Irreducibility and indecomposability in inverse limits (*)

by

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Abstract. Irreducibility, indecomposability, and hereditary indecomposability of the inverse limit of an inverse system over the positive integers are related to the bonding maps. It is shown that if there is a hereditarily indecomposable inverse limit on a sequence of nondegenerate compact metric continua, then in a certain sense almost all inverse limits on the sequence are hereditarily indecomposable.

What are the relationships between the bonding maps in an inverse limit system and various topological properties of the inverse limit space? We investigate this question for irreducibility and indecomposability and obtain characterizations of these properties in terms of the bonding maps. We then show that if there is a hereditarily indecomposable inverse limit on a sequence of nondegenerate compact metric continua, then in a certain sense almost all inverse limits on the sequence are hereditarily indecomposable.

The closure of the set A will be denoted by $\mathrm{cl}(A)$. The notation $\Pi_n(S)$ means the nth coordinate projection of a subset S of a product of spaces over the positive integers.

We assume that (X_i, f_i^{i+1}) is an inverse system such that, for each i, X_i is a compact nondegenerate metric continuum with metric d_i and f_i^{i+1} maps X_{i+1} onto X_i. Let M be the inverse limit of the system.

LEMMA. *Suppose that M is irreducible about the points $p^1, \ldots, p^k$ and $K_1, K_2, K_3, \ldots$ is a sequence of continua such that if n is a positive integer and j is a positive integer not greater than k then the n-th coordinate of P^j, p_n^j, is in K_n and K_n is a subcontinuum of X_n. Then if n is a positive integer,* $\mathrm{cl}\big(\bigcup_i f_n^{n+i}(K_{n+i})\big) = X_n$.

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Proof. Let n be a positive integer. For each positive integer j let $Y_j = \mathrm{cl}\big(\bigcup_i f_j^{j+i}(K_{j+i})\big)$; let g_j^{j+1} be the restriction of f_j^{j+1} to Y_{j+1}. For each j, Y_j is a continuum and g_j^{j+1} maps Y_{j+1} onto Y_j. The inverse limit of the system (Y_j, g_j^{j+1}) is M since it is a continuum containing p^j for each positive integer j not greater than k. This implies that $Y_n = X_n$ for each n.

THEOREM 1. *Suppose that for each positive integer i not greater than the positive integer k, p^i is a point of M. Then the following three statements are equivalent.*

(1) *M is irreducible about the points $p^1, \ldots, p^k$.*

(2) *If $K_1, K_2, K_3, \ldots$ is a sequence of continua such that for each positive integer n and positive integer i not greater than k, p_n^i is in K_n and K_n is a subcontinuum of X_n, then, for each positive integer n, $\mathrm{cl}\big(\bigcup_i f_n^{n+i}(K_{n+i})\big) = X_n$.*

(3) *If n is a positive integer and $0 < e$, there is a positive integer L, $n < L$, such that if $L < m$ and K is a subcontinuum of X_m containing p_m^i for each positive integer i not greater than k, then, for each point x in X_n, $d_n\big(x, f_n^m(K)\big) < e$.*

Proof. (3) implies (2): Suppose that (3), but not (2), holds. Then there is a sequence $K_1, K_2, K_3, \ldots$ of continua such that for each positive integer n and each positive integer i not greater than k, p_n^i is in K_n and there exists a positive integer n such that $\mathrm{cl}\big(\bigcup_i f_n^{n+i}(K_{n+i})\big) \neq X_n$. Let x be a point of $X_n - \mathrm{cl}\big(\bigcup_i f_n^{n+i}(K_{n+i})\big)$ and let e be a positive number such that if $d_n(x, y) < e$, then y is not in $\mathrm{cl}\big(\bigcup_i f_n^{n+i}(K_{n+i})\big)$. By assumption, there is a positive integer m, $n < m$, such that $d_n\big(x, f_n^m(K_m)\big) < e$, a contradiction.

(2) implies (1): Suppose that (2) holds and N is a subcontinuum of M containing p^i for each positive integer i not greater than k. If n is a positive integer then, for each positive integer i, $f_n^{n+i}\big(\Pi_{n+i}(N)\big) = \Pi_n(N)$. Therefore, for each positive integer n, $\Pi_n(N) = \bigcup_i f_n^{n+i}\big(\Pi_{n+i}(N)\big)$; thus $\mathrm{cl}\big(\Pi_n(N)\big) = X_n$; since $\Pi_n(N)$ is closed, $\Pi_n(N) = X_n$. Since $\Pi_n(N) = X_n$ for each positive integer n, $N = M$.

(1) implies (3): Suppose that (1), but not (3), holds. Then there are a positive integer n and a positive number e such that if $n < m$ then there are a positive integer L not less than m and a subcontinuum K of X_L containing p_L^i for each positive integer i not greater than k such that there is a point x in X_n such that $d_n\big(x, f_n^L(K)\big)$ is not less than e; let $K_0 = f_m^L(K)$; then K_0 is a subcontinuum of X_m and $d_n(x, f_n^m(K_0)) = d_n\big(x, f_n^L(K)\big)$, which is not less than e. For each m, $n < m$, let K_m be a subcontinuum of X_m containing p_m^i for each positive integer i not greater than k such that there is a point x in X_n with $d_n\big(x, f_n^m(K_m)\big)$ not less than e.

Suppose that R is an open set with respect to X_n. Then R contains a point of $f_n^{n+i}(K_{n+i})$ for all but finitely many i; for suppose not. Then for some increasing sequence $i_1, i_2, \ldots$ of positive integers, if r is one of them, R does not contain a point of $f_n^{n+r}(K_{n+r})$. This gives a contradiction since the inverse limit of the system $(X_{n+i_j}, f_{n+i_j}^{n+i_{j+1}})$, where j ranges over the nonnegative integers and $i_0 = 0$, is irreducible about the points $p_n^j, p_{n+i_1}^j, p_{n+i_2}^j, \ldots, j = 1, 2, \ldots, k$, implying that $\mathrm{cl}\big(\bigcup_j f_n^{n+i_j}(K_{n+i_j})\big) = X_n$.

Since X_n is compact, some finite collection $[G_1, \ldots, G_r]$ of open sets each of diameter less than e covers X_n. For each j, $1 \leqslant j \leqslant r$, there is a positive integer N_j such that if $N_j \leqslant m$ then G_j contains a point of $f_n^m(K_m)$. Let $m = N_1 + N_2 + \ldots + N_r$. Then if x is in X_n, x is in G_j for some j not greater than r and there is a point y in G_j and in $f_n^m(K_m)$; $d_n(x, y) < e$ since the diameter of G_j is less than e; thus $d_n\big(x, f_n^m(K_m)\big) < e$. This gives a contradiction.

THEOREM 2. *The following two statements are equivalent.*

(1) *M is indecomposable.*

(2) *If n is a positive integer and $0 < e$, there are a positive integer m, $n < m$, and three points of X_m such that if K is a subcontinuum of X_m containing two of them then $d_n\big(x, f_n^m(K)\big) < e$ for each x in X_n.*

Proof. (1) implies (2): There are three points between each two of which M is irreducible. Applying condition (3) of Theorem 1 to each pair of them, one sees that condition (2) holds.

(2) implies (1): Suppose that (1) does not hold. Since M is non-degenerate, there are proper subcontinua H and K of M such that $M = H \cup K$. There is a positive integer n such that $\Pi_n(H)$ and $\Pi_n(K)$ are proper subcontinua of X_n. There is a point p_1 in $X_n - \Pi_n(H)$: let e_1 be a positive number such that $d_n\big(p_1, \Pi_n(H)\big)$ is not less than e_1. There is a point p_2 in $X_n - \Pi_n(K)$; let e_2 be a positive number such that $d_n\big(p_2, \Pi_n(K)\big)$ is not less than e_2. Let $e = \min(e_1, e_2)$. If $n \leqslant m$ and a, b, and c are three points of X_m, then two of a, b, and c are in one of $\Pi_m(H)$ and $\Pi_m(K)$ and since $f_n^m\big(\Pi_m(H)\big) = \Pi_n(H)$ and $f_n^m\big(\Pi_m(K)\big) = \Pi_n(K)$, it follows that (2) does not hold.

THEOREM 3. *If K is a positive integer greater than one, then the following two statements are equivalent.*

(1) *M is irreducible about some K points.*

(2) *If n is a positive integer and $0 < e$, there are a positive integer m, $n < m$, and K points of X_m such that if H is a subcontinuum of X_m containing them then $d_n\big(x, f_n^m(H)\big) < e$ for each x in X_n.*

Proof. According to a theorem of R. H. Sorgenfrey [3], M is irreducible about some K points if and only if M is not the sum of $K+1$ subcontinua such that the sum of each K of them is a proper subcontinuum of M. Using this result, the proof is essentially the same as the proof of Theorem 2.

THEOREM 4. *The following two statements are equivalent.*

(1) *M is hereditarily indecomposable.*

(2) *If n is a positive integer and $0 < e$, there is a positive integer m, $n < m$, such that if K is a subcontinuum of X_m then there are three points of K such that if H is a subcontinuum of K containing two of them then $d_n(x, f_n^m(H)) < e$ for each x in $f_n^m(K)$.*

Proof. (2) implies (1): Suppose that (2) holds and N is a subcontinuum of M. Then N is the inverse limit on its projections and condition (2) of Theorem 2 holds for the inverse system on the projections of N. Therefore N is indecomposable.

(1) implies (2): Suppose that (1), but not (2), holds. Then there is a positive integer n such that there is a positive number e such that if m is a positive integer greater than n there is a subcontinuum K_m of X_m such that for any three points of K_m there is a subcontinuum H of K_m containing two of them with $d_n(x, f_n^m(H))$ not less than e for some point x in $f_n^m(K_m)$. Assume without loss of generality that $n = 1$. Some subsequence of the sequence $f_1^2(K_2), f_1^3(K_3), \ldots$ converges to a subcontinuum of X_1; noting that M is homeomorphic to the inverse limit on any subsequence of $X_1, X_2, X_3, \ldots$, we might as well assume that the sequence $f_1^2(K_2), f_1^3(K_3), \ldots$ converges; denote its limit by H_1. Some subsequence of the sequence $f_2^3(K_3), f_2^4(K_4), \ldots$ converges to a subcontinuum H_2 of X_2; again assume that the sequence itself converges. The sequence $f_1^2(K_2)$, $f_1^3(K_3), \ldots$, after the change in notation involved in making the above assumption, still converges to H_1. We claim that $f_1^2(K_2) = H_1$. Clearly, f_1^2 maps H_2 into H_1. Now suppose that y is a point of H_1. There is a sequence of points from a subsequence of $K_3, K_4, \ldots$, such that the sequence of images in X_1 converges to y; some subsequence of the sequence of images in X_2 converges to a point x of H_2; $f_1^2(x) = y$. Continuing this process we obtain an inverse limit system (H_i, g_i^{i+1}), where g_i^{i+1} is the restriction of f_i^{i+1} to H_{i+1}; denote by N the inverse limit of this system. By assumption, N is indecomposable. Therefore, there are a positive integer $m > 1$ and three points a_0, b_0, and c_0 of H_m such that if J_0 is a subcontinuum of H_m containing two of them, then $d_1(x, f_1^m(J_0)) < e/2$ for each point x in H_1. Taking successive subsequences and changing notation, we get the following.

(A) For each i greater than m, a_i, b_i, and c_i are three points of K_i; $f_m^{m+1}(a_{m+1}), f_m^{m+2}(a_{m+2}), \ldots$, converges to a_0, etc.; J_i is a subcontinuum of K_i containing a_i and b_i such that there is a point p_i in $f_1^i(K_i)$ such that $d_1(p_i, f_1^i(J_i))$ is not less than e.

(B) $f_m^{m+1}(J_{m+1}), f_m^{m+2}(J_{m+2}), \ldots$ converges to a subcontinuum J_0 of H_m, and $f_1^{m+1}(J_{m+1}), f_1^{m+2}(J_{m+2}), \ldots$ converges to $f_1^m(J_0)$. Note that J_0 contains a_0 and b_0.

(C) The sequence $p_{m+1}, p_{m+2}, \ldots$ converges to a point p_0 of H_1. The $e/2$ neighborhood of P_0, $N_{e/2}(p_0)$, intersects $f_1^m(J_0)$ and therefore intersects $f_1^i(J_i)$ for i sufficiently large. Also, p_i is in $N_{e/2}(p_0)$ for i sufficiently large. This implies that, for i sufficiently large, $d_1(p_i, f_1^i(J_i)) < e$, a contradiction.

Theorems similar to the following have been proved by Bing [1] and Mazurkiewicz [2]. Dr. Howard Cook suggested to me that the methods developed above could be used to prove this theorem.

THEOREM 5. *Suppose $X_1, X_2, X_3, \ldots$ is a sequence of nondegenerate compact metric continua and d_i is a metric for X_i for each i. For each positive integer i, let F_i denote the space of continuous functions from X_{i+1} onto X_i with the metric $D_i(f, g) = \max_{x \text{ in } X_{i+1}} d_i(f(x), g(x))$. Let S be the product over the positive integers of the spaces F_i. We assume that $\operatorname{diam}(X_i) = 2^{-i}$ for each i; it follows that S is metrized by taking the distance between points to be the sum of the distances between their coordinates; denote this metric by r. Let A be the set of elements of S for which the inverse limit of the corresponding system is hereditarily indecomposable. Then if A is not the empty set, A is a dense inner limiting subset of the complete metric space S.*

Proof. Suppose that A is not empty and let $f_1^2, f_2^3, \ldots$ be an element of A. As usual, for m greater than n, we denote by f_n^m the composition of the functions $f_n^{n+1}, f_{n+1}^{n+2}, \ldots, f_{m-1}^m$; we will follow this convention below for other sequences of functions. For each positive number e and positive integer n, let B_{en} be the set of elements $g_1^2, g_2^3, \ldots$ of S such that there is a positive integer m greater than n such that if K is a subcontinuum of X_m there are three points of K such that if H is a subcontinuum of K containing two of them then $d_n(x, f_n^m(H)) < e$ for each x in $f_n^m(K)$. We prove first that B_{en} is dense in S. Suppose that $h_1^2, h_2^3, \ldots$ is in S and E is a positive number. Let t be greater than n and sufficiently large so that the distance from $h_1^2, h_2^3, \ldots$ to $h_1^2, \ldots, h_{t-1}^t, f_t^{t+1}, \ldots$ is less than E. There is a number F greater than 0 such that if $d_t(x, y) < F$ then $d_n(h_n^t(x), h_n^t(y)) < e$. There is a positive integer m greater than t such that if K is a subcontinuum of X_m there are three points of K such that if H is a subcontinuum of K containing two of them then $d_t(x, f_t^m(H)) < F$ for each x in $f_t^m(K)$; for each x in $h_n^t f_t^m(K)$, $d_n(x, h_n^t f_t^m(H)) < e$. Thus B_{en} is dense in S for each positive number e and positive integer n. Now suppose that $g_1^2, g_2^3, \ldots$ is in B_{en}. Let m be a positive integer greater than n such that if K is a subcontinuum of X_m then there are three points of K such that if H is a subcontinuum of K containing two of them then $d_n(x, f_n^m(H)) < e$ for each x in $f_n^m(K)$. We show that there is a positive number $e_0 < e$ such that if K is a subcontinuum of X_m then there are three points of K such that if H is a subcontinuum of K containing two of them then $d_n(x, f_n^m(H)) < e_0$ for each x in $f_n^m(K)$; suppose this is not the case. For each positive integer i, let K_i be a subcontinuum of X_m such that if a, b, and c are three points of K_i then there is a subcontinuum H of K_i containing two of them such that there is a point x of $f_n^m(K_i)$ with $e-1/i \leqslant d_n(x, f_n^m(H))$. Some subsequence of $K_1, K_2, \ldots$ converges to a subcontinuum K_0 of X_m; for notational convenience, we assume that $K_1, K_2, \ldots$ converges to K_0. There are three points a_0, b_0, and c_0 of K_0 such that if H is a subcontinuum of K_0 containing two of them then $d_n(x, f_n^m(H)) < e$ for each x in $f_n^m(K_0)$. Taking subsequences and changing notation, we have sequences a, b, and c of points and a sequence H of continua such that, for each i, a_i, b_i, and c_i are points of K_i and H_i is a subcontinuum of K_i containing a_i and b_i such that there is a point p_i in K_i with $e-1/i \leqslant d_n(f_n^m(p_i), f_n^m(H_i))$ and such that $a_1, a_2, \ldots$ converges to a_0, etc., $H_1, H_2, \ldots$ converges to a subcontinuum H_0 of K_0, and $p_1, p_2, \ldots$ converges to a point p_0 of K_0. Note that H_0 is a subcontinuum of K_0 containing a_0 and b_0. But the fact that $d_n(f_n^m(p_i), f_n^m(H_i))$ is at least $e-1/i$ for each i implies that $d_n(f_n^m(p_0), f_n^m(H_0))$ is at least e, a contradiction.

To show that $g_1^2, g_2^3, \ldots$ is in the interior of B_{en} it remains to show that if $r(g_1^2, g_2^3, \ldots, h_1^2, h_2^3, \ldots)$ is sufficiently small then h_n^m differs from g_n^m at each point by less than $(e-e_0)/2$. We omit the proof of this.

We now have that for each positive integer n and positive number e, B_{en} is a dense open subset of S. Since S is complete, the intersection of all the sets B_{en} for e a positive rational number and n a positive integer is a dense inner limiting subset of S; but according to Theorem 4, this intersection is A.

References

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О мощности открытых покрытий топологических пространств

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Резюме. Устанавливаются оценки мощности открытых покрытий топологических пространств. Для этой цели используются инварианты: $c(X)$ — число Суслина, $cc(X)$ — наследственное число Суслина, $c_0(X) = \sup|\eta|$: η — дискретное семейство открытых в X множеств.

1) Если γ — локально $c(X)$ — кратное покрытие X, то $|\gamma| \leqslant c(X)$.

2) Если γ — локально конечное покрытие X, то $|\gamma| \leqslant c_0(X)$.

3) Если γ — точечно конечное покрытие совершенного пространства X, то $|\gamma| \leqslant cc(X)$.

4) Если γ — σ-точечно конечное покрытие бэровского пространства X, то $|\gamma| \leqslant c(X)$.

Среди других результатов отметим следующий:

5) Бэровское σ-пространство обладает плотным метризуемым подпространством.

Устанавливаются оценки мощности открытых покрытий топологических пространств. Для этой цели используются некоторые кардинально-значные инварианты пространств такие, как $c(X)$, $d(X)$, $ic(X)$. Вопросы подобного характера рассматривались и ранее, назовем для примера следующие факты:

1) Если γ — точечно счетное открытое покрытие пространства X, то $|\gamma| \leqslant s(X)$.

2) Если γ — точечно счетная база счетно компактного хаусдорфова пространства, то $|\gamma| \leqslant \aleph_0$.

3) Если γ — σ-дизъюнктное открытое покрытие X, то $|\gamma| \leqslant c(X)$.

Здесь и далее:

$|\gamma|$ — мощность семейства γ,

$s(X) = \min|S|$: S плотно в X,

$c(X) = \sup|\gamma|$: γ — дизъюнктное семейство открытых в X множеств.

Все кардинальные числа предполагаются бесконечными, все семейства и покрытия открытыми, если не оговаривается противное. На пространства налагается аксиома отделимости T_1.

Типичные обозначения:

$[A]$ — замыкание множества A,