Closed, continuous images of complete metric spaces

by

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Abstract. Every closed, continuous image of a complete metric space is shown to contain a dense subspace which is metrizable in a complete manner; hence, the Baire Category theorem is valid for closed, continuous images of a complete metric spaces. Also, if X is a closed, continuous image of a complete metric space, then there is a closed, continuous image Y of a complete metric space such that every open subset of Y contains a copy of X. Thus, a closed, continuous image of a complete metric space need not be a countable union of closed, metrizable subspaces.

Lašnev [2] has constructed a Lašnev space (Lašnev = closed, continuous image of a metric space) which is not first countable at any of its points. It is easily shown that if X is a regular space, M is a dense subset of X, and p is a point of M at which M, regarded as space, is first countable, then X is first countable at p. Hence Lašnev's space contains no dense metrizable subspace.

It is shown herein that every closed, continuous image of a *complete* metric space contains a dense subspace which is metrizable in a complete manner. It follows from this result that the Baire Category theorem is valid for closed, continuous images of complete metric spaces.

The author [7] has shown that if X is a Lašnev space which contains a dense metrizable subspace, then there is a Lašnev space Y such that every open subset of Y contains a copy of X. Thus, if X is a closed, continuous image of a complete metric space, then there is a Lašnev space Y such that every open subset of Y contains a copy of X. This result is strengthened herein by showing that there is a closed, continuous image Y of a *complete* metric space such that every open subset of Y contains a copy of X.

Fitzpatrick [1] has constructed a Lašnev space which is not a countable union of closed, metrizable subspaces. S. A. Stricklen has observed, in work as yet unpublished, that every closed, continuous image of a locally compact metric space is a countable union of closed, metrizable subspaces. It is shown herein that there is a closed, continuous image of a complete metric space which is not a countable union of closed, metrizable subspaces.

The author wishes to thank Professors David Lutzer and Frank Slaughter, Jr., for pointing out an inadequacy in the argument first given for the following theorem.

THEOREM 1. *Every closed, continuous image of a complete metric space contains a dense subspace which is metrizable in a complete manner.*

Proof. Suppose f is a closed, continuous mapping of a complete metric space X onto a topological space Y. Lašnev [2] has shown that there is a closed subset K of X such that $f(K) = Y$ and if H is a closed, proper subset of K, then $f(H)$ is a proper subset of Y. Let

$$G = \{(f|K)^{-1}(y) |\ y \in Y\}$$

and let

$$M = \{y \in Y |\ (f|K)^{-1}(y) \text{ is compact}\}.$$

Morita and Hanai [3] and Stone [4] have shown that M is a metrizable subspace of Y.

Suppose there is an element g' of G such that $f(g')$ is in $Y-M$ and g' contains an open subset of K; then g' has a boundary which is a proper subset of g'. If g is an element of G which has a boundary, let B_g denote the boundary of g; if g is an element of G which has no boundary, let B_g denote a degenerate subset of g. Let $H = \bigcup_{g \in G} B_g$: then H is a closed, proper subset of K such that $f(H) = Y$. This is a contradiction; therefore, if g is an element of G such that $f(g)$ is in $Y-M$, then g does not contain an open subset of K.

Suppose the set M is not dense in Y; then $Y-M$ contains an open subset of Y, so that $(f|K)^{-1}(Y-M)$ contains an open subset of K. Lašnev has shown that $Y-M = \bigcup_{i=1}^{\infty} N_i$, where N_i is discrete in Y; hence, $(f|K)^{-1}(Y-M) = \bigcup_{i=1}^{\infty} (f|K)^{-1}(N_i)$. Thus, $(f|K)^{-1}(Y-M)$ is a countable union of closed subsets of K, no one of which contains an open subset of K, since $(f|K)^{-1}(N_i)$ is the union of the elements of a discrete collection of closed subsets of K, no one of which contains an open subset of K. This contradicts the Baire Category theorem for complete metric spaces, since K is a complete metric subspace of X; therefore, M is dense in Y.

Vaĭnsteĭn [5] has shown that a metrizable closed, continuous image of a complete metric space is metrizable in a complete manner; hence, M is metrizable in a complete manner, since $(f|K)^{-1}(M)$ is a G_δ set in K.

COROLLARY 1. *The Baire Category theorem is valid for closed, continuous images of complete metric spaces.*

Proof. It follows from Theorem 1 that if Y is a closed, continuous image of a complete metric space, then there is a dense subspace M of Y such that the Baire Category theorem is valid for the space M. It is easily shown that if the Baire Category theorem is valid for a dense subspace of a topological space, then it is valid for the space itself.

COROLLARY 2. *If Y is a non-metrizable closed, continuous image of a complete metric space and Y is a countable union of closed, metrizable subspaces of Y, then the set of all points at which Y is not first countable is not dense in Y.*

Proof. Suppose $Y = \bigcup_{n=1}^{\infty} Y_n$, where Y_n is a closed, metrizable subspace of Y. Let H denote the set of all points of Y at which Y is not first countable. Suppose H is dense in Y. Suppose, furthermore, that there is a positive integer n such that Y_n contains an open subset U of Y; then U contains a point p of H. It is easily shown that Y is first countable at p, since Y_n is first countable at p and U is an open subset of Y which contains p and lies wholly in Y_n. This is a contradiction, since p is a point of H; therefore, if n is a positive integer, then Y_n does not contain an open subset of Y. Thus, Y is a countable union of closed subsets of Y, no one of which contains an open subset of Y. This contradicts Corollary 1; hence, H is not dense in Y.

THEOREM 2. *If X is a closed, continuous image of a complete metric space, then there is a closed, continuous image Y of a complete metric space such that every open subset of Y contains a copy of X.*

Proof. Construct an inverse sequence: $Y_1 \overset{f_1^2}{\leftarrow} Y_2 \overset{f_2^3}{\leftarrow} Y_3 \ldots$ as in the proof of Theorem 2 of [7], using $Y_1 = X =$ a closed, continuous image of a complete metric space; this is possible, since X contains a dense metrizable subspace. Furthermore, it is possible to obtain complete metric spaces for the spaces $X_1, X_2, X_3, \ldots$ Let Y' denote the subspace of the inverse limit space Y as described in the proof of Theorem 2 of [7]; then Y' is a dense subspace of Y such that every open subset of Y' contains a copy of X.

Let $G = \{\prod_{n=1}^{\infty} f_n^{-1}(y_n) |\ (y_1, y_2, y_3, \ldots) \in Y\}$; then G is upper semi-continuous, as indicated in the proof of Theorem 1 of [7]. Hence, $f^{-1}(Y)/G$ is a closed, continuous image of a complete metric space, since $f^{-1}(Y)$ is a closed subset of $\prod_{n=1}^{\infty} X_n$ and is, consequently, a complete metric space. Let $G' = \{\prod_{n=1}^{\infty} f_n^{-1}(y_n) |\ (y_1, y_2, y_3, \ldots) \in Y'\}$; then Y' is homeomorphic to $f^{-1}(Y')/G'$, so that every open subset of $f^{-1}(Y')/G'$ contains a copy of X.

Suppose $y = (y_1, y_2, y_3, \ldots)$ is a point of Y. Let $x = (x_1, x_2, x_3, \ldots)$

denote a point of $f^{-1}(y)$. For each positive integer n let $z_n = (z_{1n}, z_{2n}, z_{3n}, \ldots)$ denote the point of Y' such that $z_{kn} = y_k$ for $k = 1, 2, 3, \ldots, n$ and $z_{k+1,n} = (z_{kn}, z_{kn})$ for $k = n, n+1, n+2, \ldots$; then the sequence $z_1, z_2, z_3, \ldots$ converges to y in Y. For each positive integer n, let $w_n = (w_{1n}, w_{2n}, w_{3n}, \ldots)$ denote a point of $f^{-1}(z_n)$ such that $w_{kn} = x_n$ for $k = 1, 2, 3, \ldots, n$ and w_{kn} belongs to $f_k^{-1}(z_{kn})$ for $k = n+1, n+2, n+3, \ldots$; then the sequence $w_1, w_2, w_3, \ldots$ converges to x in $\prod_{n=1}^{\infty} X_n$. Thus, $f^{-1}(Y')/G'$ is dense in $f^{-1}(Y)/G$. It is easily shown that if every open subset of a dense subspace of a topological space contains a copy of X, then every open subset of the space itself contains a copy of X. Hence, every open subset of $f^{-1}(Y)/G$ contains a copy of X.

If $X = S$, the space described in [6], then $f^{-1}(Y)/G$ is a closed, continuous image of a complete metric space such that the set of all points at which $f^{-1}(Y)/G$ is not first countable is dense in $f^{-1}(Y)/G$. It follows from Corollary 2 that $f^{-1}(Y)/G$ is not a countable union of closed, metrizable subspaces. Therefore, closed, continuous images of complete metric spaces need not be countable unions of closed, metrizable subspaces.

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Reçu par la Rédaction le 21. 3. 1972

X^m is homeomorphic to X^n iff $m \sim n$ where $\sim$ is a congruence on natural numbers

by

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Abstract. If a congruence $\sim$ on the additive semigroup of all natural numbers is given then a locally compact separable metric space X is constructed such that X^m is homeomorphic to X^n iff $m \sim n$.

Let X be a topological space. Define an equivalence $\sim$ on the set N of all natural numbers such that $m \sim n$ iff X^m is homeomorphic to X^n. Clearly, $\sim$ is a congruence on the additive semigroup $(N, +)$. In the paper, the following theorem is proved:

THEOREM. *For every congruence $\sim$ on the additive semigroup of all natural numbers there exists a locally compact separable metric space X such that X^m is homeomorphic to X^n iff $m \sim n$.*

The analogical results for Abelian groups and modules are shown in [3] and [1] respectively, but the proofs are quite different. Concerning the terminology, see [4].

1. Productively independent spaces.

CONVENTION. Denote by N the set of all natural numbers. If X is a topological space then, as usual, $X^1 = X$, $X^{n+1} = X \times X^n$, X^0 is a one-point space.

DEFINITION. A set $\boldsymbol{X}$ of topological spaces is said to be *productively independent* if, whenever $\{k_X;\ X \in \boldsymbol{X}\}$, $\{h_X;\ X \in \boldsymbol{X}\}$ are two collections of non-negative integers and $\prod_{X \in \boldsymbol{X}} X^{k_X}$ is homeomorphic to $\prod_{X \in \boldsymbol{X}} X^{h_X}$, then $k_X = h_X$ for all $X \in \boldsymbol{X}$. We recall that a set $\boldsymbol{X}$ of topological spaces is said to be *rigid* if, whenever $f\colon X \to Y$ is a continuous mapping, $X, Y \in \boldsymbol{X}$, then either f is a constant or $X = Y$ and f is the identity. Every element of a rigid set is called a *rigid space*.

LEMMA 1. *Let $\{X, Y\}$ be a rigid set, $m, n \in N$, and $f\colon X^n \to Y^m$ a continuous mapping. Then f is a constant.*

Proof. Let $g\colon X^n \to Y$ be a continuous mapping. Choose $x \in X$. Denote by x^i the point of X^i whose coordinates are all x. Put $a = g(x^n)$. Since