N.$$

3° For $t \in (0, 1)$ the base at the point $\left[\left\{\frac{1}{n_m}\right\} \times \{t\}\right] \in A$ consists of all sets of the form

$$\left[\left\{\frac{1}{n_m}\right\} \times \left(t - \frac{1}{l}, t + \frac{1}{l}\right)\right] \cap \left[\left\{\frac{1}{n_m}\right\} \times (0, 1)\right], \quad \text{where} \quad l \in N.$$

4° The base at the point $\left[\left\{\frac{1}{n_m}\right\} \times \{1\}\right] \in A$ consists of all sets of the form

$$\bigcup_{\{n_m\} \in N^{\aleph_0}} \left(\left\{\frac{1}{n_m}\right\} \times \left(1 - \frac{1}{l}, 1\right]\right) \Big/ R, \quad \text{where} \quad l \in N.$$

It can easily be verified that X_1 is a T_3-space. The space X_1 is the union of two connected subspaces $\bigcup_{k=0}^{\infty} A_k$ and A; since each point $\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\} \in A_{k-1}$ belongs to $\overline{A}$, the space X_1 is connected.

Take $k > 1$ and assume that X_n are already defined for $n < k$. Let

$$X_{k+1} = \left[X_k \setminus \bigcup_{n_1=1}^{\infty} \ldots \bigcup_{n_k=1}^{\infty} \left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_k}\right\}\right] \cup \bigcup_{n_1=1}^{\infty} \ldots \bigcup_{n_k=1}^{\infty} [0, 1]_{n_1, \ldots, n_k}.$$

The topology in the set X_{k+1} is generated by the following neighbourhood system:

1° The base at the point $1_{n_1, \ldots, n_k} \in [0, 1]_{n_1, \ldots, n_k}$ consists of all sets of the form $\left(1 - \frac{1}{l}, 1\right]_{n_1, \ldots, n_k} \cup B \cup C$, where B and C are defined by the formula (7) and (8), and $l \in N$.

2° The base at the point $0_{n_1, \ldots, n_k} \in [0, 1]_{n_1, \ldots, n_k}$ consists of all sets of the form

$$\left\{\frac{1}{n_1}\right\} \times \ldots \times \left\{\frac{1}{n_{k-1}}\right\} \times \left(\frac{1}{n_k} - \frac{1}{l}, \frac{1}{n_k} + \frac{1}{l}\right) \cup \left[0, \frac{1}{l}\right)_{n_1, \ldots, n_k},$$

where $l \in N$ and $\frac{1}{l} < \frac{1}{n_k} - \frac{1}{n_k+1}$.

3° The base at the point $t_{n_1,\dots,n_k} \in (0,1)_{n_1,\dots,n_k}$ consists of all intervals

$$\Big(t_{n_1,\dots,n_k} - \frac{1}{l},\, t_{n_1,\dots,n_k} + \frac{1}{l}\Big)_{n_1,\dots,n_k} \cap (0,1)_{n_1,\dots,n_k}, \quad \text{where} \quad l \in N .$$

4° For $k \geqslant 2$ the base at the point $1_{n_1,\dots,n_{k-1}} \in [0,1]_{n_1,\dots,n_{k-1}}$ consists of all sets of the form

$$\Big(1-\frac{1}{l},1\Big]_{n_1,\dots,n_{k-1}} \cup \Big[\Big\{\frac{1}{n_1}\Big\}\times \dots \times\Big\{\frac{1}{n_{k-1}}\Big\}\times\Big(0,\frac{1}{l}\Big)\Big] \cup \bigcup_{n_k=l+1}^{\infty}\Big(0,\frac{1}{l}\Big)_{n_1,\dots,n_{k-1},n_k},$$

where $l \in N$.

5° Bases at all remaining points are the same as in X_k.

The space X_{k+1} defined in this way is a connected T_3-space.

We shall now prove that X_n is paracompact for each n. Observe that $X_n = A \cup (X_n \setminus A)$, where A is an open, metrizable subspace of X_n and $X_n \setminus A$ is a Lindelöf space, because it is a countable union of Lindelöf spaces. Let $\mathfrak{U}$ be an arbitrary open covering of X_n. Let $\mathfrak{U}_1$ be a countable subfamily of $\mathfrak{U}$ such that $X_n \setminus A \subset V = \bigcup \mathfrak{U}_1$. It can easily be seen that there exists an open set W satisfying $X_n \setminus A \subset W \subset \overline{W} \subset V$. Let $\mathfrak{U}_3$ be a locally finite open covering of A which is a refinement of $\mathfrak{U}_2 = \{A \cap U\}_{U \in \mathfrak{U}}$. The family $\mathfrak{U}_4 = \{U \cap (X_n \setminus \overline{W})\}_{U \in \mathfrak{U}_3}$ is a covering of $X_n \setminus \overline{W}$; moreover, $\mathfrak{U}_4$ is locally finite in X_n. Indeed, if $x \in X_n \setminus A$, then W is a neighbourhood of x disjoint with all elements of $\mathfrak{U}_4$. And for any $x \in A$ there exists, $\mathfrak{U}_4$ being locally finite in A, a set U, open in A and hence also in X_n, which contains x and meets only a finite number of elements of $\mathfrak{U}_4$. Thus $\mathfrak{U}_1 \cup \mathfrak{U}_4$ is a σ-locally finite open refinement of $\mathfrak{U}$. This proves that X_n is paracompact (see Theorem 5.1.4 in [3]).

Now define the mappings $\Pi_k^{k+1}\colon X_{k+1} \to X_k$ assuming that

$$\Pi_k^{k+1}(t_{n_1,\dots,n_k}) = \Big\{\frac{1}{n_1}\Big\}\times \dots \times \Big\{\frac{1}{n_k}\Big\} \quad \text{for} \quad t_{n_1,\dots,n_k} \in [0,1]_{n_1,\dots,n_k}$$

and

$$\Pi_k^{k+1}(x) = x \quad \text{if} \quad x \neq t_{n_1,\dots,n_k}.$$

It is easy to verify that the mappings Π_k^{k+1} are quotient, monotone (i.e. $(\Pi_k^{k+1})^{-1}(y)$ is connected for $y \in X_k$) and onto, but (except for Π_1^2) are not hereditarily quotient.

Obviously $S = \{X_n, \Pi_n^m, N\}$, where $\Pi_n^m = \Pi_n^{n+1}\Pi_{n+1}^{n+2} \dots \Pi_{m-1}^m$ for $m \geqslant n+1$, is an inverse system.

The projection $\Pi_n\colon \varprojlim S \to X_n$ is not quotient, because the set $\bigcup_{k=0}^{\infty} A_k$ is not open in X_n although the set $\Pi_n^{-1}(\bigcup_{k=0}^{\infty} A_k)$ is open in $\varprojlim S$. Indeed,

for each thread $\{x_m\} \in \Pi_n^{-1}(\bigcup_{k=0}^{\infty} A_k)$ there exists a $k \in N$ such that $x_m = x_k$ for $m \geqslant k$. If $x_k = 1_{n_1,\dots,n_l}$ for some $l \in N$, then there exists a neighbourhood U_{k+2} of the point x_{k+2} in X_{k+2} such that $\Pi_{k+2}^{-1}(U_{k+2}) \subset \Pi_n^{-1}(\bigcup_{k=0}^{\infty} A_k)$. Otherwise, we can find such a neighbourhood of x_{k+1} in X_{k+1}.

Finally, note that the limit $\varprojlim S$ is not connected, because it is the union of two sets $\Pi_n^{-1}(\bigcup_{k=0}^{\infty} A_k)$ and $\Pi_n^{-1}(A)$, open and disjoint.

4. Monotone mappings. Recall that a mapping $f\colon X \to Y$ is *monotone* if for each $y \in Y$ the set $f^{-1}(y)$ is connected.

The following theorem was proved in [4] in the case of countable systems.

THEOREM 10. *Let $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ and $S' = \{Y_{\sigma'}, \Pi_{\varrho'}^{\sigma'}, \Sigma'\}$ be two inverse systems, where all $Y_{\sigma'}$ are T_2-spaces and all $X_{\varphi(\sigma')}$ are compact. Let $\{\varphi, f_{\sigma'}\}$ be a mapping of S into S' such that $f_{\sigma'}$ is monotone for each $\sigma' \in \Sigma'$. Then $f = \varprojlim \{\varphi, f_{\sigma'}\}$ is monotone.*

The proof is exactly the same as in [4] for countable systems.

COROLLARY. *Let $S = \{X_\sigma, \Pi_\varrho^\sigma, \Sigma\}$ be an inverse system of compact spaces such that Π_ϱ^σ is monotone for $\sigma \geqslant \varrho$. Then the projection $\Pi_\sigma\colon \varprojlim S \to X_\sigma$ is monotone.*

Proof. This follows directly from Theorem 1.

If the spaces X_σ are not compact, then, as the following example shows, the above theorem does not hold even if the limit mapping is closed and onto.

EXAMPLE 7. For $n = 1, 2, 3, \dots$ let

$$X_n = \{(x, y) \in R^2\colon x \geqslant 0,\ 0 \leqslant y \leqslant 1\}$$

with the topology of a subspace of R^2 and $A_n = \{(x,y) \in X_n\colon y = 0$ or $y = 1\} \cup \{(x,y) \in X_n\colon x \geqslant n-1\}$. Denote by Y_n the quotient space X_n/R_n, where the equivalence relation R_n in X_n is defined by the formula:

$$(x,y)R_n(x',y') \Leftrightarrow [(x,y) = (x',y') \text{ or } (x,y),(x',y') \in A_n].$$

Let $f_n\colon X_n \to Y_n$ be the natural quotient mapping, $\Pi_n^m\colon X_m \to X_n$ the identity mapping and $\Pi_n^m([(x,y)]) = [(x,y)] \in Y_n$ for $[(x,y)] \in Y_m$. Then $S = \{X_n, \Pi_n^m, N\}$ and $S' = \{Y_n, \Pi_n^m, N\}$ are two inverse systems and $\{\mathrm{id}_N, f_n\}$ is a mapping of S into S' such that the mappings $\Pi_n^m\colon X_m \to X_n$, $\Pi_n^m\colon Y_m \to Y_n$ and $f_n\colon X_n \to Y_n$ are monotone, closed and onto, and the limit mapping $f = \varprojlim \{\mathrm{id}_N, f_n\}$ is closed and onto but is not monotone.

The corollary to Theorem 10 is not true if the spaces X_σ are not compact; this can be seen from Example 6. We give below a simpler example, but the bonding mappings of the given inverse system are not quotient (see the Corollary to Theorem 10).

EXAMPLE 8. Define the inverse system $S = \{X_n, \Pi_n^m, N\}$ by assuming that:

1° $X_n = A_n \cup B_n \cup C_n$, where $A_n = \{(x, y) \in R^2\colon -1 \leqslant x < 0,\ 0 \leqslant y \leqslant 1\}$, $B_n = \{(x, y) \in R^2\colon\ x = 0,\ 1 - \frac{1}{n} \leqslant y < 1\}$ and $C_n = \left\{(x, y) \in R^2\colon 0 < x \leqslant 1 - \frac{1}{n},\ y = 1 - \frac{1}{n}\right\}$ for each $n \in N$ (see Fig. 2).

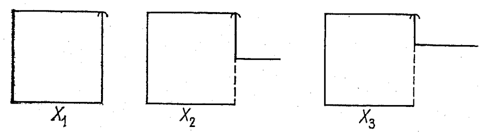

Fig. 2

2° For $m \geqslant n$ and $(x, y) \in X_m$

$$\Pi_n^m\big((x, y)\big) = \begin{cases} \left(0, 1 - \frac{1}{m} - x\right) & \text{if } (x, y) \text{ satisfies } 0 < x \leqslant \frac{1}{n} - \frac{1}{m}, \\ \left(x - \left(\frac{1}{n} - \frac{1}{m}\right), 1 - \frac{1}{n}\right) & \text{if } (x, y) \text{ satisfies } \frac{1}{n} - \frac{1}{m} < x \leqslant 1 - \frac{1}{m}, \\ (x, y) & \text{otherwise.} \end{cases}$$

The spaces X_n are connected and the bonding mappings Π_n^m are monotone and onto. The limit $\varprojlim S$ is not connected, because $\varprojlim S = U_1 \cup U_2$, where

$$U_1 = \varprojlim \{A_n, \Pi_n^m | A_m, N\}$$

and

$$U_2 = \varprojlim \{B_n \cup C_n, \Pi_n^m | B_n \cup C_n, N\}$$

are open and disjoint.

EXAMPLE 9. Let X_n and Π_n^m be as in Example 8. Let $X_0 = \{(0, 0)\}$ and $\Pi_0^m\big((x, y)\big) = (0, 0)$ for $(x, y) \in X_m$ and $m \geqslant 1$. The bonding mappings of the inverse system $S_0 = \{X_n, \Pi_n^m, N \cup \{0\}\}$ are monotone and onto, but the projection Π_0 is not monotone.

We shall now prove that the limit of the inverse system of connected spaces is connected under an additional assumption.

THEOREM 11. *Let $S = \{X_n, \Pi_n^m, N\}$ be an inverse system of connected spaces such that the bonding mappings Π_n^m are monotone, hereditarily quotient and onto. Then the limit $\varprojlim S$ is connected.*

Proof. Suppose that $\varprojlim S = U_1 \cup U_2$, where U_1 and U_2 are open, non-empty and disjoint. By Theorem 9 the mapping $\Pi_n\colon \varprojlim S \to X_n$ is hereditarily quotient. Suppose that $A_n = \Pi_n(U_1) \cap \Pi_n(U_2) = \emptyset$ for some $n \in N$. Then $U_i = \Pi_n^{-1}\Pi_n(U_i)$ for $i = 1, 2$ and $X_n = \Pi_n(U_1) \cup \Pi_n(U_2)$. Since Π_n is quotient, the sets $\Pi_n(U_1)$ and $\Pi_n(U_2)$ are open, non-empty and disjoint, but this is impossible by the connectedness of X_n; thus all sets A_n are not empty.

Clearly $\Pi_n^m(A_m) \subset A_n$ for $m \geqslant n$. We shall show that $\Pi_n^m(A_m) = A_n$. Take $x_n \in A_n$; since Π_n^m is monotone, the set $(\Pi_n^m)^{-1}(x_n)$ is connected. Let $B_i = (\Pi_n^m)^{-1}(x_n) \cap \Pi_m(U_i)$; obviously $B_1 \cup B_2 = (\Pi_n^m)^{-1}(x_n)$. To see that $B_1 \cap B_2 \neq \emptyset$ suppose the contrary. Then $\Pi_m^{-1}(B_i) = U_i \cap \Pi_m^{-1}(\Pi_n^m)^{-1}(x_n)$ and this set is open in $\Pi_m^{-1}(\Pi_n^m)^{-1}(x_n)$. Since the mapping $\Pi_m | \Pi_m^{-1}(\Pi_n^m)^{-1}(x_n)\colon \Pi_m^{-1}(\Pi_n^m)^{-1}(x_n) \to (\Pi_n^m)^{-1}(x_n)$ is quotient, the set B_i is open in $(\Pi_n^m)^{-1}(x_n)$ for $i = 1, 2$, which contradicts the assumption that $(\Pi_n^m)^{-1}(x_n)$ is connected.

The family $S' = \{A_n, \Pi_n^m | A_n, N\}$ is an inverse system of non-empty spaces with the bonding mappings onto. Thus the limit $\varprojlim S'$ is non-empty. Since the sets U_i are closed, by Theorem 2 we have $\varprojlim S' \subset U_1 \cap U_2$, which contradicts the assumption that $U_1 \cap U_2 = \emptyset$.

COROLLARY. *Let $S = \{X_n, \Pi_n^m, N\}$ be an inverse system such that the bonding mappings are monotone, hereditarily quotient and onto. Then the projection $\Pi_{n_0}\colon \varprojlim S \to X_{n_0}$ is monotone.*

Proof. Take $x_{n_0} \in X_{n_0}$; clearly $\Pi_{n_0}^{-1}(x_{n_0}) = \varprojlim \{A_n, \Pi_n^m | A_m, N\}$, where

$$A_n = \begin{cases} (\Pi_{n_0}^n)^{-1}(x_{n_0}) & \text{for} \quad n > n_0, \\ \Pi_n^{n_0}(x_{n_0}) & \text{for} \quad n \leqslant n_0. \end{cases}$$

Since each mapping Π_n^m is monotone, hereditarily quotient and onto, each A_n is connected and $\Pi_n^m | A_m\colon A_m \to A_n$ is monotone, hereditarily quotient and onto for each $m, n \in N$. Thus, by Theorem 11 the limit $\varprojlim \{A_n, \Pi_n^m | A_m, N\} = \Pi_{n_0}^{-1}(x_{n_0})$ is connected.

One can check that all the assumptions of Theorem 11 are essential.

The author does not know the answer to the following questions:

1. Does there exist an uncountable system of connected spaces with bonding mappings monotone, onto and hereditarily quotient (better: open or closed) and with a disconnected limit?

2. Does there exist an uncountable system $S = \{X_\sigma, \Pi^\sigma_\varrho, \Sigma\}$ with bonding mappings monotone, onto and hereditarily quotient (better: open or closed) such that the projection Π_n is not monotone?

Clearly a positive answer to problem 1 gives a positive answer to problem 2.

The main results of this paper are described in the following tables.

Table 1

$f_{\sigma'}$		open	closed	perfect	quotient	hereditarily quotient	monotone
$\lim\limits_{\leftarrow}\{\varphi, f_{\sigma'}\}$	uncountable systems	— + for $\{\varphi, f_{\sigma'}\}$ limit-exact	—	+ for $X_\sigma \in T_2$	—	—	— + for X_σ compact and $Y_{\sigma'} \in T_1$
	countable systems	— + for $\{\varphi, f_{\sigma'}\}$ exact	—	+ for $X_\sigma \in T_2$	—	—	— + for X_σ compact and $Y_{\sigma'} \in T_1$

Table 2

Π^σ_ϱ		open	closed	perfect	quotient	hereditarily quotient	monotone
Π_σ	uncountable systems	—	—	+ for $X_\sigma \in T_2$	—	—	— + for X_σ compact
	countable systems	+ for Π^σ_ϱ onto	+	+ for $X_\sigma \in T_2$	—	+	— + for X_σ compact + for Π^σ_ϱ onto and hereditarily quotient

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