To complete the proof of the Theorem we need only show that $f[X] \subset J^*$. Let $x \in X$, and $k_1 < \ldots < k_m$ be precisely those integers k for which $x \in U_k$; we shall prove that $f(x) \in J_{p(k_m+1)} - \{q\}$. For any coordinate mapping f_i we denote $t(i)$ by k and $i - p(k)$ by j, and note that $f_i = g_{k,j}$ by definition. If $k \neq k_h$ for all $h = 1, \ldots, m$, then $f_i(x) = g_{k,j}(x) = 0$ since $x \notin U_k$ and $G_k(x) = (0, \ldots, 0)$ for $x \in X - U_k$. Otherwise, $k = k_h$ for some $h \leqslant m$, so

$$f_i(x) = g_{k,j}(x) \leqslant 1/s(k) = 1/p\big(r(k)+1\big) \leqslant 1/p(k_m+1)$$

by (1) and since $x \in U_k \cap U_{k_m}$ implies $k_m \leqslant r(k)$. Note also that at least one integer k does exist for which $x \in U_k$, so by (i) of the Lemma there exists an integer j such that $f_{p(k)+j}(x) = g_{k,j}(x) \neq 0$, so $f(x) \neq q$. Hence $f(x) \in J_{p(k_m+1)} - \{q\}$, and since the choice of x was arbitrary we have

$$f[X] \subset \bigcup_{n=1}^{\infty} (J_n - \{q\}) = \Big(\bigcup_{n=1}^{\infty} J_n\Big) - \{q\} = J - \{q\} = J^* .$$

We close with an application of this Theorem.

COROLLARY. *If X is a separable metric locally finite-dimensional space, then X has a locally finite-dimensional completion.*

Proof. Let f be the embedding of X into J^* given by the Theorem. It is known that J is compact [cf. 8], and J^* is open in J, so there exists a complete metric for J^* [1, Theorem 9, p. 189]. Relative to this complete metric the closure Y of $f[X]$ in J^* is complete, so Y is the desired completion of X.

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A characterization of determinacy for Turing degree games ([1])

by

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Abstract. A natural method for modifying partition properties of cardinals is to require the existence of homogeneous sets having certain additional special properties. In this paper such a modification of the partition property $\omega \to (\omega)^\omega$ is shown to lead to a natural characterization of the axiom of determinacy for Turing degree games.

1. The axiom of determinacy is most appropriately described in terms of two person games of infinite length: given any set A of reals (that is, of sets of integers) a game G_A exists and is played as follows: two players, I and II, move alternately writing at each turn a 0 or a 1. In this way they build an infinite sequence of 0's and 1's, that is, a function x from ω into $\{0, 1\}$. Player I wins if x is the characteristic function of a member of A. A *winning strategy* is a function f from finite sequences of 0's and 1's into $\{0, 1\}$ such that a player making only moves as indicated by f (his move following any initial play $a_1, \ldots, a_n$ should be $f(\langle a_1, \ldots, a_n\rangle)$) wins. Clearly both players can never simultaneously possess a winning strategy. A set of reals A is said to be *determinate* if there is a winning strategy (for I or for II) associated with G_A. The axiom of determinacy (AD) is the assertion that every set of reals is determinate.

Using the axiom of choice it is fairly easy to contradict AD. One can put together a nondeterminate set simply by diagonalizing over all strategies. However, the very rich theory of AD as well as its internal appeal makes a closer examination of the question "which sets of reals are determinate?" extremely worthwhile.

One very natural approach to the question is through the various set hierarchies. It is easy to see that every open set of reals is determinate, and most recently Paris has shown that in fact every Σ_4^0 set of reals is determinate. With a large cardinal assumption one can actually do quite well here — in particular there is Martin's result ([6]) that if a measurable cardinal exists then every analytic (Σ_1^1) set of reals is determinate.

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One of the outstanding open questions in set theory at this time is whether or not every Borel set of reals is determinate.

Another approach to the question of the determinateness of various sets of reals is due to Martin ([5]). He realized that much of the power of determinacy was retained if one restricted his attention solely to sets of reals well-defined with respect to Turing degree. (A set A of reals is well-defined with respect to Turing degree if whenever a real x is in A, every real Turing equivalent to x is in A). The key to his idea lies in the following lemma (Martin [5]). Assume that every set of reals well-defined under Turing degree is determinate. Then given any set $\mathcal{E}$ of Turing degrees there exists a degree d_0 such that the set of degrees greater than or equal to d_0 is either contained in $\mathcal{E}$ or is disjoint from $\mathcal{E}$. (A cone of degrees is a set of the form $\{d|\ d \geqslant d_0\}$. d_0 is said to be the *vertex of the cone* $\{d|\ d \geqslant d_0\}$.)

Now even though the proof of Martin's lemma is simple (given $\mathcal{E}$ let E be those reals whose degree is in $\mathcal{E}$ — then the degree of any strategy for G_E is an appropriate vertex — its associated cone is contained either in $\mathcal{E}$ or in $\mathcal{E}^c$ depending upon whether the strategy is for I or II, resp.) its strength is immediate. As an example consider Martin's proof that $\aleph_1$ is a measurable cardinal (assuming Turing degree determinacy): one simply notes that the two-valued measure on the degrees given by "$\mu(\mathcal{E}) = 1$ iff $\mathcal{E}$ contains a cone" can be transferred to a measure on $\aleph_1$. Namely, if f is the function sending any degree to the least ordinal not recursive in it, a set Q of countable ordinals will have measure 1 iff $\mu(f^{-1}(Q)) = 1$.

As it turns out, the question of whether or not every Borel set of reals well-defined under Turing degree is determinate, is also open.

2. Now it is easy to see that Martin's lemma in fact lends itself to a characterization of determinacy for Turing degree games. This is that any set A of reals well-defined under Turing degree is determinate iff its associated set of degrees either contains, or is disjoint from, a cone. What we now do is give an alternate characterization of Turing degree determinacy in terms of a modification of the partition relation $\omega \to (\omega)^\omega$.

Given a set of nonnegative integers x, $[x]^\omega$ denotes the collection of infinite subsets of x. Let A be a given collection of sets of nonnegative integers. Then viewing A as a partition of $[\omega]^\omega$ into two pieces ($\{A, A^c\}$ is the partition) we say, in keeping with the usual conventions of the study of partition relations, that a set of integers x is *homogeneous* for A if $[x]^\omega$ is contained in either A or in A^c. $\omega \to (\omega)^\omega$ denotes the assertion that every set of reals has a homogeneous set.

Using the axiom of choice it is easy to contradict $\omega \to (\omega)^\omega$ (Erdös-Rado [1]) and so once again one is up against the question "which sets of reals A have homogeneous sets?". The success here has been somewhat better than with determinacy. It turns out, for instance, that (without any large cardinal assumptions) every Σ_1^1 set of reals has a homogeneous set (Silver [7]).

The connection with determinacy comes when one asks for homogeneous sets possessing certain additional properties. For example, let A be a given set of reals well-defined under Turing degree. Then there are two questions one can ask about A: (1) Is A determinate? (2) Does A have an *introreducible* [2] homogeneous set? We shall show that these two questions are equivalent.

Remark. Since it is known how to construct homogeneous sets for Borel sets of reals and since it is extremely easy to put together introreducible sets, it is tempting to think that one might incorporate the two constructions and thereby prove Borel determinacy for Turing degree games. Another tempting approach is to look for an appropriate basis result. For example, the Kondo-Addison theorem gives that every Borel set of reals has a Δ_2^1 homogeneous set. If one could improve this to Π_1^1, Borel Turing degree determinacy would follow, for Jockusch has shown ([3]) that every Π_1^1 set has an introreducible subset and so our Borel set would have an introreducible homogeneous set. Unfortunately, following work of Soare, Jockusch has exhibited ([8]) a Borel set of reals with no introreducible homogeneous set. The Borel set, however, is not well-defined under Turing degree and hence its existence does not refute Borel determinacy. But Jockusch's example does indicate that one cannot simply combine standard constructions of homogeneous and of introreducible sets nor can one use any standard basis argument to find nice homogeneous sets. In fact, a result of Friedman ([2]) indicates that *any* proof of Borel determinacy for Turing degree games must be quite strange — for instance, any such proof must use the power set axiom in an iterated fashion $\aleph_1$-many times.

3. Rather than work directly with the notion "introreducible" we shall look at what are called "rigid" sets: if x and y are sets of nonnegative integers, $y \subseteq x$, we say that y is *spread-out* in x if between any two members of y there exist at least two members of x. A set x is said to be *rigid* if it is recursive in a spread-out subset of itself. Clearly any set which is introreducible is rigid. Our main theorem is the following:

THEOREM. *Let A be a given set of reals well-defined under Turing degree. Then A is determinate iff A (viewed as a partition of $[\omega]^\omega$) has a rigid homogeneous set.*

[2] A set of integers is said to be *introreducible* if it is recursive in each of its infinite subsets.

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COROLLARY. *In the statement of the above theorem, "rigid" can be replaced with "retraceable" or "introreducible".*

Proof of theorem. ($\Rightarrow$) Suppose that A is determinate. By Martin's lemma let d_0 be a degree such that if R_{d_0} is the set of reals whose degree is at least d_0 then either $R_{d_0} \subseteq A$ or $R_{d_0} \subseteq A^c$. Let x be a set of nonnegative integers of degree d_0 and let x^* be the set of codes of initial segments of x (we assume the existence of a fixed recursive coding of the finite sequences of nonnegative integers into nonnegative integers). Then it is easy to see that x^* is of degree d_0 and is introreducible. Thus x^* is rigid, and if $y \in [x^*]^\omega$ x^* is recursive in y, i.e. $y \in R_{d_0}$. Thus x^* is a rigid set homogeneous for A.

($\Leftarrow$) Suppose that A has a rigid homogeneous set. Let A^* be the set of degrees associated with A, that is, $A^* = \{d \mid d \subseteq A\}$. By Martin's characterization it would suffice to find a degree d_0 such that the cone with vertex d_0 is either contained in, or is disjoint from, A^*. Let x be a rigid set homogeneous for A and let y be a spread-out subset of x such that x is recursive in y.

Claim. If u is any set such that y is recursive in u then there exists a subset v of x such that u and v are of the same Turing degree.

(Proof of claim. The construction here is due essentially to Soare ([9]). Let u be given. We form the subset v of x as follows: for each n let y_n be the nth largest element of y. Then for any n, the $(2n+1)$st largest element of v is y_n and the $2n$th largest element of v is either the first element in x larger than y_n or the second element in x larger than y_n depending on whether $n \in u$ or $n \notin u$, respectively. Now why is v Turing equivalent to u? Well clearly one can recover v from u for as y and x are both recursive in u (x is recursive in y recall) we can, given u, list v in increasing order simply as indicated above. On the other hand suppose that we start with v. Since the odd elements of a list of v in increasing order are precisely those in y, and since x is recursive in y we can, starting with v, make lists of x and y in increasing order. u can now be listed in increasing order simply by examining the even elements in the list of v in increasing order in light of the lists of x and y.)

Our theorem now follows. For if d_0 is the degree of y, the claim easily gives that the cone associated with d_0 is either contained in A^* or is disjoint from A^* depending upon whether $[x]^\omega \subseteq A$ or $[x]^\omega \subseteq A^c$, respectively.

In proving this theorem we also establish the corollary. The only point to observe is that the set x^* mentioned in the first half of the proof is, in fact, retraceable.

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