exists and equals 0 at each point a in $[0,1]$. This limit is equivalent to

$$\lim_{x\to a}\frac{1}{x-a}\big(F(x)G(x)-F(x)G(a)\big)-F(a)g(a)+$$
$$+\frac{1}{x-a}\Big(F(x)-F(a)G(a)-\int_a^x f(t)G(t)\,dt\Big)$$

which, since F is continuous and $G'(a)=g(a)$, is equal to

$$\lim_{x\to a}\frac{1}{x-a}\Big(\big(F(x)-F(a)\big)G(a)-\int_a^x f(t)G(t)\,dt\Big).$$

Since F is absolutely continuous and thus $F(x)-F(a)=\int_a^x f(t)\,dt$, the limit, provided it exists, is equal to

$$\lim_{x\to a}\frac{-1}{x-a}\int_a^x \big(f(t)G(t)-f(t)G(a)\big)\,dt.$$

Since $G'(a)=g(a)$, it follows that $G(t)-G(a)=\big(g(a)+\varepsilon(t)\big)(t-a)$ where $\varepsilon(t)\to 0$ as $t\to a$. Consequently

$$\Big|\frac{1}{x-a}\int_a^x f(t)G(t)-f(t)G(a)\,dt\Big|=\Big|\frac{1}{x-a}\int_a^x f(t)\big(g(a)+\varepsilon(t)\big)(t-a)\,dt\Big|$$
$$\leqslant\frac{|g(a)|+|\varepsilon(x)|}{|x-a|}\int_a^x |f(t)(t-a)|\,dt$$
$$\leqslant(|g(a)|+|\varepsilon(x)|)\int_a^x |f(t)|\,dt.$$

Since $f(t)$ is Lebesgue integrable, so is $|f(t)|$ and $\int_a^x|f(t)|\,dt\to 0$ as $x\to a$. Thus the original limit exists and equals 0 and the theorem is proved.

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Regular maps and products of p-quotient maps

by

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Abstract. It is shown that the finite product of proximity quotient maps between separated proximity spaces is a proximity quotient map. A theorem on products of topological quotient maps follows from this result. Also, the regular maps of Poljakov are shown to preserve the semi-metrizability of proximity spaces.

1. Introduction. Much current research in general topology has been concerned with generalizations of topological quotient maps. Two of these generalizations are hereditarily quotient maps, which have been considered by Archangel'skiĭ [2] and Michael [8], and the bi-quotient maps of Michael [8] and Hájek [6]. These two mappings have analogues in proximity spaces which, in general, preserve more structure than their topological counterparts.

Our notation will follow [3] and [15]. In particular, $A \Subset B$ will mean A nonδ $(X-B)$ and proximity maps and proximity quotient maps will be called p-maps and p-quotient maps respectively. If δ and δ' are two proximities on a set X, δ' is said to be *finer* than δ iff $A\ \delta' B$ implies $A\ \delta\ B$ for all subsets A and B of X. If $f\colon (X,\delta)\to Y$ is a function from a proximity space (X,δ) onto a set Y, the *quotient proximity* is the finest proximity on Y for which f is a p-map.

2. Regular maps. Poljakov in [13] introduced regular maps and asked if the regular image of a metrizable proximity space is metrizable. In [14] he showed that a proximity space can be "determined by sequences" iff it is the regular image of the disjoint union of metrizable proximity spaces. However, since the disjoint union of metrizable proximity spaces might not be a metrizable proximity space (although the induced topology is, of course, metrizable), this did not answer the original question. The purpose of this section is to show that the image of a semi-metrizable proximity space under a regular map is semi-metrizable, thus giving a partial solution to the problem. We begin with the definition and a characterization due to Poljakov [13].

2.1. Definition. A p-map $f\colon (X,\delta)\to(Y,\delta')$ is *regular* iff $A\,\delta' B$ implies $f^{-1}(A)\,\delta\, f^{-1}(B)$.

2.2. Theorem. *Let $f\colon (X,\delta)\to(Y,\delta')$ be a p-map. Then the following are equivalent*:

(1) *f is regular*,

(2) $f^{-1}(A)\Subset U\Rightarrow A\Subset f(U)$,

(3) $f|_{f^{-1}(S)}\colon f^{-1}(S)\to S$ *is p-quotient for all $S\subseteq Y$.*

2.3. Remarks. (a) The equivalence of (2) and (3) above is similar to Archangel'skiĭ's result in [2] which states that a map is hereditarily quotient iff whenever U is a neighborhood of $f^{-1}(x)$, $f(U)$ is a neighborhood of x.

(b) Theorem 3.5 of [3] implies that every p-open map is regular and (3) above shows that regular maps are p-quotient. Poljakov [13] gives examples to show that the reverse implications are not true in general.

2.4. Definition. A *semi-metric* on a set X is a real-valued function d on $X\times X$ such that for all x and y in X,

(a) $d(x,y)=d(y,x)\geqslant 0$, and

(b) $d(x,y)=0$ iff $x=y$.

2.5. Definition. A proximity space (X,δ) is *semi-metrizable* iff there is a semi-metric d on X such that $A\,\delta\,B$ iff $d(A,B)=0$.

We shall need a lemma which may be of some interest in itself. It is a proximity analogue of a topological semi-metrization theorem due independently to C. M. Pareek [11] and C. C. Alexander [1]. Gagrat and Naimpally [4] have also recently considered the semi-metrization of proximity spaces. The emphasis of their research, however, was to use proximities to obtain topological results.

2.6. Lemma. *A separated proximity space (X,δ) is semi-metrizable iff there is a countable family $\{V_i\}_{i=1}^{\infty}$ of symmetric subsets of $X\times X$ satisfying*:

(a) $\bigcap\limits_{i=1}^{\infty} V_i=\Delta$ (*the diagonal*), *and*

(b) *for each closed subset A of X, $\{V_i[A]\}_{i=1}^{\infty}$ forms a δ-neighborhood base for A ($A\Subset V_i[A]$ for all i and if $A\Subset B$, then $A\Subset V_N[A]\subset B$ for some N).*

Proof. $\Leftarrow$ Assume $V_{i+1}\subset V_i$ and let

$$d(x,y)=0 \quad\text{iff}\quad (x,y)\in V_i \text{ for all } i,$$
$$d(x,y)=1 \quad\text{iff}\quad (x,y)\notin V_i \text{ for any } i,$$
$$d(x,y)=\frac{1}{i+1} \quad\text{iff}\quad (x,y)\in V_i-V_{i+1}.$$

Then d is a semi-metric. Now, let A nonδ B, i.e. $A\Subset X-B$. By our assumption, $A\Subset V_N[A]\subset X-B$ for some N. For each pair $(a,b)\in A\times B$, it must be true that $d(a,b)\geqslant 1/N$, since if $d(a_0,b_0)<1/N$ for some $(a_0,b_0)\in A\times B$, then $b_0\in V_N[A]$ — a contradiction. Therefore, $d(A,B)\geqslant 1/N>0$.

Conversely, if $d(A,B)=\varepsilon>0$, pick a positive integer N such that $1/N<\varepsilon$. Then $V_N[A]\cap B=\emptyset$, so that $A\Subset V_N[A]\subset X-B$, and A nonδ B.

$\Rightarrow$ Let (X,δ) be semi-metrizable with semi-metric d. Let $V_i=\{(x,y)\in X\times X|\ d(x,y)<1/i\}$. Clearly, $\{V_i\}_{i=1}^{\infty}$ has the required properties.

2.7. Corollary. *A separated proximity space (X,δ) is semi-metrizable iff there is a countable family $\{U_i\}_{i=1}^{\infty}$ of covers of X such that*

(a) $U_{i+1}<U_i$ *and*

(b) *for each closed subset A of X, $\{\mathrm{St}(A,U_i)\}_{i=1}^{\infty}$ forms a δ-neighborhood base for A.*

2.8. Theorem. *Let f be a regular map from a semi-metrizable proximity space (X,δ) onto the separated proximity space (Y,δ'). Then (Y,δ') is semi-metrizable.*

Proof. Since (X,δ) is semi-metrizable, there is a sequence $\{V_i\}_{i=1}^{\infty}$ of covers of X such that the star at any closed subset of X forms a δ-neighborhood base. Let A be a closed subset of Y. Clearly, $f^{-1}(A)$ is a closed subset of X and $f^{-1}(A)\Subset \mathrm{St}(f^{-1}(A),V_i)$ for all i. Since f is regular, it follows from Theorem 2.2 that

$$A\Subset f\big(\mathrm{St}(f^{-1}(A),V_i)\big)=\mathrm{St}(A,f(V_i))\quad \text{for all } i.$$

Also, if $A\Subset B$ then $f^{-1}(A)\Subset f^{-1}(B)$, and thus $f^{-1}(A)\Subset \mathrm{St}(f^{-1}(A),V_N)\subseteq f^{-1}(B)$ for some N. It follows as before that $A\Subset\mathrm{St}(A,f(V_N))\subseteq B$. Now, if $U_i=f(V_i)$, the conditions of Corollary 2.7 are satisfied, so (Y,δ') is semi-metrizable.

2.9. Proposition. *Let $f\colon (X,\delta)\to(Y,\delta')$ be regular and let $f^{-1}(y)$ be compact for all $y\in Y$. Then f is a (topological) quotient map.*

Proof. Since f is continuous, $\zeta(\delta')\subseteq a$ where a is the quotient topology on Y. Now, let U be a-open; that is, let $f^{-1}(U)$ be open. If $y\in U$, then $y\Subset U$ iff $f^{-1}(y)\Subset f^{-1}(U)$, since f is regular. But $f^{-1}(y)$ is compact and is contained in the open set $f^{-1}(U)$, so it is easy to find a set W for which $f^{-1}(y)\Subset W\subset f^{-1}(U)$. The result follows.

Remark. Poljakov [13] gives an example of a regular map which is not a quotient map, so the condition "$f^{-1}(y)$ compact" cannot be eliminated, in 2.9. He also states that a regular, perfect map is hereditarily quotient. If the domain has the elementary proximity or is metrizable, a slightly stronger result is true. We omit the easy proof for part (a). Part (b) follows from Theorem 2.3 of [3] and Remark 2.3 (a).

2.10. Proposition. *Let $f\colon (X,\delta)\to(Y,\delta')$ be a regular map onto a separated proximity space. If either*

(a) $f^{-1}(y)$ *is compact for all* $y \in Y$ *and* (X, δ) *is metrizable,*

or

(b) δ *is the elementary proximity and the quotient topology is completely regular,*

then f *is hereditarily quotient.*

3. Products of p-quotient maps. Michael [8], Hajek [6], and others have recently considered products of quotient maps on topological spaces. It is well-known that the product of quotient maps is not, in general, a quotient map. In this section it will be shown that products of p-quotient maps behave better than products of (topological) quotient maps and we will use our result on proximities to obtain a theorem about topological quotients.

Throughout, X^* will denote the Smirnov Compactification of a proximity space (X, δ) and $h^*\colon X^* \to Y^*$ the unique extension of a p-map h which maps X to Y. For any space Z, i_z will denote the identity map on Z. We will need to make use of the following characterization of the quotient proximity:

3.1. THEOREM [3]. *Let* $f\colon (X, \delta) \to Y$ *be a function from a proximity space* (X, δ) *onto a set* Y. *The quotient proximity* δ' *on* Y *is given by:* $A \operatorname{non}\delta' B$ *iff there is some function* $g\colon Y \to I$ *such that* $g(A) = 0$, $g(B) = 1$ *and* $g \circ f$ *is a* p*-map.*

3.2. DEFINITION. A map (p-map) f of X onto Y is *bi-quotient* (p-bi-quotient) iff for every topological space (separated proximity space) Z, $f \times i_z\colon X \times Z \to Y \times Z$ is a quotient (p-quotient) map.

Before proving the main result of this section, we shall need two lemmas.

3.3. LEMMA. *Let* $\{f_a\}$ *be a collection of* p*-maps such that* $f_a\colon X_a \to Y_a$ *for all* a. *Then* $f\colon \Pi X_a \to \Pi Y_a$ *defined by* $[f(x)]_a = f_a(x_a)$ *is a* p*-map.*

Proof. It is sufficient to show $\pi_a \circ f$ is a p-map for all a. But $(\pi_a \circ f)(x) = f_a(x_a) = f_a \circ \pi_a(x)$, and the result follows.

3.4. LEMMA. *Let* $f_1\colon X_1 \to Y_1$ *and* $f_2\colon X_2 \to Y_2$ *be a* p*-quotient maps and let* $Y_1 \times Y_2$ *have the quotient proximity induced by* $F = f_1 \times f_2$. *Then if* $x_1 \in X_1$ *and* $x_2 \in X_2$, $F|_{X_1 \times \{x_2\}}\colon X_1 \times \{x_2\} \to Y_1 \times \{f_2(x_2)\}$ *and* $F|_{\{x_1\} \times X_2}\colon \{x_1\} \times X_2 \to \{f_1(x_1)\} \times Y_2$ *are* p*-quotient maps.*

Proof. Let δ' be the quotient proximity on $Y_1 \times Y_2$, δ_s the restriction of δ' to $S = Y_2 \times \{f_2(x_2)\} = Y_2 \times \{y_2\}$, and δ_s' the quotient proximity on S induced by $F|_{X_1 \times \{x_2\}}$. Then since the restriction of a p-map is a p-map, $\delta_s < \delta_s'$. To show $\delta_s' < \delta_s$, let $A \operatorname{non}\delta_s' B$, where $A, B \subseteq S$. We must prove that $A \operatorname{non}\delta_s B$. By the definition of the quotient proximity δ_s' on S, there is some $g_s\colon S \to I$ such that $g_s(A) = 0$, $g_s(B) = 1$ and $G_s = g_s \circ (F|_{X_1 \times \{x_2\}})$ is a p-map.

Extend G_s to $X_1 \times X_2$ as follows: $G(x, y) = G_s(x, x_2)$. Further, let $g\colon Y_1 \times Y_2 \to I$ be defined by $g(u, z) = g_s(u, y_2)$. Now $(g \circ F)(x, y) = G(x, y)$ since

$$(g \circ F)(x, y) = g(F(x, y)) = g(f_1(x), f_2(y)) = g_s(f_1(x), y_2)$$
$$= g_s(F|_{X_1 \times \{x_2\}})(x, x_2) = G_s(x, x_2) = G(x, y)\,.$$

Since X_1 is p-isomorphic to $X_1 \times \{x_2\}$, say by $\psi(x) = (x, x_2)$, G is equal to $G_s \circ \psi \circ \pi_{X_1}$, and thus is a p-map. Therefore, $g(A) = 0$, $g(B) = 1$ and $g \circ F = G$ is a p-map, so by the definition of the quotient proximity on $Y_1 \times Y_2$, $A \operatorname{non}\delta' B$. But since δ_s is the restriction of δ' to S, $A \operatorname{non}\delta_s B$. Hence, $\delta_s = \delta_s'$.

The corresponding result with $x_1 \in X_1$ follows similarly.

Note that this result implies that $Y_1 \times \{y_2\}$ as a subspace of $(Y_1 \times Y_2, \delta')$ is p-isomorphic to Y_1.

3.5. THEOREM. *Let* $f_1\colon X_1 \to Y_1$ *and* $f_2\colon X_2 \to Y_2$ *be* p*-quotient maps between separated proximity spaces. Then* $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ *is a* p*-quotient map.*

Proof. Let δ and P be the product proximities on $X_1 \times X_2$ and $Y_1 \times Y_2$ respectively and let δ' be the quotient proximity on $Y_1 \times Y_2$ induced by $F = f_1 \times f_2$. By Lemma 3.3 and the definition of the quotient proximity, $P < \delta'$. If $P \neq \delta'$, $Y_1 \times Y_2$ contains two subsets A and B such that APB but $A \operatorname{non}\delta' B$.

Since APB, there is a point in $\mathrm{Cl}_{(Y_1 \times Y_2)^*} A \cap \mathrm{Cl}_{(Y_1 \times Y_2)^*} B$, where $(Y_1 \times Y_2)^*$ is the Smirnov Compactification of $(Y_1 \times Y_2, P)$. By a result of Leader [7], $(X_1 \times X_2)^* = X_1^* \times X_2^*$ and $(Y_1 \times Y_2)^* = Y_1^* \times Y_2^*$, so say (y^*, z^*) is the point in the intersection of $\mathrm{Cl}_{(Y_1 \times Y_2)^*} A$ and $\mathrm{Cl}_{(Y_1 \times Y_2)^*} B$. If f_i^*, $i = 1, 2$ is the extension of f_i to X_i^*, then $F^* = f_1^* \times f_2^*$ must be the unique extension of $F\colon (X_1 \times X_2, \delta) \to (Y_1 \times Y_2, P)$ to $X_1^* \times X_2^*$.

Consider $W = F^{*-1}(y^*, z^*) = f_1^{*-1}(y^*) \times f_2^{*-1}(z^*)$.

(i) *We claim that* $\mathrm{Cl}_{(X_1 \times X_2)^*} F^{-1}(A) \cap W = \emptyset$.

Assume not. Then there is an open set U containing W such that $U \cap F^{-1}(A) = \emptyset$. But since $f_1^{*-1}(y^*)$ and $f_2^{*-1}(z^*)$ are compact, open sets U_1 in X_1^* and U_2 in X_2^* exist such that $W \subseteq U_1 \times U_2 \subseteq U$. Now, f_1^* and f_2^* are closed maps, hence easily hereditarily quotient. It follows from Remark 2.3 (a) that $f_1^*(U_1)$ is a neighborhood of y^* and $f_2^*(U_2)$ is a neighborhood of z^*; hence, $F^*(U_1 \times U_2) = f_1^*(U_1) \times f_2^*(U_2)$ is a neighborhood of (y^*, z^*). But $U_1 \times U_2 \subseteq U$ and $F^*(U) \cap A = \emptyset$ — which contradicts the fact that $(y^*, z^*) \in \mathrm{Cl}_{(Y_1 \times Y_2)^*} A$. *This establishes the claim.* Similarly, $\mathrm{Cl}_{(X_1 \times X_2)^*} F^{-1}(B) \cap W \neq \emptyset$.

Let (a, b) be a point in $\mathrm{Cl}_{(X_1 \times X_2)^*} F^{-1}(A) \cap W$ and let (c, d) be a point in $\mathrm{Cl}_{(X_1 \times X_2)^*} F^{-1}(B) \cap W$.

Since A non $\delta' B$, there is a function $g\colon Y\to I$ such that $g(A)=0$, $g(B)=1$ and $g\circ F$ is a p-map. Our objective is to show that the extension, $(g\circ F)^*$, of $g\circ F$ to $(X_1\times X_2)^*$ takes (a,b) and (c,d) to the same point in I. But first, let $x_2\in X_2$.

(ii) *We claim that* $(g\circ F)^*(a,x_2)=(g\circ F)^*(c,x_2)$.

If we consider F as a map onto $(Y_1\times Y_2,\delta')$, then by Lemma 9.4, $Y_2\times\{y_2\}=Y_2\times\{f_2(x_2)\}$ has the quotient proximity induced by $F|_{X_1\times\{x_2\}}$. Then, since $(g\circ F)|_{X_1\times\{x_2\}}=(g|_{Y_1\times\{y_2\}})\circ(F|_{X_1\times\{x_2\}})$, it follows from Theorem 1 of [10] that $g|_{Y_1\times\{y_2\}}$ is a p-map. But the extension of a p-map to the Smirnov Compactification is unique, hence

(iii) $\big((g\circ F)|_{X_1\times\{x_2\}}\big)^*=(g\circ F)^*|_{X_1^*\times\{x_2\}}=(g|_{Y_1\times\{y_2\}})^*\circ(F|_{X_1\times\{x_2\}})^*$

where $\big((g\circ F)|_{X_1\times\{x_2\}}\big)^*$ is the extension of $(g\circ F)|_{X_1\times\{x_2\}}$ to $X_1^*\times\{x_2\}$ and $(g|_{Y_1\times\{y_2\}})^*$ is the extension of the p-map $g|_{Y_1\times\{y_2\}}$ to $(Y_1\times\{y_2\})^*$ (which is $Y_1^*\times\{y_2\}$ by the remark following Lemma 9.4). Now, $f_1^*(a)=f_1^*(c)=y^*$, so $(F|_{X_1\times\{x_2\}})^*(a,x_2)=(F|_{X_1\times\{x_2\}})^*(c,x_2)$ and hence

$$(g|_{Y_1\times\{y_2\}})^*\circ(F|_{X_1\times\{x_2\}})^*(a,x_2)=(g|_{Y_1\times\{y_2\}})^*\circ(F|_{X_1\times\{x_2\}})^*(c,x_2)\,.$$

It follows from equation (iii) above that $(g\circ F)^*(a,x_2)=(g\circ F)^*(c,x_2)$, *establishing claim* (ii). If we repeat the above argument with the roles of X_1 and X_2 interchanged, then it follows similarly that $(g\circ F)^*(x_1,b)=(g\circ F)^*(x_1,d)$ for any $x_1\in X_1$.

(iv) We now use a limiting process to show $(g\circ F)^*(a,b)=(g\circ F)^*(c,d)$.

Pick a net $\langle(a_\alpha,b_\alpha)\rangle$ in $F^{-1}(A)$ converging to (a,b). Then $\langle b_\alpha\rangle\subseteq X_2$ converges to b, so $\langle(a,b_\alpha)\rangle\to(a,b)$ and $\langle(c,b_\alpha)\rangle\to(c,b)$. But for each α, it follows from (ii) that $(g\circ F)^*(a,b_\alpha)=(g\circ F)^*(c,b_\alpha)$, hence in the limit,

$$(g\circ F)^*(a,b)=(g\circ F)^*(c,d)\,.$$

If we pick a net in $F^{-1}(B)$ converging to (c,d), a similar argument will show

$$(g\circ F)^*(a,d)=(g\circ F)^*(c,d)\,.$$

Again, we can easily interchange the roles of X_1 and X_2 to show that

$$(g\circ F)^*(a,d)=(g\circ F)^*(a,b)\quad\text{and}\quad(g\circ F)^*(c,d)=(g\circ F)^*(c,b)\,.$$

Putting these together we easily verify (iv).

However, since $g(A)=0$, $(g\circ F)\big(F^{-1}(A)\big)=0$, so that

$$(g\circ F)^*\big(\mathrm{Cl}_{(X_1\times X_2)^*}F^{-1}(A)\big)=0.\ \text{Similarly}\ (g\circ F)^*\big(\mathrm{Cl}_{(X_1\times X_2)^*}F^{-1}(B)\big)=1\,.$$

In particular, $(g\circ F)^*(a,b)=0$ and $(g\circ F)^*(c,d)=1$ — which contradicts (iv).

It follows that $\delta'=P$ and $f_1\times f_2$ is a p-quotient map. This completes the proof.

3.6. Corollary. *If $f_i\colon X_i\to Y_i$, $i=1,\dots,n$ are p-quotient maps between separated proximity spaces, then* $F\colon\prod_{i=1}^n X_i\to\prod_{i=1}^n Y_i$ *defined by* $F(x_1,\dots,x_n)=\big(f_1(x),\dots,f_n(x_n)\big)$ *is a p-quotient map.*

3.7. Corollary. *A p-quotient map between separated proximity spaces is p-bi-quotient.*

The next lemma follows from a result of Hager [5].

3.8. Definition. A topological space X is *pseudocompact* iff every real-valued continuous function on X is bounded.

3.9. Lemma. *Let X and Y be infinite separated proximity spaces each with the fine proximity. Then the product proximity is the fine proximity on $X\times Y$ iff $X\times Y$ is pseudocompact.*

3.10. Theorem. *Let $f_1\colon X_1\to Y_1$ and $f_2\colon X_2\to Y_2$ be (topological) quotient maps between $T_{3^1/_2}$ spaces and let X_1 and X_2 be infinite. Then if $X_1\times X_2$ is pseudocompact, $f_1\times f_2$ is a quotient map.*

Proof. Let X_1, X_2, Y_1 and Y_2 each have the fine proximity. Then by Lemma 3.9, the product proximity on $X_1\times X_2$ is the fine proximity and since $Y_1\times Y_2$ is $T_{3^1/_2}$, it follows from Theorem 2.3 of [3] that the quotient topology is equal to $\zeta(\delta')$. Now, by Theorem 3.4, δ' is the product proximity P on $Y_1\times Y_2$ so $\zeta(P)=\zeta(\delta')$ is equal to the quotient topology and hence, $f\times g$ is a quotient map.

4. Examples and relationships. In this section we explore the relationships between the maps introduced in sections 2 and 3 and their topological analogues.

4.1. Example. A bi-quotient, p-quotient map which is not regular.

Let $X=[0,2]$ and identify $[\frac12,\frac32]$ to a point. Then the natural map is easily bi-quotient and p-quotient. But $f[0,\frac12]\,\delta' f(\frac32,2]$, although $[0,\frac12)$ non $\delta\,(\frac32,2]$.

Example 4.1 contrasts with topological quotient maps since bi-quotient maps are hereditarily quotient.

4.2. Example. A regular map which is not quotient.

Let $X=R\times R$ with the product proximity. For $(x,y)\in R\times R$, let $f(x,y)$ be the point on R where a line through (x,y) at a positive angle of 45° with the x-axis crosses that axis. The quotient topology is clearly the usual topology. It is not hard to show that the quotient proximity is the trivial proximity and hence does not induce the quotient topology although f is regular.

4.3. EXAMPLE. A hereditarily quotient, p-quotient map which is not bi-quotient.

Let X be the disjoint union of countably many copies of $[0, 1]$ and let X have the elementary proximity. Identify all 0's to a common point. Then δ with the quotient topology is completely regular, so the natural map f is quotient and p-quotient. f is also easily hereditarily quotient. But Michael has shown [8, Example 8.1] that f is not bi-quotient.

In [8], Michael proves a theorem for bi-quotient maps that has an analogue for proximity spaces. His topological result might lead us to expect that 4.4 (a) below is equivalent to f being p-bi-quotient, and hence that 4.4 (a) holds for every p-quotient map. However, since 4.4 (a) easily implies that f is regular, Example 4.1 shows that this is not the case. The proof of the proposition is similar to Michael's.

4.4. PROPOSITION. *If Y is a separated proximity space and $f\colon X\to Y$ is a p-map, then the following are equivalent:*

(a) *For A, a closed subset of Y, $\mathfrak{U}$ a p-cover of $f^{-1}(A)$, then finitely many $f(U)$, $U\in\mathfrak{U}$ cover some p-neighborhood of A.*

(b) *For A, a closed subset of Y, $\mathfrak{U}$ a p-cover of X, then finitely many $f(U)$, $U\in\mathfrak{U}$, cover some p-neighborhood of A.*

(A cover $U=\{U_a\}_{a\in A}$ *is a p-cover* of X iff there is a cover $V=\{V_a\}_{a\in A}$ such that for all a $V_a\Subset U_a$.)

Proof. $a\Rightarrow b$ is clear. To show $b\Rightarrow a$, let A be a closed subset of Y and $\mathfrak{U}$ a p-cover of $f^{-1}(A)$. Since Y is T_2, for each $x\notin A$ pick W_x and V_x such that $W_x\Subset V_x$ and V_x nonδ A. Let $\mathfrak{V}=\{V_x\}_{x\notin A}$ and $\mathfrak{W}=\mathfrak{U}\cup f^{-1}(\mathfrak{V})$. Then $\mathfrak{W}$ is a p-cover of X, so there is a set $N=C\cup D$ such that $A\Subset N$, where C is the union of finitely many $f(U)$, $U\in\mathfrak{U}$ and D is the union of finitely many V_x. But then A nonδ D implies $A\Subset C$.

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