Однако заметим, что справедливо следующее

Предложение 5. *Пусть X удовлетворяет первой аксиоме счетности и $s(X) \leqslant \aleph_1$. Тогда X сепарабельно тогда и только тогда, когда оно обладает свойством* (C).

Доказательство несложно.

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A universal separable metric locally finite-dimensional space

by

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Abstract. The author has previously defined a topological space to be *locally finite-dimensional* if every point has a neighborhood of finite covering dimension [cf. Pacific J. Math. 42 (1972), pp. 267–276]. In this communication we introduce a locally finite-dimensional subset of the Hilbert Cube in which every separable metric locally finite-dimensional space can be topologically embedded. From this it follows readily that every separable metric locally finite-dimensional space has a locally finite-dimensional complete extension.

By a space *universal in the class* $\mathcal{C}$ we shall understand a member of $\mathcal{C}$ in which every member of $\mathcal{C}$ can be topologically embedded (X is said to be *topologically embedded in* Y iff there exists a homeomorphism from X onto a subspace of Y). Well-known universal spaces have been obtained in the classes of separable metric n-dimensional spaces (by K. Menger [2] and G. Nöbeling [6]), separable metric countable-dimensional spaces (by J. Nagata [4]), and separable metric strongly countable-dimensional spaces (by J. Nagata [4] and Ju. M. Smirnov [7]), respectively. In this communication we shall describe a space universal in the class of separable metric locally finite-dimensional spaces.

The dimension of a space X will be denoted by $\dim X$, and will be interpreted as the covering dimension of Lebesgue [cf. 5]. A space X is said to be *locally finite-dimensional* iff every point of X has a finite-dimensional neighborhood. We denote the Hilbert Cube by I^∞, and for each $n = 1, 2, \ldots$ we define a subset

$$J_n = \{(x_i) \in I^\infty\colon\ 0 \leqslant x_i \leqslant 1/n \text{ for } i = 1, \ldots, n, \text{ and } x_i = 0 \text{ for } i > n\},$$

and

$$J = \bigcup_{n=1}^{\infty} J_n .$$

Let q denote the origin of I^∞, and define $J^* = J - \{q\}$. The author has shown in an earlier work [8, Lemma 3] that J^* is locally finite-dimensional, and J^* is a separable metric space since it is a subset of I^∞. In order to prove that any separable metric locally finite-dimensional space can be topologically embedded in J^* we shall require the following Lemma.

LEMMA. *Let X be a separable metric space, U an open subset of X with $\dim \overline{U} \leqslant k$, and let $\varepsilon > 0$. Define $E(k, \varepsilon) = \{(x_1, \ldots, x_{2k+2}) \in E^{2k+2}\colon 0 \leqslant x_i \leqslant \varepsilon$ for all $i = 1, \ldots, 2k+2\}$. Then there exists a continuous mapping $G\colon X \to E(k, \varepsilon)$ such that*

(i) *$G(x) = (0, \ldots, 0)$ iff $x \in X - U$, and*

(ii) *the restriction of G to U is a homeomorphism.*

Proof. This is essentially a special case of a lemma due to Smirnov [7, Lemma 2]. In the terminology of that lemma we let $A = \overline{U} - U$ and $R = B = \overline{U}$, and note that Smirnov's $Q^{n(k)}$ is homeomorphic to $E(k, \varepsilon)$, so Smirnov's mapping F can be considered to be a mapping into $E(k, \varepsilon)$ with the same properties. If by F' we denote the constant mapping of $X - U$ onto the origin of $E(k, \varepsilon)$, we clearly obtain the desired mapping $G\colon X \to E(k, \varepsilon)$ by letting G take on the values given by F on $\overline{U}$ and by F' on $X - U$ (note that F and F' agree on the boundary of U).

THEOREM. *Let X be a separable metric locally finite-dimensional space. Then X can be topologically embedded in J^*.*

Proof. Since X is locally finite-dimensional we can use regularity to find an open cover consisting of sets whose closures are finite-dimensional. A separable metric space is Lindelöf, so there exists an open starfinite cover $\{U_k\colon k = 1, 2, \ldots\}$ of X for which $\dim \overline{U}_k = n(k) < \infty$ for all $k = 1, 2, \ldots$ [3, Theorem 10]. We now define the following four functions on the positive integers by means of the formulae:

$$p(k) = \sum_{i=1}^{k-1} \big(2n(i)+2\big), \quad p(1) = 0,$$

$$r(k) = \max\{j\colon U_j \cap U_k \neq \emptyset\},$$

$$s(k) = p\big(r(k)+1\big), \quad \text{and}$$

$$t(k) = \max\{j\colon p(j) < k\}.$$

The following assertions are easily verified:

(1) $\qquad j < k$ implies $p(j) < p(k)$,

(2) $\qquad j < k$ implies $t(j) \leqslant t(k)$,

(3) $\qquad t\big(p(k)\big) < k \leqslant t\big(p(k+1)\big)$ for all $k = 1, 2, \ldots$, and

(4) $\qquad t\big(p(k)+j\big) = k$ for all $k = 1, 2, \ldots$ and all $j \leqslant 2n(k)+2$.

For each $k = 1, 2, \ldots$ there exists a continuous mapping $G_k\colon X \to E\big(n(k), 1/s(k)\big)$ which satisfies conditions (i) and (ii) of the Lemma with respect to the open set U_k. For each $j = 1, \ldots, 2n(k)+2$ let $g_{k,j}$ denote the composition of G_k with the jth projection; i.e., for any $x \in X$, $G_k(x) = \big(g_{k,1}(x), \ldots, g_{k,2n(k)+2}(x)\big)$. It follows from the definitions that

(5) $\qquad 0 \leqslant g_{k,j}(x) \leqslant 1/s(k)$ for all $x \in X$, $k = 1, 2, \ldots$, $j = 1, \ldots, 2n(k)+2$.

The desired homeomorphism $f\colon X \to I^\infty$ is the function for which $f(x) = \big(f_1(x), f_2(x), \ldots\big)$, where $f_k = g_{t(k), k-p(t(k))}$ for all $k = 1, 2, \ldots$; intuitively, the coordinate functions for f are obtained by lining up in order the coordinate functions of successive G_k. Alternatively,

(6) $\qquad g_{k,j} = f_{p(k)+j}$ for all $k = 1, 2, \ldots$ and all $j = 1, \ldots, 2n(k)+2$,

since

$$g_{k,j} = g_{k,(p(k)+j)-p(k)} = g_{t(p(k)+j),\,(p(k)+j)-p(t(p(k)+j))} \underset{\text{(by (4))}}{=} f_{p(k)+j}$$

(the last equality follows from the definition of the coordinate function $f_{p(k)+j}$).

Clearly f is continuous, since each coordinate mapping f_i is simply a coordinate mapping of some continuous mapping G_k. To see that f is one-to-one, we let x and y be distinct points of X and choose k such that $x \in U_k$. If $y \in U_k$ we note that G_k is a homeomorphism on U_k, so $G_k(y) \neq G_k(x)$; hence there exists $j \leqslant 2n(k)+2$ such that $f_{p(k)+j}(y) = g_{k,j}(y) \neq g_{k,j}(x) = f_{p(k)+j}$ (by (6)), which implies that $f(y) \neq f(x)$. On the other hand, if $y \notin U_k$ then $G_k(y) = (0, \ldots, 0) \neq G_k(x)$ (by condition (i) of the Lemma), so there exists $j \leqslant 2n(k)+2$ for which $f_{p(k)+j}(x) = g_{k,j}(x) \neq 0 = g_{k,j}(y) = f_{p(k)+j}(y)$, so again $f(x) \neq f(y)$.

To show f is open we shall prove that if $x \in X$ and U is an open neighborhood of x, there exists an open neighborhood V of $f(x)$ such that $V \cap f[X] \subset f[U]$. Given x and U, let k be such that $x \in U_k$. For each $j \leqslant 2n(k)+2$ there exists an open neighborhood V_j (relative to the jth coordinate space of $E\big(2n(k)+2, 1/s(k)\big)$) of the coordinate $g_{k,j}(x)$ such that

(7) $$\Big(\prod_{j=1}^{2n(k)+2} V_j\Big) \cap G_k[X] \subset G_k[U \cap U_k],$$

by the definition of the product topology and the fact that G_k is open on U_k. We define the desired neighborhood in I^∞ by

$$V = \bigcap_{j=1}^{2n(k)+2} (\pi_{p(k)+j})^{-1}[V_j],$$

where by π_i we denote the ith projection into I^∞. Now let $y \in V \cap f[X]$. Then there exists $z \in X$ such that $f(z) = y \in V$, so $f_i(z) \in V_i$ for all $i = p(k)+1, \ldots, p(k)+\big(2n(k)+2\big) = p(k+1)$; i.e., $f_{p(k)+j}(z) = g_{k,j}(z) \in V_j$ for all $j = 1, \ldots, 2n(k)+2$. By (7) this implies that $z \in U \cap U_k$, so $y = f(z) \in f[U \cap U_k] \subset f[U]$, and f is open.

To complete the proof of the Theorem we need only show that $f[X] \subset J^*$. Let $x \in X$, and $k_1 < \ldots < k_m$ be precisely those integers k for which $x \in U_k$; we shall prove that $f(x) \in J_{p(k_m+1)} - \{q\}$. For any coordinate mapping f_i we denote $t(i)$ by k and $i - p(k)$ by j, and note that $f_i = g_{k,j}$ by definition. If $k \neq k_h$ for all $h = 1, \ldots, m$, then $f_i(x) = g_{k,j}(x) = 0$ since $x \notin U_k$ and $G_k(x) = (0, \ldots, 0)$ for $x \in X - U_k$. Otherwise, $k = k_h$ for some $h \leqslant m$, so

$$f_i(x) = g_{k,j}(x) \leqslant 1/s(k) = 1/p\big(r(k)+1\big) \leqslant 1/p(k_m+1)$$

by (1) and since $x \in U_k \cap U_{k_m}$ implies $k_m \leqslant r(k)$. Note also that at least one integer k does exist for which $x \in U_k$, so by (i) of the Lemma there exists an integer j such that $f_{p(k)+j}(x) = g_{k,j}(x) \neq 0$, so $f(x) \neq q$. Hence $f(x) \in J_{p(k_m+1)} - \{q\}$, and since the choice of x was arbitrary we have

$$f[X] \subset \bigcup_{n=1}^{\infty} (J_n - \{q\}) = \Big(\bigcup_{n=1}^{\infty} J_n\Big) - \{q\} = J - \{q\} = J^* .$$

We close with an application of this Theorem.

COROLLARY. *If X is a separable metric locally finite-dimensional space, then X has a locally finite-dimensional completion.*

Proof. Let f be the embedding of X into J^* given by the Theorem. It is known that J is compact [cf. 8], and J^* is open in J, so there exists a complete metric for J^* [1, Theorem 9, p. 189]. Relative to this complete metric the closure Y of $f[X]$ in J^* is complete, so Y is the desired completion of X.

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A characterization of determinacy for Turing degree games (1)

by

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Abstract. A natural method for modifying partition properties of cardinals is to require the existence of homogeneous sets having certain additional special properties. In this paper such a modification of the partition property $\omega \to (\omega)^{\omega}$ is shown to lead to a natural characterization of the axiom of determinacy for Turing degree games.

1. The axiom of determinacy is most appropriately described in terms of two person games of infinite length: given any set A of reals (that is, of sets of integers) a game G_A exists and is played as follows: two players, I and II, move alternately writing at each turn a 0 or a 1. In this way they build an infinite sequence of 0's and 1's, that is, a function x from ω into $\{0, 1\}$. Player I wins if x is the characteristic function of a member of A. A *winning strategy* is a function f from finite sequences of 0's and 1's into $\{0, 1\}$ such that a player making only moves as indicated by f (his move following any initial play $a_1, \ldots, a_n$ should be $f(\langle a_1, \ldots, a_n\rangle)$) wins. Clearly both players can never simultaneously possess a winning strategy. A set of reals A is said to be *determinate* if there is a winning strategy (for I or for II) associated with G_A. The axiom of determinacy (AD) is the assertion that every set of reals is determinate.

Using the axiom of choice it is fairly easy to contradict AD. One can put together a nondeterminate set simply by diagonalizing over all strategies. However, the very rich theory of AD as well as its internal appeal makes a closer examination of the question "which sets of reals are determinate?" extremely worthwhile.

One very natural approach to the question is through the various set hierarchies. It is easy to see that every open set of reals is determinate, and most recently Paris has shown that in fact every Σ^0_4 set of reals is determinate. With a large cardinal assumption one can actually do quite well here — in particular there is Martin's result ([6]) that if a measurable cardinal exists then every analytic (Σ^1_1) set of reals is determinate.

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