On the product of derivatives

by

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Abstract. The purpose of this paper is to prove that the product of any derivative with an absolutely continuous function is a derivative. From this result it follows that the product of two derivatives is a derivative if for each point x there is a neighborhood of x in which one of the two derivatives is absolutely continuous. An example of a continuous function and a derivative whose product is not a derivative is given.

A sufficient condition under which the product of two derivatives is a derivative is given in the theorem below.

Without loss, functions will be considered defined on the closed interval $[0, 1]$. If A is the class of functions whose product with any derivative is a derivative, it is shown here that all absolutely continuous functions belong to A. The generalization to the product of derivatives follows without difficulty, i.e., given two derivatives $f(x)$ and $g(x)$, if at each point t there is a neighborhood of t in which either $f(x)$ or $g(x)$)is absolutely continuous, then $f \cdot g(x)$ is a derivative.

It does not appear to be known whether the functions in A must be continuous. An example of a continuous function not in A is obtained readily. On $(0, 1]$ let

$$f(x) = x^{1/2}\sin(1/x), \quad g(x) = x^{-1/2}\sin(1/x), \quad \text{and} \quad h(x) = x^{3/2}\cos(1/x).$$

Let $f(0) = g(0) = h(0) = 0$. That $f(x)$ is a continuous function, $h(x)$ a differentiable function and $h'(x) - g(x)$ a continuous function is easily checked. Consequently $g(x)$ is a derivative; but $f \cdot g(x)$ is not a derivative. (For a short proof of this, see [2].)

THEOREM. *If on* $[0, 1]$ $F(x)$ *is an absolutely continuous function and* $g(x)$ *is a derivative, then* $F \cdot g(x)$ *is a derivative.*

Proof. Let $g(x) = G'(x)$. It will be shown that $F \cdot g(x)$ is the derivative of $F(x)G(x) - \int_0^x f(t)G(t)\,dt$ where $f(t) = F'(t)$ almost everywhere.

In order to establish the last assertion it suffices to show that

$$\lim_{x \to a} \frac{1}{x - a}\Big(F(x)G(x) - F(a)G(a) - \int_a^x f(t)G(t)\,dt\Big) - F(a)g(a)$$

exists and equals 0 at each point a in $[0,1]$. This limit is equivalent to

$$\lim_{x\to a}\frac{1}{x-a}\big(F(x)G(x)-F(x)G(a)\big)-F(a)g(a)+$$
$$+\frac{1}{x-a}\Big(F(x)-F(a)G(a)-\int_a^x f(t)G(t)\,dt\Big)$$

which, since F is continuous and $G'(a)=g(a)$, is equal to

$$\lim_{x\to a}\frac{1}{x-a}\Big(\big(F(x)-F(a)\big)G(a)-\int_a^x f(t)G(t)\,dt\Big).$$

Since F is absolutely continuous and thus $F(x)-F(a)=\int_a^x f(t)\,dt$, the limit, provided it exists, is equal to

$$\lim_{x\to a}\frac{-1}{x-a}\int_a^x \big(f(t)G(t)-f(t)G(a)\big)\,dt\,.$$

Since $G'(a)=g(a)$, it follows that $G(t)-G(a)=\big(g(a)+\varepsilon(t)\big)(t-a)$ where $\varepsilon(t)\to 0$ as $t\to a$. Consequently

$$\Big|\frac{1}{x-a}\int_a^x f(t)G(t)-f(t)G(a)\,dt\Big|=\Big|\frac{1}{x-a}\int_a^x f(t)\big(g(a)+\varepsilon(t)\big)(t-a)\,dt\Big|$$
$$\leqslant\frac{|g(a)|+|\varepsilon(x)|}{|x-a|}\int_a^x |f(t)(t-a)|\,dt$$
$$\leqslant\big(|g(a)|+|\varepsilon(x)|\big)\int_a^x |f(t)|\,dt\,.$$

Since $f(t)$ is Lebesgue integrable, so is $|f(t)|$ and $\int_a^x |f(t)|\,dt\to 0$ as $x\to a$. Thus the original limit exists and equals 0 and the theorem is proved.

References

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Regular maps and products of p-quotient maps

by

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Abstract. It is shown that the finite product of proximity quotient maps between separated proximity spaces is a proximity quotient map. A theorem on products of topological quotient maps follows from this result. Also, the regular maps of Poljakov are shown to preserve the semi-metrizability of proximity spaces.

1. Introduction. Much current research in general topology has been concerned with generalizations of topological quotient maps. Two of these generalizations are hereditarily quotient maps, which have been considered by Archangel'skiĭ [2] and Michael [8], and the bi-quotient maps of Michael [8] and Hájek [6]. These two mappings have analogues in proximity spaces which, in general, preserve more structure than their topological counterparts.

Our notation will follow [3] and [15]. In particular, $A\Subset B$ will mean A non$\,\delta\,(X-B)$ and proximity maps and proximity quotient maps will be called p-maps and p-quotient maps respectively. If δ and δ' are two proximities on a set X, δ' is said to be *finer* than δ iff $A\,\delta' B$ implies $A\,\delta\,B$ for all subsets A and B of X. If $f\colon (X,\delta)\to Y$ is a function from a proximity space (X,δ) onto a set Y, the *quotient proximity* is the finest proximity on Y for which f is a p-map.

2. Regular maps. Poljakov in [13] introduced regular maps and asked if the regular image of a metrizable proximity space is metrizable. In [14] he showed that a proximity space can be "determined by sequences" iff it is the regular image of the disjoint union of metrizable proximity spaces. However, since the disjoint union of metrizable proximity spaces might not be a metrizable proximity space (although the induced topology is, of course, metrizable), this did not answer the original question. The purpose of this section is to show that the image of a semi-metrizable proximity space under a regular map is semi-metrizable, thus giving a partial solution to the problem. We begin with the definition and a characterization due to Poljakov [13].