The connections between different notions of a Boolean elementary subsystem may be visualised by means of the following diagram.

$$\begin{array}{ccc} \mathfrak{S} \precsim_s^s \mathfrak{S}' & \rightarrow & \mathfrak{S} \precsim_s \mathfrak{S}' \\ \downarrow & & \downarrow \not\uparrow \\ \mathfrak{S} \precsim^s \mathfrak{S}' & \rightarrow & \mathfrak{S} \precsim \mathfrak{S}' \\ \downarrow & & \downarrow \\ \mathfrak{S} \precsim_w^s \mathfrak{S}' & \leftrightarrow & \mathfrak{S} \precsim_w \mathfrak{S}' \end{array}$$

PROBLEM 1. How to complete the diagram? Which implications hold and which fail?

We have proved the upper and lower Löwenheim-Skolem-Tarski theorems for $\precsim$. Thus from Theorem 5 it follows that both theorems hold for $\precsim_w$ and $\precsim_w^s$. From Guzicki's example it follows that the lower Löwenheim-Skolem-Tarski theorem fails for $\precsim_s$ and therefore for $\precsim_s^s$. However, if we introduce into the lower Löwenheim-Skolem-Tarski theorem some stronger assumptions (see [1] Theorem 4.3.1), it will hold for $\precsim_s$.

PROBLEM 2. Does the lower Löwenheim-Skolem-Tarski theorem hold for $\precsim^s$? Does the upper Löwenheim-Skolem-Tarski theorem hold for $\precsim^s$, $\precsim_s$ and $\precsim_s^s$?

A partial answer to Problem 2 is given in [1] (see page 63). Namely, for any Boolean-valued relational system $\mathfrak{S}$ and any cardinal number μ there is a Boolean-valued relational system $\mathfrak{S}'$ of the power at least μ such that $\mathfrak{S} \precsim_s \mathfrak{S}'$.

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A non-symmetric generalization of the Borsuk-Ulam theorem (*)

by

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Abstract. The following generalization of the well-known Borsuk-Ulam theorem is proved. Theorem: Let X be a compact subset of the Euclidean space R^{n+1} which disconnects R^{n+1} in such a way that the origin is in a bounded component of $R^{n+1}-X$ and let $f\colon X \to R^n$ be a map. Then there exist two points x, y in X, lying on opposite rays from the origin (i.e. $y = -\lambda x$ for some $\lambda > 0$), such that $f(x) = f(y)$. This provides an affirmative answer to a question of Borsuk. The proof is based on P. A. Smith's theory of the index of a periodic transformation acting on a topological space, Yang's result about maps from such spaces to the Euclidean spaces and the technique of approximating the set X by a special class of polyhedra, the so-called "regular polyhedra" defined in the paper. The special cases $n = 1$ or 2 of the theorem were proved earlier by Siekłucki by a different argument.

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1. Introduction. The classic Borsuk-Ulam theorem [2] states that if f is a map from the n-dimensional sphere S^n into the n-dimensional Euclidean space R^n then there exists a pair of antipodal points $\{x, -x\}$ on S^n such that $f(x) = f(-x)$. Several generalizations of this theorem, proceeding in various directions, have been obtained among others by Agoston [1], Jaworowski [8], Yang [14], Granas [6] who extended the result to infinite-dimensional Banach spaces and Munkholm [9] who considered Z_p-actions on a homology sphere for a prime p. In many of

(*) The main result of this paper was included in the author's dissertation submitted to Indiana University in partial fulfillment of the requirements for the degree Doctor of Philosophy in April 1972. The author would like to thank his advisor, Prof. Jan Jaworowski for his help.

these generalizations (see for example, [14], [8] and [1]) the sphere is replaced by a more general space on which some suitable notion of antipodality can be defined. The generalization to be proved in this paper is also of this kind. We replace the n-sphere S^n by an arbitrary compact subset X of R^{n+1} which disconnects the latter in such a way that the origin O of R^{n+1} lies in a bounded component of $R^{n+1}-X$ (we abbreviate this to say that X separates O from ∞). For our purpose, two points in X are antipodal if they lie on opposite rays from the origin. Thus our main theorem is as follows.

THEOREM A. *Let X be a compact subset of R^{n+1} which disconnects R^{n+1} in such a way that the origin lies in a bounded component of $R^{n+1}-X$. Then given any map $f\colon X\to R^n$ there exist two points x and y in X lying on opposite rays from the origin* (*that is*, $y=-\lambda x$ *for some* $\lambda>0$) *such that* $f(x)=f(y)$.

It should be noted that the points x and y whose existence is asserted in this theorem are related to each other only by the geometric condition that they lie on opposite rays from the origin and not by any functional relation. In particular, they need not be the images of each other under the action of some involution on X. In fact although we shall work with a space with an involution which is closely related to X, the space X itself may not have any involution acting upon it. In this sense Theorem A is "non-symmetric" in nature and bears some resemblance to the results obtained by Hopf [7], Noguchi [11] and Agoston [1].

Theorem A provides an affirmative answer to a question raised by Borsuk at the International Symposium on Topology and its Applications, held in Herceg-Novi in 1968 (see [4], p. 344). The special cases when $n=1$ or 2 have been proved by Sieklucki in [12]. However, the proof there depends heavily on the fact that S^{n-1} can be given the structure of a topological abelian group for these values of n. Since the non-existence of such a group structure on S^{n-1} is well-known for other values of n, Sieklucki's proof cannot be applied for any other value of n.

Our line of approach can be described as follows. Suppose X is a compact subset of R^{n+1} separating O from ∞. We consider the space $A(X)$ called the antipodal space of X and defined by $A(X)=\{(x,y)\in X\times X\mid y=-\lambda x \text{ for some } \lambda>0\}$. There is a fixed-point-free involution T on $A(X)$ defined by $T(x,y)=(y,x)$. A map $f\colon X\to R^n$ induces a map $g\colon A(X)\to R^n$ defined by $g(x,y)=f(x)$. The desired result about f can be obtained by an application of a theorem of Yang ([14], p. 270) to the map g. But in order to do the latter we must first show that the Smith index of $(A(X);T)$ is at least n. We have not succeeded in doing this in general. However if X happens to be what we call a regular polyhedron (defined in Section 3) then it can be shown that the Smith index of $(A(X);T)$ is n. Fortunately the class of regular polyhedra turns out to be large enough so that any given neighborhood of X always contains a regular polyhedron. Thus, in this sense an arbitrary compact subset of R^{n+1} which separates O from ∞ can be approximated by regular polyhedra. The desired result about such a set is established by first proving it in the case of a regular polyhedron and then applying the limiting process. The technique of approximation by polyhedra was introduced by Sieklucki [12]. What we use here is a sharper version of it. Namely, not only are our approximating sets polyhedra, but they are polyhedra of a very special kind.

In section 2 we list facts from P. A. Smith's theory of fixed-point-free involutions as well as Yang's result which are vital for our work. In section 3 we introduce the notion of a regular polyhedron and prove that if X is a regular polyhedron then the Smith index of $(A(X);T)$ is n. In section 5 we show how an arbitrary compact subset of R^{n+1} separating O from ∞ can be approximated by regular polyhedra. Section 4 is preparatory for section 5. In the last section we combine the machinery developed in sections 3 and 5 with Yang's result to obtain the desired generalization of the Borsuk-Ulam theorem.

2. Smith theory and Yang's result. Throughout this and the remaining sections by a space we shall mean a compact metric space. The homology groups will always have coefficients in Z_2 and consequently Z_2 may be suppressed from the notation. Unless otherwise stated, we shall use Čech homology. Of course, if X has the same homotopy type as a compact polyhedron, then it does not matter which homology theory is used. By abuse of language we shall sometimes denote certain cycles on a space by subsets of that space. For example, when we say, "consider the n-cycle S^n on S^n" we mean "consider a suitable triangulation of S^n and consider the simplicial n-cycle whose carrier is S^n". Similarly if $f\colon X\to Y$ is a map and z is a cycle on X, we shall denote by $f(z)$ the image cycle on Y. Also by abuse of language, we shall use the same symbol, when confusion is not likely, to denote a Euclidean complex and its polyhedron. A similar statement holds when we consider cell complexes. The cell complexes we consider are a very special case of Whitehead's CW complexes, in that the cells are homeomorphic to Euclidean disks and the attaching maps are embeddings. If K is a triangulation of a polyhedron X and $Y\subset X$ corresponds to a subcomplex of K, then this subcomplex will be denoted by $K|Y$. A similar notation will be used if K is a cell structure on X.

A homeomorphism T of a space X onto itself is said to be of *period* n, if the order of T in the group of all homeomorphisms of X onto itself is n. The theory of homeomorphisms of finite period was developed by P. A. Smith and a treatment of this can be found in [13]. A homeomorphism of period 2 is called an *involution*. We shall be exclusively concerned with involutions which have no fixed point (i.e. a point which is mapped onto itself by the involution). Smith's theory takes a particularly simple form when applied to involutions. Despite this, in the interest of space it is not possible to give here a complete account of the results we shall need from Smith theory. A detailed exposition is given by Yang, [14]. We shall therefore list down only those definitions and theorems which will be directly used in this work and refer the reader to [14] for details.

A *T-space* is a pair $(X; T)$ where X is a space and T is an involution on X. A subset A of X is said to be *T-invariant* if $T(A) \subset A$ (and hence $T(A) = A$).

The same symbol T can be used without confusion to denote involutions on different spaces at the same time. If $(X; T)$ is a T-space and $A \subset X$ is T-invariant, then there is an involution on A, induced by the involution T on X. This induced involution will again be denoted by T itself. Occasionally when the involution on X is of a very special kind, a different symbol may be used to emphasize this fact.

If $(X; T)$, $(Y; T)$ are T-spaces, a map $f\colon X \to Y$ is said to be a *T-map* or an *equivariant map* if $Tf = fT$.

A T-space $(X; T)$ is said to be *simplicial* if there is a triangulation K of X such that T maps each simplex of K onto some other simplex of K. Obviously when this is the case, simplices of a given dimension are permuted among themselves under the action of T. If $T\colon X \to X$ is simplicial with respect to a triangulation K of X, we say K is a *symmetric triangulation of* X. When this is the case, we also say that *T acts simplicially*. Note that if T acts simplicially on X, then the T-space $(X; T)$ is simplicial, but the converse is not true. For example, let $X = [0,1] \cup [2,3]$ and define T by

$$T(x) = \begin{cases} x^2+2 & \text{if } x \in [0,1], \\ (x-2)^{1/2} & \text{if } x \in [2,3]. \end{cases}$$

Clearly $(X; T)$ is simplicial although it is impossible to triangulate X in such a way that T itself acts simplicially.

In developing the theory of special homology groups it suffices to deal with simplicial T-spaces. Whatever is true for such T-spaces is of course true for those where T acts simplicially.

Let $(X; T)$ be a simplicial T-space. We recall that the homology groups are assumed to have coefficients in Z_2. For an integer $p \geqslant 0$, let $C_p(X)$, $Z_p(X)$, $B_p(X)$ and $H_p(X)$ be respectively the pth chain, cycle, boundary and homology groups of X (in some triangulation given by the definition of a simplicial T-space). Since T permutes the p-dimensional simplices of X among themselves it induces a map, also denoted by T, $T\colon C_p(X) \to C_p(X)$. A chain c in $C_p(X)$ is called a *T-chain* if $T(c) = c$. It is not hard to show that c is a T-chain if and only if $c = d + T(d)$ for some d in $C_p(X)$. The T-chains in $C_p(X)$ form a subgroup of $C_p(X)$. This subgroup is denoted by $C_p(X; T)$. Note that T commutes with the boundary operator ∂ and so ∂ maps $C_p(X; T)$ into $C_{p-1}(X; T)$.

We define $Z_p(X; T) = C_p(X; T) \cap Z_p(X)$ and $B_p(X; T) = \partial C_{p+1}(X; T)$. Then $B_p(X; T) \subset Z_p(X; T)$ and we define $H_p(X; T) =$ the factor group $Z_p(X; T)/B_p(X; T)$. Elements of $Z_p(X; T)$ are called *p-dimensional T-cycles*, or *invariant p-cycles* or *symmetric p-cycles*; those of $B_p(X; T)$ are called *p-dimensional T-boundaries* or *invariant p-boundaries* or *symmetric p-boundaries*. $H_p(X; T)$ is called the *p-dimensional, special Smith homology group of* $(X; T)$. The term "equivariant" is also used in place of "invariant".

If $(X; T)$ and $(Y; T)$ are simplicial T-spaces and if $f\colon X \to Y$ is a simplicial, equivariant map, then f induces a homomorphism $(f_*)_p\colon H_p(X; T) \to H_p(Y; T)$. We shall often drop the subscript p from $(f_*)_p$ and write $f_*\colon H_p(X; T) \to H_p(Y; T)$. The operation of associating f_* to f is functorial.

To define the special Smith homology groups of an arbitrary T-space $(X; T)$, we consider the inverse system of the nerves of certain coverings of X and take the inverse limit. These groups are also denoted by $H_p(X; T)$ etc., and they turn out to be isomorphic to $H_p(X; T)$ as defined before in case $(X; T)$ is simplicial to start with. Again, an equivariant map $f\colon X \to Y$ between two T-spaces $(X; T)$ and $(Y; T)$ induces homomorphisms $f_*\colon H_p(X; T) \to H_p(Y; T)$ in a functorial way. We omit the details of these definitions since they will not be needed directly.

If $(S^n; T)$ is the n-sphere S^n with T the antipodal involution, then it can be shown that

$$H_p(S^n; T) \begin{cases} \cong Z_2 & \text{for } 0 \leqslant p \leqslant n, \\ = 0 & \text{for } p > n. \end{cases}$$

Next we come to the important concept of the so-called index homomorphism. For every T-space $(X; T)$ and for every integer $p \geqslant 0$, there is a homomorphism $\nu\colon H_p(X; T) \to Z_2$ called the index homomorphism. (A better notation for this would be $\nu_{X,T,p}$. But ν alone will not cause any confusion.) Again we omit the definition of ν. The index homomorphism is natural in the sense of the following theorem.

(2.1). THEOREM. *If $f\colon X \to Y$ is an equivariant map between T-spaces then $\nu f_* = \nu$, where on the left hand side ν means $\nu\colon H_p(Y; T) \to Z_2$ and on the right hand side ν means $\nu\colon H_p(X; T) \to Z_2$.*

(2.2). THEOREM. *For every T-space $(X; T)$ there is an integer n such that $\nu\colon H_p(X; T) \to Z_2$ is an epimorphism for $0 \leqslant p \leqslant n$ and is the zero map for $p > n$.*

It is interesting to note that this theorem holds even when X itself is not finite dimensional. (We, of course, assume, as always that X is compact and metric.) The integer n is called the *Smith index* (or simply the *index*) of the T-space $(X; T)$. We shall denote it by $\mathrm{Ind}(X; T)$.

An immediate consequence of (2.1) is,

(2.3). THEOREM. *If $f\colon X\to Y$ is an equivariant map between two T-spaces then* $\mathrm{Ind}(X; T)\leqslant \mathrm{Ind}(Y; T)$.

If T is the antipodal involution on S^n, then it turns out that the index of $(S^n; T)$ is n. This fact, combined with the last theorem gives,

(2.4). THEOREM. *If $f\colon X\to S^n$ is an equivariant map, then* $\mathrm{Ind}(X; T)\leqslant n$. *Equality holds if and only if the map $f_*\colon H_n(X; T)\to H_n(S^n; T)$ induced by f is non-zero. If, moreover, $(X; T)$ and the map f are simplicial then equality holds if and only if there is a symmetric n-cycle z on X which is mapped onto the non-trivial n-cycle on S^n.*

It should be noted that the word "symmetric" in the statement of the last theorem cannot be dropped. It may happen that the map $f_*\colon H_n(X)\to H_n(S^n)$ is non-zero but still the map $f_*\colon H_n(X; T)\to H_n(S^n; T)$ is the zero map. As an example let $X = S^n\times\{-1, 1\}$. Define $T\colon X\to X$ by $T(x, t) = (-x, -t)$ and $f\colon X\to S^n$ by $f(x, t) = x$. Clearly f is equivariant and takes the n-cycle carried by $S^n\times\{1\}$ on X to the non-trivial n-cycle on S^n. But since the coefficients are in Z_2, the only non-zero symmetric n-cycle on X, which is carried by X itself, goes to the zero n-cycle on S^n under f. Thus $f_*\colon H_n(X)\to H_n(S^n)$ is non-zero but the map $f_*\colon H_n(X; T)\to H_n(S^n; T)$ is zero and so the index of $(X; T)$ is less than n. Actually it is not hard to show that in this example the index of $(X; T)$ is zero.

We conclude this section by stating the following theorem due to Yang ([14], p. 270) which will be needed in section 6.

(2.5) THEOREM. *Let $(X; T)$ be a T-space of index n and let f be a map of X into the Euclidean k-space R^k, where $0\leqslant k\leqslant n$. Let $X_k = \{x\in X\mid f(x) = f(Tx)\}$. Then X_k is T-invariant, compact and $(X_k; T)$ is of index $\geqslant n-k$. In particular if $n = k$ then X_k is non-empty, i.e. there exists x in X such that $f(x) = f(Tx)$.*

Since the Smith index of the n-sphere with the antipodal involution is n, the Borsuk-Ulam theorem follows immediately from (2.5). This theorem will also play a crucial role in our proof of the desired generalization of the Borsuk-Ulam theorem.

3. Regular polyhedra. We are concerned with compact subsets X of R^{n+1} which separate the origin O from ∞ (i.e. for which the origin lies in a bounded component of the complement). If X is any compact subset of R^{n+1} not containing O, then there is a map, often called the Borsuk map, $p\colon X\to S^n$ defined by $p(x) = x/\|x\|$. Borsuk [3] proved that the origin is in a bounded component of $R^{n+1}-X$ if and only if the map p is essential. This, in turn, is equivalent to the condition that the induced map on the n-dimensional Čech homology groups $p_*\colon H_n(X)\to H_n(S^n)$ be non-zero. Given any such set X we construct a space $A(X)$, called the *antipodal space of* X and defined as $\{(x, y)\in X\times X\mid y = -\lambda x$, for some $\lambda > 0\}$. Equivalently, $A(X) = \{(x, y)\in X\times X\mid p(x) = -p(y)\}$. Clearly $A(X)$ is a closed subset of $X\times X$ and consequently is compact. There is a natural, fixed-point-free involution T on $A(X)$ defined by $T(x, y) = (y, x)$.

We are interested in the Smith index of the T-space $(A(X); T)$. There is a map $q\colon A(X)\to S^n$ defined by $q(x, y) = p(x)$. Clearly this map is equivariant with respect to T and the antipodal involution on S. Hence by (2.4) we certainly know that the Smith index of $(A(X); T)$ cannot exceed n. We would like to know if it is equal to n. The importance of knowing this is that a map $f\colon X\to R^n$ induces a map $g\colon A(X)\to R^n$, defined by $g(x, y) = f(x)$ and in order to apply Yang's result to the map g, we must know that the Smith index of $(A(X); T)$ is at least n.

We have not succeeded in showing that the Smith index of $(A(X); T)$ is necessarily n. We remark that from the fact that $p_*\colon H_n(X)\to H_n(S^n)$ is non-zero, it is not hard to show that $q_*\colon H_n(A(X))\to H_n(S^n)$ is also non-zero. However, as remarked after the statement of (2.4), from this alone, we cannot conclude that $q_*\colon H_n(A(X); T)\to H_n(S^n; T)$ is non-zero.

In some special cases we can show that the index of $(A(X); T)$ is n by actually constructing an invariant n-cycle on $A(X)$ which is taken by q to the non-trivial invariant n-cycle on S^n. An important instance in which such a construction is possible is when X is a regular polyhedron, a concept we shall define shortly. Before doing so we put the argument in a slightly different but equivalent framework. This will allow us to handle the maps p and q with more ease. We emphasize, however, that the change is purely technical and not conceptual. At the end of section 5 we shall come back to the original framework.

If X is a compact subset of R^{n+1} not containing the origin, then there exist real numbers $0 < r < R$ (depending on X) such that X is contained in the interior of the annulus $A(r, R) = \{x\in R^{n+1}\mid r\leqslant\|x\|\leqslant R\}$. There is a homeomorphism $\varphi\colon A(r, R)\to S^n\times I$ defined by

$$\varphi(x) = \left(\frac{x}{\|x\|}, \frac{R-\|x\|}{R-r}\right).$$

By means of φ we regard X as a subset of $S^n\times I$; in other words we identify X with $\varphi(X)$. Then the Borsuk map p becomes merely the restriction of the projection of $S^n\times I$ onto S^n. Moreover, O is in a bounded component of $R^{n+1}-X$ if and only if X (that is, $\varphi(X)$) separates the top and the bottom of the cylinder $S^n \times I$. This leads to the following definition.

(3.1). DEFINITION. A subset X of $S^n \times I$ is said to be a *Borsuk set* if X is compact and separates the top $X^n \times \{1\}$ from the bottom $S^n \times \{0\}$.

In this new formulation Borsuk's characterization [3] can be rephrased by saying that a compact subset X of $S^n \times I$, not intersecting either the top or the bottom, is a Borsuk set if and only if the map $p\colon X \to S^n$, which is the restriction of the projection, is essential. This is so if and only if the map $p_*\colon H_n(X) \to H_n(S^n)$ on Čech homology groups is non-zero. Notice that the set $A(X)$ now becomes $\{((x, s), (-x, t)) \in X \times X|\ x \in S^n, s, t \in I\}$ and the map $q\colon A(X) \to S^n$ becomes $q((x, s), (-x, t)) = x$.

To define regular polyhedra we first make the following definition.

(3.2). DEFINITION. Let Y be a topological space and let $\mathrm{pr.}\colon Y \times I \to Y$ be the projection. A subset S of $Y \times I$ is said to be *vertical* if the map $\mathrm{pr.}|S\colon S \to Y$ is not injective, that is, if there exist y in Y and s, t in I such that (y, s) and (y, t) are both in S and $s \neq t$.

Clearly, if Y is Hausdorff and S is compact and not vertical then the map $p\colon S \to \mathrm{pr.}(S)$ defined by the projection is a homeomorphism. In this case we say that S *falls on* or *projects onto* or *lies above* $p(S)$.

Now we are ready to define regular polyhedra.

(3.3). DEFINITION. A compact polyhedron $P \subset S^n \times I$ is said to be a *regular polyhedron* if there exists a triangulation K of P and a symmetric triangulation L of S^n such that,

(i) the map $p\colon P \to S^n$, which is the restriction of the projection, is simplicial with respect to the triangulations K and L,

(ii) no simplex of P is vertical,

(iii) P intersects neither the top nor the bottom of the cylinder $S^n \times I$,

(iv) the totality of n-simplices of P form an n-cycle z (with coefficients in Z_2 as usual),

(v) $p_*(\{z\}) = a$, the non-trivial homology class in $H_n(S^n)$.

First we draw some immediate consequences from the definition. Conditions (i) and (ii) imply that every simplex of P falls on a simplex of S^n of the same dimension. In particular, $\dim P \leqslant n$. Conditions (iii), (iv) and (v) imply that P is a Borsuk set and also that $\dim P \geqslant n$. Hence $\dim P = n$. Condition (iv) implies that every $(n-1)$-simplex of P is a face of an even number (possibly 0) of n-simplices of P. Conditions (iv) and (v) together imply that for any n-simplex σ of S^n, the number of n-simplices of P which lie above σ is odd.

Despite the large number of restrictions P has to satisfy in order to be a regular polyhedron, it turns out that the class of regular polyhedra is large enough, so that given a Borsuk set X and a neighborhood U of X in $S^n \times I$, we can find a regular polyhedron P contained in U. We shall prove this in section 5. This section is aimed at proving the following important property of regular polyhedra.

(3.4). THEOREM. *Let P be a regular polyhedron in $S^n \times I$ and let K, L be the triangulations of P and S^n respectively as in the definition above. Then the antipodal space $A(P)$ has a triangulation such that,*

(a) *the map $q\colon A(P) \to S^n$ is simplicial, where on S^n we have the symmetric triangulation L,*

(b) *the involution T on $A(P)$ acts simplicially with respect to this triangulation,*

(c) *the totatility of all n-simplices of $A(P)$ forms an invariant n-cycle ζ,*

(d) *$q_*(\{\zeta\}) = a$, the non-zero homology class in $H_n(S^n)$.*

Consequently, the map $q_*\colon H_n(A(P); T) \to H_n(S^n; T)$ is non-zero and thus the Smith index of $(A(P); T)$ is n.

Proof. First we introduce some notation.

Given a simplex σ of S^n, by $-\sigma$ we denote the image of σ under the antipodal involution on S^n. We note that $-\sigma$ is also a simplex of S^n since the triangulation of S^n is symmetric. Clearly σ and $-\sigma$ have the same dimension and $-(-\sigma) = \sigma$. We say that σ and $-\sigma$ are opposite to each other. The minus sign should cause no confusion, even when we consider a chain in which σ appears. The reason for this is that since our coefficient group is Z_2, the negative of the chain σ in the group of chains is σ itself; and we shall never use $-\sigma$ to denote this negative.

If C and D are subsets of P we define $\mathrm{Ant}(C, D)$ to be the set $\{((x, s), (-x, t)) \in P \times P|\ (x, s) \in C \text{ and } (-x, t) \in D\}$. Clearly, $\mathrm{Ant}(C, D) \subset A(P)$ and $A(P)$ is in fact equal to $\mathrm{Ant}(P, P)$. In the following lemma we list down some simple properties of the binary operation $\mathrm{Ant}(\ ,\ )$ just defined.

(3.5). LEMMA. *The following facts hold.*

Fact 1. *If C, D are compact, so is $\mathrm{Ant}(C, D)$.*

Fact 2. *If C_1, C_2, D_1, D_2 are subsets of P then*

$$\mathrm{Ant}(C_1, D_1) \cap \mathrm{Ant}(C_2, D_2) = \mathrm{Ant}(C_1 \cap C_2, D_1 \cap D_2).$$

Fact 3. *The involution T on $A(P)$ maps $\mathrm{Ant}(C, D)$ onto $\mathrm{Ant}(D, C)$.*

Fact 4. *If τ_1, τ_2 are any two simplices of P then q maps $\mathrm{Ant}(\tau_1, \tau_2)$ injectively into S^n. Moreover, $q(\mathrm{Ant}(\tau_1, \tau_2)) = p(\tau_1) \cap (-p(\tau_2))$. Consequently, $\mathrm{Ant}(\tau_1, \tau_2)$ is homeomorphic to a simplex of S^n.*

Fact 5. *If τ_1, τ_2 are simplices of P, then $\dim(\mathrm{Ant}(\tau_1, \tau_2)) = n$ if and only if τ_1 and τ_2 both have dimension n and $p(\tau_1) = -p(\tau_2)$.*

Fact 6. *Let $\theta_1, \theta_2, \theta_1', \theta_2'$ be n-simplices of P such that $\dim(\mathrm{Ant}(\theta_1, \theta_2)) = n$. Then, $\mathrm{Ant}(\theta_1, \theta_2) = \mathrm{Ant}(\theta_1', \theta_2')$ if and only if $\theta_1 = \theta_1'$ and $\theta_2 = \theta_2'$.*

Proof. The first three facts are obvious.

To prove Fact 4, we recall that no simplex of P is vertical. Hence the restriction of q to $\mathrm{Ant}(\tau_1, \tau_2)$ is evidently one-to-one. Also by Fact 1 above, $\mathrm{Ant}(\tau_1, \tau_2)$ is compact. Hence, $\mathrm{Ant}(\tau_1, \tau_2)$ is homeomorphic to its image under q. Next, we show that, $q(\mathrm{Ant}(\tau_1, \tau_2)) = p(\tau_1) \cap (-p(\tau_2))$. First suppose $((x, s), (-x, t)) \in \mathrm{Ant}(\tau_1, \tau_2)$. Then $x \in p(\tau_1)$ and $-x \in p(\tau_2)$, and so, $x \in p(\tau_1) \cap (-p(\tau_2))$. But $x = q((x, s), (-x, t))$. Thus $q(\mathrm{Ant}(\tau_1, \tau_2))$ is contained in $p(\tau_1) \cap (-p(\tau_2))$. Conversely suppose $x \in p(\tau_1) \cap (-p(\tau_2))$. Then there exist $s, t \in I$ such that $(x, s) \in \tau_1$ and $(-x, t) \in \tau_2$. But then, $((x, s), (-x, t)) \in \mathrm{Ant}(\tau_1, \tau_2)$ and so $x = q((x, s), (-x, t)) \in q(\mathrm{Ant}(\tau_1, \tau_2))$. Thus $p(\tau_1) \cap (-p(\tau_2))$ is contained in $q(\mathrm{Ant}(\tau_1, \tau_2))$. Hence $q(\mathrm{Ant}(\tau_1, \tau_2)) = p(\tau_1) \cap (-p(\tau_2))$. Since the latter is a simplex of S^n the proof of Fact 4 is complete.

Fact 5 follows easily from Fact 4. The simplex $p(\tau_1) \cap (-p(\tau_2))$ of S^n has dimension n if and only if $p(\tau_1)$ and $P(\tau_2)$ are both n-dimensional and $p(\tau_1) = -p(\tau_2)$. The result now follows because p maps simplices of P onto simplices of S^n of the same dimension, as we observed after stating the definition of a regular polyhedron.

To prove Fact 6, we need only to prove that under the given conditions, $\mathrm{Ant}(\theta_1, \theta_2) = \mathrm{Ant}(\theta_1', \theta_2')$ implies that $\theta_1 = \theta_1'$ and $\theta_2 = \theta_2'$. So suppose $\mathrm{Ant}(\theta_1, \theta_2) = \mathrm{Ant}(\theta_1', \theta_2')$. Then by Fact 2 above,

$$\mathrm{Ant}(\theta_1 \cap \theta_1', \theta_2 \cap \theta_2') = \mathrm{Ant}(\theta_1, \theta_2) \cap \mathrm{Ant}(\theta_1', \theta_2') = \mathrm{Ant}(\theta_1, \theta_2).$$

Therefore $\dim \mathrm{Ant}(\theta_1 \cap \theta_1', \theta_2 \cap \theta_2') = n$. Hence by Fact 5, $\dim(\theta_1 \cap \theta_1') = \dim(\theta_2 \cap \theta_2') = n$. Since the dimension of the intersection of two distinct n-simplices is less than n, we must conclude that $\theta_1 = \theta_1'$ and $\theta_2 = \theta_2'$. The proof of the lemma is now complete.

The importance of Facts 5 and 6 in the last lemma is that they give a precise description of the form and the number of the n-simplices of $A(P)$ in the triangulation which we are about to construct. Fact 6 precludes the possibility that two distinct pairs (θ_1, θ_2) and (θ_1', θ_2') may give rise to the same n-simplex of $A(P)$ and thereby establishes a one-to-one correspondence between the set of n-simplices of $A(P)$ and the set of pairs (θ_1, θ_2) for which $\dim \theta_1 = n = \dim \theta_2$ and $p(\theta_1) = -p(\theta_2)$. This correspondence is of great help whenever we want to find the number of n-simplices of $A(P)$ which satisfy a certain condition.

Now consider the collection $\mathcal{F}$ of all sets of the form $\mathrm{Ant}(\tau_1, \tau_2)$ where (τ_1, τ_2) runs over all pairs of simplices of P. From Fact 2 above, it follows that the intersection of any two members of $\mathcal{F}$ is again a member of $\mathcal{F}$. By Fact 4 in (3.5), each member of $\mathcal{F}$ is homeomorphic to a simplex (possibly the empty simplex) and the homeomorphisms agree on the intersections, since they are all induced by q. Thus, the family $\mathcal{F}$ will be a triangulation of $A(P)$ provided we show that $A(P)$ is covered by members of $\mathcal{F}$. But this is obvious. In fact, if $((x, s), (-x, t)) \in A(P)$ then $((x, s), (-x, t)) \in \mathrm{Ant}(\tau_1, \tau_2)$ where τ_1 and τ_2 are any two simplices of P such that $(x, s) \in \tau_1$ and $(-x, t) \in \tau_2$.

It is clear that the map $q\colon A(P) \to S^n$ is simplicial with respect to the triangulation just constructed. Moreover, Fact 3 mentioned above shows that T acts simplicially with respect to this triangulation. Thus we have proved parts (a) and (b) of our theorem.

Now we turn to the proof of (c). If the totality of n-simplices of $A(P)$ at all forms a cycle ζ then this cycle must be invariant because T acts simplicially. We have therefore merely to show that the totality of all n-simplices of $A(P)$ forms a cycle. Since the coefficients are in Z_2, this amounts to showing that every $(n-1)$-simplex of $A(P)$ is in the boundary of an even number (possibly zero) of n-simplices. If an $(n-1)$-simplex of $A(P)$ is not a face of any n-simplex of $A(P)$ this assertion is obvious for that $(n-1)$-simplex. Hence we have to consider only those $(n-1)$-simplices of $A(P)$ which occur in the boundary of at least one n-simplex of $A(P)$. Consider an n-simplex $\mathrm{Ant}(\tau_1, \tau_2)$ of $A(P)$. By Fact 5 in (3.5) $p(\tau_1) = -p(\tau_2)$ and so $q(\mathrm{Ant}(\tau_1, \tau_2)) = p(\tau_1)$. Therefore an $(n-1)$-face of $\mathrm{Ant}(\tau_1, \tau_2)$ corresponds to a unique $(n-1)$-face of $p(\tau_1)$ and hence to a unique $(n-1)$-face of τ_1. It follows that every $(n-1)$-face of $\mathrm{Ant}(\tau_1, \tau_2)$ can be written in the form $\mathrm{Ant}(\sigma_1, \sigma_2)$ for some $(n-1)$-face σ_1 of τ_1, σ_2 being the unique $(n-1)$-face of τ_2 satisfying the condition $p(\sigma_2) = -p(\sigma_1)$. Consider a particular $(n-1)$-face $\mathrm{Ant}(\sigma_1, \sigma_2)$. We have to show that the number of all n-simplices of $A(P)$ which have $\mathrm{Ant}(\sigma_1, \sigma_2)$ as a face is even. Let R_{σ_1,σ_2} be the set of all n-simplices of $A(P)$ which have $\mathrm{Ant}(\sigma_1, \sigma_2)$ as a face. Let S_{σ_1,σ_2} be the set defined by

$$S_{\sigma_1,\sigma_2} = \{(\theta_1, \theta_2) |\ \theta_1, \theta_2\ n\text{-simplices of } P,\ \sigma_1 \lessdot \theta_1,\ \sigma_2 \lessdot \theta_2,\ -p(\theta_2) = p(\theta_1)\}$$

(the symbol $\lessdot$ denotes the relation "is a face of"). The preceding discussion about the $(n-1)$-faces of an n-simplex of $A(P)$ combined with Fact 6 in (3.5) shows that the sets R_{σ_1,σ_2} and S_{σ_1,σ_2} are in one-to-one correspondence with each other and therefore have the same cardinality. Thus, the proof of (c) will be complete once we prove the following lemma.

(3.6). LEMMA. *Cardinality of the set S_{σ_1,σ_2} defined above is even.*

Proof. $p(\sigma_1)$ is an $(n-1)$-simplex of S^n and consequently is in the boundary of exactly two n-simplices of S^n, say τ and θ. Since $p(\sigma_2) = -p(\sigma_1)$ it follows that $p(\sigma_2)$ is in the boundary of exactly two n-simplices of S^n, $-\tau$ and $-\theta$. If $\sigma_1 \lessdot \theta_1$ then $p(\sigma_1)$ is a face of $p(\theta_1)$ because p is simplicial. Therefore $p(\theta_1)$ is either τ or θ. Similarly, if $\sigma_2 \lessdot \theta_2$ then $p(\theta_2) = -\tau$ or $-\theta$.

Let

$a =$ number of n-simplices of P which have σ_1 as a face and which fall on τ.

$b =$ number of n-simplices of P which have σ_1 as a face and which fall on θ.

$c =$ number of n-simplices of P which have σ_2 as a face and which fall on $-\tau$.

$d =$ number of n-simplices of P which have σ_2 as a face and which fall on $-\theta$.

The set S_{σ_1,σ_2} decomposes into two disjoint sets defined by

$$A_{\sigma_1,\sigma_2} = \{(\theta_1, \theta_2) \in S_{\sigma_1,\sigma_2} |\ p(\theta_1) = \tau, p(\theta_2) = -\tau\}$$

and

$$B_{\sigma_1,\sigma_2} = \{(\theta_1, \theta_2) \in S_{\sigma_1,\sigma_2} |\ p(\theta_1) = \theta, p(\theta_2) = -\theta\}.$$

Evidently the cardinality of A_{σ_1,σ_2} is ac while that of B_{σ_1,σ_2} is bd. To prove the lemma we must show that $(ac+bd)$ is even. Now, $(a+b)$ is the number of n-simplices of P which have σ_1 as a face. Hence $(a+b)$ is even as observed after the definition of a regular polyhedron. Similarly $(c+d)$, being the number of n-simplices of P which have σ_2 as a face, is even. From this it follows by an elementary argument that $(ac+bd)$ is even. This proves the lemma and consequently (c).

Finally we prove (d). First we observe that if $\mathrm{Ant}(\tau_1, \tau_2)$ is an n-simplex of $A(P)$, then $q(\mathrm{Ant}(\tau_1, \tau_2)) = p(\tau_1)$. Hence, again using Fact 6 in (3.5), the number of n-simplices of $A(P)$ which are mapped by q onto a given n-simplex σ of S^n is equal to the cardinality of the set S_σ defined by

$$S_\sigma = \{(\tau_1, \tau_2) |\ \tau_1, \tau_2\ n\text{-simplices of } P,\ p(\tau_1) = \sigma, p(\tau_2) = -\sigma\}.$$

Let $k =$ number of n-simplices of P which fall on σ and let $j =$ number of n-simplices of P which fall on $-\sigma$. Then k, j are both odd, as we observed after the definition of a regular polyhedron. Hence kj is odd. But evidently kj is the cardinality of the set S_σ defined above. Therefore, in the n-cycle $q(\zeta)$ on S^n every n-simplex of S^n occurs an odd number of times; and so $q_*(\{\zeta\}) = a$, the non-zero homology class in $H_n(S^n)$. Thus (d) is proved.

Since ζ is invariant, so is $q(\zeta)$ and thus we have shown that the map $q_*\colon H_n(A(P); T) \to H_n(S^n; T)$ induced by the map $q\colon A(P) \to S^n$ is non-zero. This shows that the Smith index of $(A(P); T)$ is n and completes the proof of the theorem.

4. Admissible cell structures on cylinders on manifolds. This section is preparatory for the next section. In the next section we shall show that if $X \subset S^n \times I$ is a Borsuk set and U is a neighborhood of X in $S^n \times I$ then there exists a regular polyhedron P contained in U. The polyhedron P is obtained from the n-skeleton of a subcomplex of a certain "admissible" cell structure on $S^n \times I$. The purpose of this section is to define the concept of admissible cell structures and to give a method for their construction. The only property of S^n which is needed crucially is that it is a compact, combinatorial n-manifold without boundary. Therefore, we give the construction for the more general case of a cylinder on a compact, combinatorial n-manifold without boundary.

Let Y be a compact polyhedron with a fixed metric d_0. We shall put the metric d on $Y \times I$ defined by $d((x, s), (y, t)) = d_0(x, y) + |s-t|$. By the *mesh* $\mu(L)$ of a triangulation L of Y we mean the maximum of the diameters (with respect to d_0) of simplices of L. Similarly by the *mesh* $\mu(K)$ of a cell structure K on $Y \times I$ we mean the maximum of the diameters (with respect to d) of the closed cells of K.

A cell structure K on a space S is said to be *simplicial* if all closed cells of K are simplices and the attaching maps are simplicial embeddings. This is equivalent to requiring that K be a triangulation of S. We recall that if K is a cell structure on a space S and if $S' \subset S$ corresponds to a subcomplex of K then by $K|S'$ we denote the induced cell structure on S'.

(4.1). Definition. A cell structure K on $Y \times I$ is said to be *n-admissible* if there exists a triangulation L of Y such that

(i) K^n, the n-skeleton of K is simplicial,

(ii) the map $\mathrm{pr.}|K^n\colon K^n \to Y$ (where pr. is the projection of $Y \times I$ onto Y) is simplicial with respect to K^n and L,

(iii) no simplex of K^n is vertical.

Obviously an n-admissible cell structure is also m-admissible for $m \leqslant n$. When the integer n is understood an n-admissible cell structure may simply be called admissible. The main theorem of this section is the following.

(4.2). Theorem. *Suppose Y is a compact, combinatorial n-manifold without boundary. Then given $\varepsilon > 0$, there exists an n-admissible cell structure K on $Y \times I$ whose mesh is less than ε. If, moreover, T is a simplicial involution on Y then we may require, in addition, that the triangulation L of Y appearing in the definition be symmetric with respect to T.*

Proof. First we shall show how to construct an n-admissible cell structure on $Y \times I$, starting from a given triangulation of Y. (By a triangulation of Y we always mean a triangulation which makes Y into a combinatorial n-manifold, i.e. the star of every vertex is a polyhedral n-disk whose boundary is the link of the vertex.) Later we shall show how this construction can be modified so as to give arbitrarily fine n-admissible cell structures on $Y \times I$.

Let L_0 be a given triangulation of Y.

Let L be the first barycentric subdivision of L_0. Then, Y is also a combinatorial manifold with respect to L. The advantage of taking L instead of L_0 is that the vertices of any simplex of L have a canonical ordering. The $n+1$ vertices of an n-simplex σ of L can be written uniquely as $v_0, v_1, \ldots, v_n$ where v_i is the barycenter of an $(n-i)$-dimensional simplex of L_0 $(0 \leqslant i \leqslant n)$. We call v_1 the *i-th vertex* of σ. Note that if σ' is another n-simplex of L which has v_i as a vertex then v_i must also be the ith vertex of σ'. Hence, without ambiguity we can say that v_i is an ith vertex of L. (This terminology is somewhat misleading in the beginning, because there are more than one vertices of L each of which is an ith vertex for the same i. Perhaps a term such as "vertex of order i" instead of "ith vertex" would be better. But we choose the latter for its brevity.)

If x is a point of σ, then for $0 \leqslant i \leqslant n$, the barycentric coordinate of x with respect to the ith vertex of σ will be denoted by x_i. Thus $0 \leqslant x_i \leqslant 1$ for every i and $\sum_{i=0}^{n} x_i = 1$. Note that if $x \in \sigma \cap \sigma'$, then all the barycentric coordinates of x regarded as a point of σ are equal to the corresponding barycentric coordinates of x regarded as a point of σ', where σ, σ' are any two n-simplices of L. Thus we can speak of the *i-th barycentric coordinate* of a point of Y without ambiguity. For each i, $0 \leqslant i \leqslant n$, taking the ith barycentric coordinate defines a continuous function from Y to the unit interval. Of course it may happen that two distinct points of Y have all the corresponding barycentric coordinates equal.

We construct $n+2$ continuous functions from Y to the unit interval $[0,1]$ as follows.

f_0 is the function which is identically 0.

For $1 \leqslant i \leqslant n+1$, define $f_i(x) = \sum_{k=0}^{i-1} x_k$ where x_k is the kth barycentric coordinate of x for $k = 0, \ldots, n$.

Then for every x, $f_i(x) \leqslant f_{i+1}(x)$ for $0 \leqslant i \leqslant n$ and f_{n+1} is the function identically equal to 1. Note that $f_i(x) = f_{i+1}(x)$ if and only if x is on the face of some n-simplex of L which is opposite to the ith vertex of that simplex. Each f_i is linear when restricted to any simplex of L.

For each i let $A_i = \{(x, t) \in Y \times I \mid f_i(x) \leqslant t \leqslant f_{i+1}(x)\}$. Then clearly $Y \times I = \bigcup_{i=0}^{n} A_i$.

A triangulation for $Y \times I$ can be given in which the $(n+1)$-simplices are of the form $\{(x, t) \mid x \in \sigma, f_i(x) \leqslant t \leqslant f_{i+1}(x)\}$ for some n-simplex σ of L and some $0 \leqslant i \leqslant n$. This triangulation coincides with the standard product triangulation (see, for example, [5], p. 67) on $Y \times I$; for the $(n+1)$-simplex $\{(x, t) \mid x \in \sigma, f_i(x) \leqslant t \leqslant f_{i+1}(x)\}$ is precisely the $(n+1)$-simplex spanned by the $n+2$ points, $(v_0, 1), \ldots, (v_i, 1), (v_i, 0)(v_{i+1}, 0), \ldots, (v_n, 0)$, where v_k is the kth vertex of σ, $0 \leqslant k \leqslant n$.

Unfortunately there are some n-simplices in this triangulation of $Y \times I$ which are vertical. Our intention is to match together such vertical n-simplices. It is precisely at this point that we need the hypothesis that Y is a combinatorial n-manifold w.r.t. the triangulation L.

Let v_i be an ith vertex of L. Let Δ_{v_i} and Σ_{v_i} be respectively the star and the link of v_i in L. Then the pair $(|\Delta_{v_i}|, |\Sigma_{v_i}|)$ is homeomorphic to (D^n, S^{n-1}) where D^n is the n-dimensional disk.

For a point x in $|\Delta_{v_i}|$, $f_i(x) = f_{i+1}(x)$ if and only if x is in $|\Sigma_{v_i}|$. Hence if $x \in |\Delta_{v_i}| - |\Sigma_{v_i}|$ then $f_i(x) < f_{i+1}(x)$.

It follows that the set $B_{v_i} = \{(x, t) \mid x \in |\Delta_{v_i}|, f_i(x) \leqslant t \leqslant f_{i+1}(x)\}$ is homeomorphic to the $(n+1)$-disk D^{n+1}. Indeed, let D^{n+1} and D^n be the standard Euclidean disks of dimension $n+1$ and n respectively with $D^n \subset D^{n+1}$, in such a way that D^n bounds the equatorial S^{n-1} on S^n and let $g\colon |\Delta_{v_i}| \to D^n$ be a homeomorphism. Then a homeomorphism $h\colon B_{v_i} \to D^{n+1}$ can easily be defined in such a way that for each $x \in |\Delta_{v_i}|$, the segment $\{(x, t) \mid f_i(x) \leqslant t \leqslant f_{i+1}(x)\}$ is mapped onto the line segment which is the intersection with D^{n+1} of the line through $g(x)$, perpendicular to the equatorial plane. h is well-defined because if $f_i(x) = f_{i+1}(x)$, then $g(x)$ must be on S^{n-1}. The boundary of B_{v_i} consists of n-simplices of $Y \times I$, of the form $\{(x, t) \mid x \in \sigma, f_i(x) = t\}$ or of the form $\{(x, t) \mid x \in \sigma, f_{i+1}(x) = t\}$ for $\sigma \in \Delta_{v_i}$. None of these simplices is vertical. Moreover the restriction of the projection map to each such simplex is simplicial. Also $A_i = \bigcup B_{v_i}$ as v_i ranges over all the ith vertices of L. It follows that $Y \times I$ has a cell structure K whose n-skeleton consists of faces of simplices of the form $\{(x, t) \mid x \in \sigma, f_i(x) = t\}$ for some n-simplex σ of L and for some $0 \leqslant i \leqslant n+1$. None of these simplices is vertical. Moreover $\mathrm{pr.}|K^n\colon K^n \to Y$ is simplicial. Thus K is an n-admissible cell structure on $Y \times I$.

To get sufficiently fine n-admissible cell structures on $Y \times I$ we start with a sufficiently fine triangulation of Y. We divide the interval $[0,1]$ into r equal parts where r is sufficiently large. Then we put cell structures on each of the slices $Y \times \left[\frac{k}{r}, \frac{k+1}{r}\right]$ for $k = 0, 1, \ldots, r-1$ by the construction given above in such a way that two adjacent slices induce the same cell structure on their common face.

To make this precise, let $\varepsilon > 0$ be given. Take a triangulation L_0 of Y such that $\mu(L_0) < \frac{1}{4}\varepsilon$. L_0 may be taken to be an iterated barycentric subdivision of a given triangulation of Y. Since L is the first barycentric subdivision of L_0 we have $\mu(L) < \frac{1}{4}\varepsilon$. Let r be a positive integer such that $1/r < \frac{1}{4}\varepsilon$. For $k = 0, 1, \ldots, r-1$ define homeomorphisms $h_k\colon Y \times [k/r, (k+1)/r] \to Y \times I$ by

$$h_k(y, t) = \begin{cases} (y, rt-k) & \text{if } k \text{ is even}, \\ (y, k+1-rt) & \text{if } k \text{ is odd}. \end{cases}$$

Note that for each k, h_k and h_{k+1} agree on $Y\times\{(k+1)/r\}$. Thus we get a map $h\colon Y\times I\to Y\times I$ whose restriction to each slice $Y\times[k/r,(k+1)/r]$ is a homeomorphism.

Let K^* be the cell structure on $Y\times I$ obtained from L by the construction above. Then h_k gives a cell structure K_k on $Y\times[k/r,(k+1)/r]$. If k is even then $K_k|Y\times\{(k+1)/r\}$ and $K_{k+1}|Y\times\{(k+1)/r\}$ coincide since each of the corresponds to $K^*|Y\times\{1\}$ under h. Similarly if k is odd then $K_k|Y\times\{(k+1)/r\}$ and $K_{k+1}|Y\times\{(k+1)/r\}$ both correspond to $K^*|Y\times\{0\}$ under h and so coincide with each other. It follows that K_k's define a cell structure K on $Y\times I$. The n-skeleton K^n of K consists of the union of the n-skeletons K_k^n's of the K_k's. The n-skeleton of each K_k is simplicial with no vertical simplices; since K_k is obtained from K^* via h_k and h_k preserves verticality. It follows that K^n is simplicial with no vertical simplex. Also the map $\mathrm{pr.}|K^n\colon K^n\to Y$ is simplicial.

It only remains to show that $\mu(K)<\varepsilon$. Since every cell of K is contained in a closed $(n+1)$-cell of K it suffices to show that the diameter of each closed $(n+1)$-cell of K is less than ε. But by construction of K, every closed $(n+1)$-cell of K is of the form $h_k^{-1}(B_{v_i})$ for some k, $0\leqslant k\leqslant r-1$; some i, $0\leqslant i\leqslant n$ and some ith vertex v_i of L, where B_{v_i} is as in the construction above. Therefore, we only need to show that for each $k=0,1,\ldots,r-1$ each $i=0,1,\ldots,n$ and for each ith vertex v_i of L, $\delta\big(h_k^{-1}(B_{v_i})\big)<\varepsilon$ where δ denotes the diameter. Now, obviously $B_{v_i}\subset|\Delta_{v_i}|\times[0,1]$ where Δ_{v_i} is the star of v_i in L. Hence $h_k^{-1}(B_{v_i})\subset|\Delta_{v_i}|\times\times[k/r,(k+1)/r]$. The set $|\Delta_{v_i}|$ is the union of a finite number of n-simplices of L, all having v_i as a vertex. Since the diameter of each simplex of L is less than $\frac14\varepsilon$ it follows that $\delta(|\Delta_{v_i}|)<\frac12\varepsilon$. By choice of r we also have that $1/r<\frac14\varepsilon$. Combining these together we see that the diameter of $|\Delta_{v_i}|\times[k/r,(k+1)/r]$ is less than $(\frac12\varepsilon+\frac14\varepsilon)$. In particular $\delta\big(h_k^{-1}(B_{v_i})\big)<\varepsilon$.

Finally, we observe that if T is a simplicial involution on Y then L_0 may be so chosen that T acts simplicially with respect to L_0. Then L is also symmetric with respect to T. The theorem is now completely proved.

Remark 1. The hypothesis that Y be a combinatorial n-manifold cannot be replaced by the weaker one, say, $\dim Y=n$. For example if Y is the space homeomorphic to the letter Y with y_0 as the common point of the three rays then it is easy to show that in any cell structure on $Y\times I$, $\{y_0\}\times I$ is contained in the 1-skeleton. Therefore, there is no 1-admissible cell structure on $Y\times I$.

Remark 2. Since Y is a polyhedron, so is $Y\times I$. We actually constructed a triangulation of $Y\times I$ in the construction above. However, we have to content ourselves with the structure of a cell complex rather than that of a simplicial complex, on $Y\times I$. This is so because, although the sets B_{v_i} in the construction above are homeomorphic to D^{n+1} and their boundaries are polyhedral n-spheres, the sets B_{v_i} themselves are not necessarily $(n+1)$-simplices. In fact, they may not even be convex.

5. Approximation by regular polyhedra. In this section we apply the construction given in the last section to prove the "approximation theorem" which asserts the existence of a regular polyhedron within a given neighborhood of a Borsuk set in $S^n\times I$. At the end of the section we define the terms "Borsuk set" and "regular polyhedron" for subsets of R^{n+1} and reformulate the results proved for subsets of $S^n\times I$. The intention of doing this is to have the results in a form in which they can be immediately applied in the next section.

(5.1). THEOREM. *Let X be a Borsuk set in $S^n\times I$ and U a neighborhood of X in $S^n\times I$. Then there exists a regular polyhedron P contained in U.*

Proof. We may assume that U is open and does not intersect $S^n\times\{0\}$ and $S^n\times\{1\}$. Let $2\varepsilon=d(X,S^n\times I-U)$ where d is the metric on $S^n\times I$ obtained from some given metric on S^n as in the last section. Let K be an n-admissible cell structure on $S^n\times I$ for which the triangulation L of S^n is symmetric with respect to the antipodal involution on S^n and whose mesh $\mu(K)$ is less than ε. Such a triangulation exists by (4.2).

Let K_X be the sub-cell-complex of K consisting of those cells of K which intersect X. Clearly $|K_X|$ is a neighborhood of X and $|K_X|\subset U$ by choice of ε. Since X separates $S^n\times\{0\}$ and $S^n\times\{1\}$ so does $|K_X|$. Hence by Borsuk's characterization, the map $\mathrm{pr.}||K_X|\colon|K_X|\to S^n$ induces a non-zero map on the n-dimensional Čech homology group (with coefficients in Z_2 as usual). But since K_X is a finite cell complex, $|K_X|$ has the same homotopy type as a compact polyhedron (actually if we trace back the constructions then $|K_X|$ is a polyhedron itself); while S^n is a compact polyhedron by itself. We conclude that

(i) $(\mathrm{pr.}||K_X|)_*\colon H_n(|K_X|)\to H_n(S^n)$ is an epimorphism in the singular homology theory.

Let P' be the n-skeleton of K_X and $i\colon P'\to|K_X|$ be the inclusion map. It is well-known that,

(ii) $i_*\colon H_n(P')\to H_n(|K_X|)$ is an epimorphism in the singular homology theory.

Let $p'\colon P'\to S^n$ be the composite $(\mathrm{pr.}||K_X|)\circ i$. Then p' is also the restriction of the projection map pr. to P'. From (i) and (ii) we get

(iii) $p'_*\colon H_n(P')\to H_n(S^n)$ is an epimorphism in the singular homology theory.

P' is a subcomplex of the n-skeleton K^n of K. Since K is n-admissible, K^n is simplicial and consequently P' is simplicial. Thus P' is a polyhedron. No simplex of P' is vertical because no simplex of K^n is vertical.

Moreover the map p' is simplicial since pr.$|K^n$: $K^n \to S^n$ is simplicial. Since on the category of finite simplicial complexes and simplicial maps the singular homology theory is naturally equivalent to the simplicial homology theory, we conclude from (iii) that

(iv) p'_*: $H_n(P') \to H_n(S^n)$ is an epimorphism in the simplicial homology theory.

Clearly $P' \subset U$ and so P' does not intersect $S^n \times \{0\}$ or $S^n \times \{1\}$. Thus P' is very close to be a regular polyhedron, except that the totality of all n-simplices of P' may not form a cycle whose image under p' is the non-zero n-cycle on S^n. To remedy this, let $z = \sigma_1 + \sigma_2 + \ldots + \sigma_k$ be an n-cycle on P', where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are n-simplices of P' such that $p'_*(\{z\}) = \alpha$, the non-zero element in $H_n(S^n)$. Such a cycle z exists by (iv). Let P be the polyhedron formed by $\sigma_1, \sigma_2, \ldots, \sigma_k$ and their faces. Then P is a subcomplex of P' and satisfies all the conditions in the definition of a regular polyhedron. Finally, since $P \subset U$, the theorem is proved.

Now we go back to the original framework in which X is a subset of R^{n+1} rather than a subset of $S^n \times I$.

(5.2). Definition. A subset X of R^{n+1} is called a *Borsuk set* if X is compact and disconnects R^{n+1} in such a way that the origin O is in a bounded component of $R^{n+1} - X$ (that is, X separates O from ∞ in our terminology).

Given real numbers $0 < r < R$ by $A(r, R)$ we denote the closed annulus $\{x \in R^{n+1} |\ r \leqslant \|x\| \leqslant R\}$ and by $A^0(r, R)$ its interior $\{x \in R^{n+1} |\ r < \|x\| < R\}$. Such an annulus is canonically homeomorphic to $S^n \times I$ by the homeomorphism $h_{r,R}$: $A(r, R) \to S^n \times I$ defined by

$$h_{r,R}(x) = \big(x/\|x\|, (R - \|x\|)/(R - r)\big) .$$

The following proposition is obvious.

(5.3). Proposition. *Let X be a compact subset of R^{n+1} not containing the origin. Then the following statements are equivalent.*

(i) *X is a Borsuk set.*

(ii) *If $A(r, R)$ is any annulus such that $X \subset A^0(r, R)$ then $h_{r,R}(X)$ is a Borsuk set in $S^n \times I$.*

(iii) *There is some annulus $A(r, R)$ which contains X in its interior and is such that $h_{r,R}(X)$ is a Borsuk set in $S^n \times I$.*

Next, we define the notion of a regular polyhedron for subsets of R^{n+1}.

(5.4). Definition. A subset P or R^{n+1} is said to be a *regular polyhedron* if,

(i) P is a Borsuk set and

(ii) for every annulus $A(r, R)$ containing X in its interior, $h_{r,R}(P)$ is a regular polyhedron in $S^n \times I$.

If $A(r, R)$ and $A(s, S)$ are two annuli and Y is a subset of each of them, then it is easy to see that the sets $h_{r,R}(Y)$ and $h_{s,S}(Y)$ are homeomorphic to each other by a homeomorphism h which is compatible with the projection pr.: $S^n \times I \to S^n$ and consequently preserves verticality. In view of this, the following proposition is also clear.

(5.5). Proposition. *A Borsuk set P in R^{n+1} is a regular polyhedron if and only if there is some annulus $A(r, R)$ containing P in its interior such that $h_{r,R}(P)$ is a regular polyhedron in $S^n \times I$.*

The following theorem is a reformulation of (3.4).

(5.6). Theorem. *Let P be a regular polyhedron in R^{n+1}. Then the Smith index of the T-space $\big(A(P); T\big)$ is n.*

Proof. Let $A(r, R)$ be an annulus such that $X \subset A^0(r, R)$. Let $Q = h_{r,R}(P)$. Then Q is a regular polyhedron in $S^n \times I$ and so by (3.4) the Smith index of $\big(A(Q); T\big)$ is n. Define k: $A(P) \to A(Q)$ by $k(x, y) = \big(h_{r,R}(x), h_{r,R}(y)\big)$. Clearly k is an equivariant homeomorphism and so the Smith index of $\big(A(P); T\big)$ is also n.

Finally we reformulate (5.1).

(5.7). Theorem. *Let X be a Borsuk set in R^{n+1} and U a neighborhood of X in R^{n+1}. Then there exists a regular polyhedron P contained in U.*

Proof. Let $A(r, R)$ be an annulus containing X in its interior. We may assume that U is open and contained in $A^0(r, R)$. Let $V = h_{r,R}(U)$. Then V is a neighbourhood of the Borsuk set $h_{r,R}(X)$ in $S^n \times I$. By (5.1) there is a regular polyhedron Q in $S^n \times I$ such that $Q \subset V$. Let $P = h_{r,R}^{-1}(Q)$. Then by (5.5) P is a regular polyhedron. Also $P \subset U$. This completes the proof.

6. The main result. In this section we apply the machinery developed in the previous sections together with Yang's result, (2.5) to derive the desired generalization of the Borsuk-Ulam theorem. We first prove the result in the case of a regular polyhedron and then for an arbitrary Borsuk set by "approximating" the latter by regular polyhedra and applying the limiting process.

(6.1). Theorem. *Let X be a regular polyhedron in R^{n+1} and let f: $X \to R^n$ be a map. Then there exist two points x and y in X such that $y = -\lambda x$ for some $\lambda > 0$ and $f(x) = f(y)$.*

Proof. Define the space $A(X)$ and the involution T on $A(X)$ as in the previous sections. By (5.6) the index of $\big(A(X); T\big)$ is n. Define g: $A(X) \to R^n$ by $g(x, y) = f(x)$. Then by (2.5), there exists a point (x, y) in $A(X)$ such that $g(x, y) = g(y, x)$. This of course gives the desired result.

(6.2). THEOREM. *Let X be a Borsuk set in R^{n+1} and let $f\colon X \to R^n$ be a map. Then there exist two points x and y in X such that $y = -\lambda x$ for some $\lambda > 0$ and $f(x) = f(y)$.*

Proof. First by Tietze's extension theorem extend f to a map $F\colon R^{n+1} \to R^n$. Fix real numbers $0 < r < R$ such that X is contained in the interior of the annulus $A(r, R)$ with the notation of the last section. Let $U_1 \supset U_2 \supset U_3 \supset \ldots$ be a sequence of open neighborhoods of X in R^{n+1} such that $U_1 \subset A^0(r, R)$ and $\bigcap_{k=1}^{\infty} U_k = X$. By (5.7) for every k there exists a regular polyhedron P_k contained in U_k. Applying (6.1) to P_k and the restriction of F to P_k, we get for each k points x_k and y_k in P_k and positive real numbers λ_k such that $y_k = -\lambda_k x_k$ and $F(x_k) = F(y_k)$. Note that since both x_k, y_k are in $A(r, R)$ and $\|y_k\| = \lambda_k \cdot \|x_k\|$, it follows that $r/R \leqslant \lambda_k \leqslant R/r$ for each k. Hence by compactness of $A(r, R)$ and of the closed interval $[r/R, R/r]$ we can find integers $0 < k_1 < k_2 < k_3 < \ldots$ such that the sequences $\{x_{k_i}\}_{i=1}^{\infty}$, $\{y_{k_i}\}_{i=1}^{\infty}$ and $\{\lambda_{k_i}\}_{i=1}^{\infty}$ all converge, say, to x, y and λ respectively. Then we have $y = -\lambda x$ and $\lambda > 0$ (in fact, $\lambda \geqslant r/R$). Moreover, $x \in \bigcap_{k=1}^{\infty} U_k$ and so $x \in X$. Similarly $y \in X$. But by continuity of F we also have that $F(x) = F(y)$ and hence that $f(x) = f(y)$. This completes the proof.

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