is separated by no subcontinuum, what is a necessary and sufficient condition (or conditions) in order that M have a monotone, upper semi-continuous decomposition, each element of which has void interior and such that the quotient space is a simple closed curve?

References

[1] R. H. Bing, *Some characterizations of arcs and simple closed curves*, Amer. J. Math. 70 (1948), pp. 497–506.

[2] H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Properties of the set function T^n*, Portugal. Math. 21 (1962), pp. 113–133.

[3] R. W. FitzGerald and P. M. Swingle, *Core decompositions of continua*, Fund. Math. 61 (1967), pp. 33–50.

[4] G. R. Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua*, Pacific J. Math. 36 (1971), pp. 647–658.

[5] F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. J. Math. 67 (1941), pp. 545–553.

[6] H. H. Stratton, *On continua which resemble simple closed curves*, Fund. Math. 68 (1970), pp. 121–128.

[7] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Matematyczne 50 (1966), pp. 1–74.

[8] G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Public. 28 (1942).

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The Whitehead Theorem in the theory of shapes

by

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Abstract. The purpose of this paper is to establish in the theory of shapes a theorem, which is an analogue of the Whitehead Theorem. We start with proving some statements concerning category theory (Section 1); they could not be found by the author in the literature. These statements enable us to prove the exactness property for homotopy systems (§ 1 of Section 2). Next, we establish some propositions on inverse systems of polyhedra (§ 2 of Section 2); they are needed in a proof of Theorem 3.5, which is referred to as the Whitehead Theorem for inverse sequences of polyhedra (§ 3 of Section 2). At last, applying the Freudental Theorem, the Holsztyński theorem on the fundamental dimension and the results of [6] and [10]-[13], we obtain the main theorem (Th.4.3).

Contents

Introduction. As proved by J. H. C. Whitehead in [16] (see also [15]), if two spaces X and Y are homotopically dominated by some connected CW-complexes of dimension $\leqslant n_0$ (of infinite dimension), then for a map $f\colon X \to Y$ to be a homotopy equivalence it is sufficient that f induces isomorphisms of homotopy groups, $f_n\colon \pi_n(X) \to \pi_n(Y)$, for $n = 1, \ldots, n_0$ (for $n = 1, 2, \ldots$).

Thus, for spaces with nice local properties (e.g. ANR's) the homotopy groups are the most important homotopy invariants. However, for arbitrary compact metric spaces, the homotopy groups lose their validity. For this reason K. Borsuk introduced the notion of fundamental groups. As proved in [1], p. 253, the fundamental groups are shape invariants; for ANR's they are isomorphic to the homotopy groups.

The main result of this paper (Th. 4.3 of Section 2) shows that, as regards movable compacta of a finite fundamental dimension, the role of fundamental groups in the shape theory is quite the same as the role of homotopy groups in the homotopy theory.

For the notion of movability and uniform movability, see [2], [7] and [12].

The author would like to express her gratitude to Mr. S. Nowak for his valuable remarks.

Section 1

1. Exact diagrams in categories with zero-objects. Let us consider an arbitrary category $\mathcal{K}$. We use the symbols $X, Y, \ldots, A, B, \ldots$ to denote objects of $\mathcal{K}$ and $f, g, \ldots, \varphi, \psi, \ldots$ to denote morphisms. If f belongs to $\mathrm{Mor}(X, Y)$ we write $f\colon X \to Y$; the symbol $1_X\colon X \to X$ denotes the identity.

Let us recall some definitions (see [9]).

/1/ $f\colon X\to Y$ is a *monomorphism* $\underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\varphi,\varphi'\colon Z\to X} f\varphi = f\varphi' \Rightarrow \varphi = \varphi'$.

/2/ $f\colon X\to Y$ is an *epimorphism* $\underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{\psi,\psi'\colon Y\to Z} \psi f = \psi' f \Rightarrow \psi = \psi'$.

/3/ $f\colon X\to Y$ is a *bimorphism* $\underset{\mathrm{Df}}{\Leftrightarrow}$ f is a monomorphism $\wedge$ f is an epimorphism.

We have

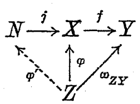
1.1. *Every isomorphism is a bimorphism.*

Obviously the converse is false.

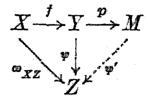
/4/ A family of morphisms, $\{\omega_{XY} \in \mathrm{Mor}(XY)\}_{X,Y}$, is a *class of zero-morphisms* $\underset{\mathrm{Df}}{\Leftrightarrow} \bigwedge_{f\colon A\to X} \bigwedge_{g\colon Y\to B} [\omega_{XY} f = \omega_{AY}] \wedge [g\omega_{XY} = \omega_{XB}]$.

If the category $\mathcal{K}$ has a class of zero-morphisms, then the following three notions can be defined in $\mathcal{K}$:

/5/ Take $N \xrightarrow{j} X \xrightarrow{f} Y$. The pair (N, j) is a *kernel* of f (in symbols $\mathrm{Ker} f$) $\underset{\mathrm{Df}}{\Leftrightarrow} [j$ is a monomorphism$] \wedge [fj = \omega_{NY}] \wedge \bigwedge_{\varphi\colon Z\to X} [f\varphi = \omega_{ZY} \Rightarrow \bigvee_{\varphi'\colon Z\to N} \varphi = j\varphi']$



/6/ Take $X \xrightarrow{f} Y \xrightarrow{p} M$. The pair (M, p) is a *cokernel* of f (in symbols $\mathrm{Coker} f$) $\underset{\mathrm{Df}}{\Leftrightarrow} [p$ is an epimorphism$] \wedge [pf = \omega_{XM}] \wedge \bigwedge_{\psi\colon Y\to Z} [\psi f = \omega_{XZ} \Rightarrow \bigvee_{\psi'\colon M\to Z} \psi = \psi' p]$



/7/ Let $\mathrm{Coker} f = (M, p)$. Then

$$\mathrm{Im} f \underset{\mathrm{Df}}{=} \mathrm{Ker} p\,.$$

Now one can define the exactness of a diagram as follows:

/8/ The diagram $\ldots \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$ is *exact* $\underset{\mathrm{Df}}{\Leftrightarrow}$ all f_i have kernels and images and $\mathrm{Im} f_i = \mathrm{Ker} f_{i+1}$ for every i.

Thus, the notion of exact diagram can be introduced for an arbitrary category $\mathcal{K}$ with zero-morphisms. Now, let us recall the definition of zero-object:

/9/ X is a *zero-object* $\underset{\mathrm{Df}}{\Leftrightarrow} \mathrm{Mor}(X, X) = \{1_X\}$.

It is known that (see [9]),

1.2. *All the zero-objects in $\mathcal{K}$ are isomorphic.*

1.3. *If $\mathcal{K}$ has zero-objects, then it has a class of zero-morphisms as well.*

1.4. *X is a zero-object* $\Leftrightarrow \mathrm{Mor}(X, X) = \{\omega_{XX}\} \Leftrightarrow \bigwedge_{Y} [\mathrm{Mor}(X, Y) = \{\omega_{XY}\}] \wedge [\mathrm{Mor}(Y, X) = \{\omega_{YX}\}]$.

By 1.3, the exactness of a diagram can be defined for any category $\mathcal{K}$ with zero-objects. The statement 1.2 enables us to denote by 0 an arbitrary zero-object.

Assuming $\mathcal{K}$ to have zero-objects, let us establish the following statements 1.5–1.8 [1].

1.5. *$f\colon X\to Y$ is a monomorphism* $\Rightarrow \mathrm{Ker} f = (0, \omega_{0X})$.

Proof. Let $\mathrm{Ker} f = (N, j)$. For $j, \omega_{NX}\colon N \to X$, we have $fj = \omega_{NY} = f\omega_{NX}$; thus, by /1/, $j = \omega_{NX}$.

Take an arbitrary $\varphi\colon N\to N$. We have $j\varphi = \omega_{NX}\varphi = \omega_{NX} = j\omega_{NN}$, where j is a monomorphism; so $\varphi = \omega_{NN}$. Hence $\mathrm{Mor}(N, N) = \{\omega_{NN}\}$ and then, by 1.2, N is a zero-object. ∎

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[1] Some of these statements can be found in the literature, however only under additional assumptions on $\mathcal{K}$.

1.6. *$f\colon X \to Y$ is an epimorphism* $\Rightarrow \operatorname{Coker} f = (0, \omega_{Y0})$.

Proof. This statement is dual to 1.5. ■

1.7. $\operatorname{Coker} f = (0, \omega_{Y0}) \Leftrightarrow \operatorname{Im} f = (N_p, j_p)$, *$j_p$ being an isomorphism.*

Proof. $\Rightarrow$. Let $\operatorname{Coker} f = (0, p)$, where $p = \omega_{Y0}$. By /7/, $\operatorname{Im} f = \operatorname{Ker} p = (N_p, j_p)$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{p} & M = 0 \\ & & \uparrow {\scriptstyle j_p} & & \\ & & N_p & & \end{array}$$

By /5/, since $p = \omega_{YM}$, we have

$$\bigwedge_{\varphi\colon Z\to Y} \bigvee_{\varphi'\colon Z \to N_p} \varphi = j_p\varphi' .$$

Take $\varphi = 1_Y\colon Y \to Y$; there is $\varphi'\colon Y \to N_p$ such that $j_p\varphi' = 1_Y$. Hence, $j_p(\varphi' j_p) = 1_Y j_p = j_p = j_p 1_{N_p}$, where j_p is a monomorphism; thus, by /1/, $\varphi' j_p = 1_{N_p}$. So φ' is an inverse of j_p, therefore j_p is an isomorphism.

$\Leftarrow$. Let $\operatorname{Coker} f = (M, p)$ and $\operatorname{Im} f = (N_p, j_p)$, j_p being an isomorphism. By /5/, $pj_p = \omega_{N_pM} = \omega_{YM} j_p$, where j_p is an epimorphism; thus, by /2/, $p = \omega_{YM}$. For any $\varphi\colon M \to M$, we have $\varphi p = \varphi\omega_{YM} = \omega_{YM} = \omega_{MM} p$, where p is an epimorphism; thus $\varphi = \omega_{MM}$. Hence, by 1.2, M is a zero-object. ■

1.8. $\operatorname{Ker} f = (N, j)$, *$j$ being an isomorphism* $\Rightarrow \operatorname{Im} f = (0, \omega_{0Y})$.

Proof. We have

$$\begin{array}{ccccccc} N & \xrightarrow{j} & X & \xrightarrow{f} & Y & \xrightarrow{p} & M , \\ & & & & \uparrow {\scriptstyle j_p} & & \\ & & & & N_p & & \end{array}$$

where j is an isomorphism, $\operatorname{Im} f = \operatorname{Ker} p = (N_p, j_p)$ and $\operatorname{Ker} f = (N, j)$. By /5/, $fj = \omega_{NY} = \omega_{XY} j$; so, by 1.1 and /2/, $f = \omega_{XY}$. Then, by /6/,

$$\bigwedge_{\psi\colon Y\to Z} \bigvee_{\psi'\colon M \to Z} \psi = \psi' p .$$

Take $\psi = 1_Y\colon Y \to Y$; there is $\psi'\colon M \to Y$ such that $\psi' p = 1_Y$. By /5/, $pj_p = \omega_{N_pM}$; so $j_p = \psi' p j_p = \omega_{N_pY}$. For any $\varphi\colon N_p \to N_p$, we have $j_p = \omega_{N_pY} = j_p\omega_{N_pN_p}$, where j_p is a monomorphism; thus $\varphi = \omega_{N_pN_p}$. Hence, N_p is a zero-object and $j_p = \omega_{N_pY} = \omega_{0Y}$. ■

Let us prove now

1.9. Proposition. *Given the exact diagram $X \xrightarrow{\tau} Y \xrightarrow{\xi} Z \xrightarrow{\partial} 0$ in a category with zero-objects, let τ be an epimorphism. Then $Z = 0$.*

Proof. By 1.6 and 1.7, since τ is an epimorphism, we get $\operatorname{Im}\tau = (N_p, j_p)$, j_p being an isomorphism. By the exactness of our diagram

and by 1.8, we get $\operatorname{Im}\xi = (N, \omega_{NZ}) = \operatorname{Ker}\partial$, where N is a zero-object. Take an arbitrary $\varphi\colon Z \to Z$. Since $\partial\varphi \in \operatorname{Mor}(Z, 0)$, it follows by 1.2 that $\partial\varphi = \omega_{Z0}$. So, by /5/, there is a $\varphi'\colon Z \to N$ such that $\varphi = j\varphi'$. But N is a zero-object, thus $\varphi' = \omega_{ZN}$ and therefore $\varphi = \omega_{ZZ}$. Hence Z is a zero-object. ■

2. Exact diagrams in weak additive categories. Let $\mathcal{K}$ be a category with a class of zero-morphisms $\{\omega_{XY}\}_{X,Y}$. To every two objects X, Y let us assign an operation

$$+_{XY}\colon \operatorname{Mor}(X, Y) \times \operatorname{Mor}(X, Y) \to \operatorname{Mor}(X, Y) .$$

The system $\langle \mathcal{K}, \{+_{XY}\}, \{\omega_{XY}\}\rangle$ is said to be a *weak additive category* whenever the following two conditions are satisfied:

(i) $\langle \operatorname{Mor}(X, Y), +_{XY}, \omega_{XY}\rangle$ is a group (with the group operation $+_{XY}$ and the neutral element ω_{XY}) [2].

(ii) $g(f_1 +_{XY} f_2) = gf_1 +_{XZ} gf_2$ and $(g_1 +_{YZ} g_2)f = g_1 f +_{XZ} g_2 f$ for any $f, f_i \in \operatorname{Mor}(X, Y)$, $g, g_i \in \operatorname{Mor}(Y, Z)$.

Obviously $-_{XY} f$ denotes the unique morphism f' satisfying the condition $f +_{XY} f' = \omega_{XY}$; we define as usually $f -_{XY} g \underset{\mathrm{Df}}{=} f +_{XY} (-_{XY} g)$. In the sequel we shall write often $+$, $-$ instead of $+_{XY}$, $-_{XY}$. Assuming $\mathcal{K}$ to be a weak additive category with zero-objects, let us establish the following statements 2.1-2.3.

2.1. *For every $f\colon X \to Y$,*

$$\operatorname{Ker} f = (0, \omega_{0X}) \Rightarrow f \textit{ is a monomorphism} .$$

Proof. In order to prove f to be a monomorphism, let us take $\varphi_1, \varphi_2\colon Z \to X$ and assume $f\varphi_1 = f\varphi_2$. Since $\operatorname{Ker} f = (0, \omega_{0X})$, by 1.2 and by definition /5/

$$\bigwedge_{\varphi\colon Z\to X} [f\varphi = \omega_{ZY} \Rightarrow \varphi = \omega_{ZX}] .$$

Setting $\varphi = \varphi_1 - \varphi_2$, by (i), (ii) we have

$$f\varphi = f(\varphi_1 - \varphi_2) = f\varphi_1 - f\varphi_2 = \omega_{ZY} ;$$

so $\varphi_1 - \varphi_2 = \omega_{ZX}$, and then $\varphi_1 = \varphi_2$. Hence f is a monomorphism. ■

2.2. *For every $f\colon X \to Y$,*

$$\operatorname{Ker} f = (0, \omega_{0X}) \wedge \operatorname{Im} f = (0, \omega_{0Y}) \Rightarrow X = 0 .$$

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[2] If, moreover, $\langle \operatorname{Mor}(X, Y), +_{X,Y}, \omega_{X,Y}\rangle$ is an abelian group, then we get an additive category.

Proof. Take an arbitrary $\varphi\colon X\to X$. We have

$$\begin{array}{c} X\xrightarrow{f} Y\xrightarrow{p} M\,, \\ \quad\uparrow j_p \\ \quad 0 \end{array}\qquad \operatorname{Im} f=\operatorname{Ker} p=(0,j_p),\ \ j_p=\omega_{0Y}\,,$$

thus it follows by 2.1 that p is a monomorphism. Besides $pf=\omega_{XM}$. Hence, $p(f\varphi)=\omega_{XM}\varphi=\omega_{XM}=p\omega_{XY}$ implies $f\varphi=\omega_{XY}$. Furthermore, since $\operatorname{Ker} f=(0,\omega_{0X})$, there is $\varphi'\colon X\to 0$ such that $\varphi=\omega_{0X}\varphi'$. Thus $\varphi=\omega_{XX}$ and therefore $X=0$. ■

Now, let us prove

2.3. PROPOSITION. *Given the exact diagram*

$$X\overset{\tau}{\to} Y\overset{\xi}{\to} Z\overset{\partial}{\to} X'\overset{\tau'}{\to} Y'$$

in a weak additive category with zero-objects, let τ be an epimorphism and τ' be a monomorphism. Then $Z=0$.

Proof. Since τ' is a monomorphism, it follows by 1.5 that $\operatorname{Ker}\tau'=(0,\omega_{0X'})$. Since τ is an epimorphism, it follows by 1.6 that $\operatorname{Coker}\tau=(0,\omega_{Y0})$; and then, by 1.7, $\operatorname{Im}\tau=(N_p,j_p)$, j_p being an isomorphism.

By the exactness of the diagram we obtain

$$\operatorname{Ker}\xi=\operatorname{Im}\tau=(N_p,j_p),\ j_p \text{ being an isomorphism;}$$

so, by 1.8, $\operatorname{Im}\xi=(0,\omega_{0Z})$.

By the exactness, we have

$$\operatorname{Ker}\partial=\operatorname{Im}\xi=(0,\omega_{0Z})\quad\text{and}\quad \operatorname{Im}\partial=\operatorname{Ker}\tau'=(0,\omega_{0X'})\,.$$

Hence, applying 2.2, we obtain $Z=0$. ■

3. Categories of inverse systems. We are concerned with the inverse systems over $(A,\leqslant)$ in a category $\mathcal{K}$. Let us recall the definition.

Take a *closure finite directed set*, i.e. a pair $(A,\leqslant)$, where the relation $\leqslant$ is reflexive, transitive, for every $a,a'\in A$ there exists an $a''\in A$ such that $a,a'\leqslant a''$, and for every $a\in A$ there is at most finite number of predecessors (see [5]). The system $X=(X_a,p_a^{a'},A)$ is said to be an *inverse system over* $(A,\leqslant)$ in the category $\mathcal{K}$, whenever

$$X_a\in \operatorname{Ob}_{\mathcal{K}}\quad\text{for every } a\in A\,,$$
$$p_a^{a'}\in \operatorname{Mor}_{\mathcal{K}}(X_{a'},X_a)\quad\text{for}\quad a'\geqslant a\,,$$
$$a\leqslant a'\leqslant a''\Rightarrow p_a^{a'}p_{a'}^{a''}=p_a^{a''}\,,$$

and

$$p_a^a=1_{X_a}\quad\text{for every } a\in A\,.$$

(We write also $a'\geqslant a$ instead of $a\leqslant a'$).

Given two inverse systems in $\mathcal{K}$,

$$X=(X_a,p_a^{a'},A)\quad\text{and}\quad Y=(Y_a,q_a^{a'},A)\,,$$

the system $f=(\varphi,f_a)$ is said to be a *cofinal map* of systems whenever $\varphi\colon A\to A$ is an increasing function, $\varphi(A)$ is cofinal with A, $f_a\in \operatorname{Mor}_{\mathcal{K}}(X_{\varphi(a)},Y_a)$ and the diagram

$$\begin{array}{ccc} X_{\varphi(a)} & \xleftarrow{p_{\varphi(a)}^{\varphi(a')}} & X_{\varphi(a')} \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_{a'}} \\ Y_a & \xleftarrow[q_a^{a'}]{} & Y_{a'} \end{array}\qquad\text{is commutative for every } a'\geqslant a\,.$$

If, in particular, $\varphi=1_A$, then f is referred to as an *ordinary map*; in symbols $f=(f_a)$.

Let $\mathcal{K}^*$ be the category with inverse systems in $\mathcal{K}$ over $(A,\leqslant)$ as objects and cofinal maps of systems as morphisms (see [12]) (³). Define the equivalence relation $\simeq$ in $\operatorname{Mor}_{\mathcal{K}^*}(X,Y)$ as follows: let $f=(\varphi,f_a)$, $f'=(\varphi',f'_a)$, then

$$f\simeq f'\underset{\mathrm{Df}}{\Leftrightarrow}\bigvee_{\delta\colon(A,\leqslant)\to(A,\leqslant)}\ \bigwedge_a\big(\delta(a)\geqslant\varphi(a),\varphi'(a)\big)\wedge\big(f_a p_{\varphi(a)}^{\delta(a)}=f'_a p_{\varphi'(a)}^{\delta(a)}\big)\,.$$

Since A is assumed closure-finite, it follows by Lemma 5 of [5] that the above definition of $\simeq$ is equivalent to that used in [12] and [13].

The quotient category $\mathcal{K}^*|_{\simeq}$ will be denoted by $\hat{\mathcal{K}}^*$ (⁴).

Let us prove that

3.1. *For any morphism $f=(\varphi,f_a)$ in $\mathcal{K}^*$ the following two implications hold*:

f_a is a monomorphism (epimorphism) in $\mathcal{K}$ for every $a\Rightarrow$ $[f]$ is a monomorphism (epimorphism) in $\hat{\mathcal{K}}^*$.

Proof. Take $X=(X_a,p_a^{a'},A)$, $Y=(Y_a,q_a^{a'},A)$ and $f=(\varphi,f_a)\colon X\to Y$. Let f_a be a monomorphism in $\mathcal{K}$ for every $a\in A$. Take $Z=(Z_a,r_a^{a'},A)$ and $g,g'\colon Z\to X$, $g=(\psi,g_a)$, $g'=(\psi',g'_a)$. If $fg\simeq fg'$, then there exists $\delta\colon(A,\leqslant)\to(A,\leqslant)$ such that $\delta(a)\geqslant\psi\varphi(a)$, $\delta(a)\geqslant\psi'\varphi(a)$ and

(1) $$f_a g_{\varphi(a)} r_{\psi\varphi(a)}^{\delta(a)}=f_a g'_{\varphi(a)} r_{\psi'\varphi(a)}^{\delta(a)}\,;$$

thus we get

(2) $$g_{\varphi(a)} r_{\psi\varphi(a)}^{\delta(a)}=g'_{\varphi(a)} r_{\psi'\varphi(a)}^{\delta(a)}\quad\text{for every } a\in A\,.$$

(³) In [12] we did not restrict $\operatorname{Mor}_{\mathcal{K}^*}$ to the cofinal maps.

(⁴) In [12] and [13] the more general case was considered. Here $\sim$ is an identity relation.

Take any $\beta \in A$ and let $\varphi(\alpha) \geqslant \beta$. By (2) we have

$$p_\beta^{\varphi(\alpha)} g_{\varphi(\alpha)} r^{\delta(\alpha)}_{\psi\varphi(\alpha)} = p_\beta^{\varphi(\alpha)} g'_{\varphi(\alpha)} r^{\delta(\alpha)}_{\psi'\varphi(\alpha)},$$

so

$$g_\beta r^{\psi\varphi(\alpha)}_{\psi(\beta)} r^{\delta(\alpha)}_{\psi\varphi(\alpha)} = g'_\beta r^{\psi'\varphi(\alpha)}_{\psi'(\beta)} r^{\delta(\alpha)}_{\psi'\varphi(\alpha)},$$

whence

(3) $$g_\beta r^{\delta(\alpha)}_{\psi(\beta)} = g'_\beta r^{\delta(\alpha)}_{\psi'(\beta)}.$$

By (3) it follows immediately that $g \cong g'$; hence $[f]$ is a monomorphism in $\hat{\mathcal{K}}^*$.

Now, let f_α be epimorphisms, $\alpha \in A$. Take $Z = (Z_\alpha, r_\alpha^{\alpha'}, A)$ and $g, g'\colon Y \to Z$; let $g = (\psi, g_\alpha)$ and $g' = (\psi', g'_\alpha)$. If $gf \cong g'f$, then there exists $\delta\colon (A, \leqslant) \to (A, \leqslant)$ such that $\delta(\alpha) \geqslant \varphi\psi(\alpha)$, $\delta(\alpha) \geqslant \varphi\psi'(\alpha)$ and

(4) $$g_\alpha f_{\psi(\alpha)} p^{\delta(\alpha)}_{\varphi\psi(\alpha)} = g'_\alpha f_{\psi'(\alpha)} p^{\delta(\alpha)}_{\varphi\psi'(\alpha)} \quad \text{for every } \alpha \in A.$$

It is easy to see that

(5) $$\bigwedge_\alpha \bigvee_\beta [\beta \geqslant \psi(\alpha), \psi'(\alpha)] \wedge [\varphi(\beta) \geqslant \delta(\alpha)].$$

By (4) and (5) we get

(6) $$\bigwedge_\alpha \bigvee_{\beta \geqslant \psi(\alpha), \psi'(\alpha)} g_\alpha f_{\psi(\alpha)} p^{\varphi(\beta)}_{\varphi\psi(\alpha)} = g'_\alpha f_{\psi'(\alpha)} p^{\varphi(\beta)}_{\varphi\psi'(\alpha)}.$$

By the commutativity of the diagrams for the map f, the condition (6) implies

(7) $$\bigwedge_\alpha \bigvee_\beta g_\alpha q^\beta_{\psi(\alpha)} f_\beta = g'_\alpha q^\beta_{\psi'(\alpha)} f_\beta.$$

Since f_β is an epimorphism, we get $g \cong g'$, whence $[f]$ is an epimorphism in $\hat{\mathcal{K}}^*$. ■

Let us notice that

3.2. *If the collection $\{\omega_{X_\alpha Y_\alpha}\}$ is a class of zero-morphisms in $\mathcal{K}$, then the collection $\{\omega_{XY} = (\omega_{X_\alpha Y_\alpha})\}$ is a class of zero-morphisms in $\mathcal{K}^*$ and the collection $\{[\omega_{XY}]\}$ of equivalence classes with respect to $\cong$ is a class of zero-morphisms in $\hat{\mathcal{K}}^*$.* ■

In turn, we are interested in kernels and cokernels of ordinary morphisms in the category $\hat{\mathcal{K}}^*$. Let us prove

3.3. PROPOSITION. *Given an ordinary map $f = (f_\alpha)\colon X \to Y$ in $\mathcal{K}^*$, let $\operatorname{Ker} f_\alpha = (N_\alpha, j_\alpha)$ for any $\alpha \in A$. Then there exists a collection of morphisms in $\mathcal{K}$, $(n_\alpha^{\alpha'})_{\alpha' \geqslant \alpha}$, such that*

(i) $$(N_\alpha, n_\alpha^{\alpha'}, A) \in \operatorname{Ob}_{\mathcal{K}^*} \quad \text{and} \quad (1_A, j_\alpha) \in \operatorname{Mor}_{\mathcal{K}^*},$$

(ii) $$\operatorname{Ker}[f] = (N, [j]), \quad \text{where} \quad N = (N_\alpha, n_\alpha^{\alpha'}, A) \text{ and } j = (j_\alpha).$$

Proof. Let $X = (X_\alpha, p_\alpha^{\alpha'}, A)$ and $Y = (Y_\alpha, q_\alpha^{\alpha'}, A)$. By the assumption, $\operatorname{Ker} f_\alpha = (N_\alpha, j_\alpha)$, i.e.

(1) $$j_\alpha \text{ is a monomorphism},$$

(2) $$f_\alpha j_\alpha = \omega_{N_\alpha Y_\alpha}$$

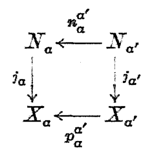
and

(3) $$\bigwedge_{\zeta\colon Z \to X_\alpha} [f_\alpha \zeta = \omega_{Z Y_\alpha} \Rightarrow \bigvee_{\zeta'\colon Z \to N_\alpha} \zeta = j_\alpha \zeta'].$$

Let $\alpha' \geqslant \alpha$. Take $Z = N_{\alpha'}$ and $\zeta = p_\alpha^{\alpha'} j_{\alpha'}\colon N_{\alpha'} \to X_\alpha$; by (3) there exists $n_\alpha^{\alpha'}\colon N_{\alpha'} \to N_\alpha$, such that

(4) $$j_\alpha n_\alpha^{\alpha'} = p_\alpha^{\alpha'} j_{\alpha'},$$

i.e. the diagram



is commutative for every $\alpha' \geqslant \alpha$.

Thus, $\alpha \leqslant \alpha' \leqslant \alpha''$ implies

$$j_\alpha n_\alpha^{\alpha''} = p_\alpha^{\alpha''} j_{\alpha''} = p_\alpha^{\alpha'}(p_{\alpha'}^{\alpha''} j_{\alpha''}) = p_\alpha^{\alpha'} j_{\alpha'} n_{\alpha'}^{\alpha''} = j_\alpha(n_\alpha^{\alpha'} n_{\alpha'}^{\alpha''});$$

since j_α is a monomorphism, we get

(5) $$\alpha \leqslant \alpha' \leqslant \alpha'' \Rightarrow n_\alpha^{\alpha''} = n_\alpha^{\alpha'} n_{\alpha'}^{\alpha''}.$$

In turn, we have

$$j_\alpha n_\alpha^\alpha = p_\alpha^\alpha j_\alpha = j_\alpha,$$

which implies the condition

(6) $$n_\alpha^\alpha = 1_{N_\alpha} \quad \text{for any } \alpha.$$

By (4), (5) and (6) we obtain (i).

Let $N = (N_\alpha, n_\alpha^{\alpha'}, A)$ and $j = (j_\alpha)\colon N \to X$. By 3.1, the condition (1) implies

(7) $$[j] \text{ is a monomorphism in } \hat{\mathcal{K}}^*.$$

By 3.2, the condition (2) implies

(8) $$[f][j] = [\omega_{NY}] \quad (\text{moreover } fj = \omega_{NY}).$$

Take any $Z = (Z_a, r_a^{a'}, A)$ and a morphism $g = (\psi, g_a)\colon Z \to X$ in $\mathcal{K}^*$. Let $fg \simeq \omega_{Z,Y}$, i.e. there is a $\delta\colon (A, \leqslant) \to (A, \leqslant)$ such that

(9) $$f_a g_a r^{\delta(a)}_{\psi(a)} = \omega_{Z_{\delta(a)} Y_a} \quad \text{for} \quad a \in A .$$

By (3), for any $a \in A$ there exists $g'_a\colon Z_{\delta(a)} \to N_a$ such that

(10) $$g_a r^{\delta(a)}_{\psi(a)} = j_a g'_a .$$

In order to show that (δ, g'_a) is a map of systems, consider the diagram

$$\begin{array}{ccc} Z_{\delta(a)} & \xleftarrow{\;r^{\delta(a')}_{\delta(a)}\;} & Z_{\delta(a')} \\ {\scriptstyle g'_a}\downarrow & & \downarrow{\scriptstyle g'_{a'}} \\ N_a & \xleftarrow[\;n_a^{a'}\;]{} & N_{a'} \end{array}$$

By (10), we have

$$j_a g'_a r^{\delta(a')}_{\delta(a)} = g_a r^{\delta(a')}_{\psi(a)} ;$$

on the other hand, by (10), (4) and by the commutativity of the diagrams for g,

$$j_a n_a^{a'} g'_{a'} = p_a^{a'} j_{a'} g'_{a'} = p_a^{a'} g_{a'} r^{\delta(a')}_{\psi(a')} = g_a r^{\delta(a')}_{\psi(a)} ;$$

thus

$$j_a (g'_a r^{\delta(a')}_{\delta(a)}) = j_a (n_a^{a'} g'_{a'}) .$$

Hence, by (1),

$$g'_a r^{\delta(a')}_{\delta(a)} = n_a^{a'} g'_{a'} ,$$

i.e. the diagram commutes. So, we get

$$g' = (\delta, g'_a)\colon Z \to N .$$

The condition (10) implies

(11) $$[g] = [j][g'] .$$

By (7), (8) and (11) we obtain (ii). ■

3.4. Proposition. *Given an ordinary map $f = (f_a)\colon X \to Y$ in $\mathcal{K}^*$, let $\mathrm{Coker} f_a = (M_a, p_a)$ for any $a \in A$. Then there exists a collection of morphism in $\mathcal{K}$, $(m_a^{a'})_{a' \geqslant a}$, such that*

(i) $$(M_a, m_a^{a'}, A) \in \mathrm{Ob}_{\mathcal{K}^*} \quad \textit{and} \quad (1_A, p_a) \in \mathrm{Mor}_{\mathcal{K}^*} ,$$

(ii) $\mathrm{Coker}[f] = (M, [p])$, *where* $M = (M_a, m_a^{a'}, A)$ *and* $p = (p_a)$.

Proof. Let $X = (X_a, p_a^{a'}, A)$ and $Y = (Y_a, q_a^{a'}, A)$. By the assumption, $\mathrm{Coker} f_a = (M_a, p_a)$, i.e.

(1) $$p_a \text{ is an epimorphism} ,$$

(2) $$p_a f_a = \omega_{X_a M_a}$$

and

(3) $$\bigwedge_{\xi\colon Y_{a'} \to Z} [\xi f_{a'} = \omega_{X_{a'} Z} \Rightarrow \bigvee_{\xi'\colon M_{a'} \to Z} \xi = \xi' p_{a'}] .$$

Let $a \leqslant a'$. Take $Z = M_a$ and $\xi = p_a q_a^{a'}\colon Y_{a'} \to M_a$; by (3) there exists $m_a^{a'}\colon M_{a'} \to M_a$ such that

(4) $$p_a q_a^{a'} = m_a^{a'} p_{a'} ,$$

i.e. the diagram

$$\begin{array}{ccc} Y_a & \xleftarrow{\;q_a^{a'}\;} & Y_{a'} \\ {\scriptstyle p_a}\downarrow & & \downarrow{\scriptstyle p_{a'}} \\ M_a & \xleftarrow[\;m_a^{a'}\;]{} & M_{a'} \end{array}$$

is commutative for every $a \leqslant a'$.

Thus, $a \leqslant a' \leqslant a''$ implies

$$m_a^{a''} p_{a''} = p_a q_a^{a''} = (p_a q_a^{a'}) q_{a'}^{a''} = m_a^{a'} p_{a'} q_{a'}^{a''} = (m_a^{a'} m_{a'}^{a''}) p_{a''} ;$$

since $p_{a''}$ is an epimorphism, we get

(5) $$a \leqslant a' \leqslant a'' \Rightarrow m_a^{a''} = m_a^{a'} m_{a'}^{a''} .$$

In turn, we have

$$m_a^a p_a = q_a^a p_a = p_a ,$$

which implies the condition

(6) $$m_a^a = 1_{M_a} \quad \text{for any} \quad a .$$

By (4), (5) and (6) we get (i). Let $M = (M_a, m_a^{a'}, A)$ and $p = (p_a)\colon Y \to M$. By 3.1, the condition (1) implies

(7) $$[p] \quad \text{is an epimorphism in } \hat{\mathcal{K}}^*.$$

By 3.2, the condition (2) implies

(8) $$[p][f] = [\omega_{XM}] \quad (\text{moreover } pf = \omega_{XM}) .$$

Take any $Z = (Z_a, r_a^{a'}, A)$ and a morphism $g = (\psi, g_a)\colon Y \to Z$ in $\mathcal{K}^*$. Let $gf \simeq \omega_{XZ}$, i.e. there is a $\delta\colon (A, \leqslant) \to (A, \leqslant)$ such that

(9) $$g_a f_{\psi(a)} p^{\delta(a)}_{\psi(a)} = \omega_{X_{\delta(a)} Z_a} \quad \text{for} \quad a \in A .$$

Then, by the commutativity of the diagrams for $\boldsymbol{f}$, we have

(9′) $$(g_a q_{\varphi(a)}^{\delta(a)}) f_{\delta(a)} = \omega_{X_{\delta(a)} Z_a} \quad \text{for} \quad a \in A .$$

By (3), for any $a \in A$, there exists $g'_a\colon M_{\delta(a)} \to Z_a$ such that

(10) $$g_a q_{\varphi(a)}^{\delta(a)} = g'_a p_{\delta(a)} .$$

In order to show that (δ, g'_a) is a map of systems, consider the diagram

$$\begin{array}{ccc} M_{\delta(a)} & \xleftarrow{m_{\delta(a)}^{\delta(a')}} & M_{\delta(a')} \\ {\scriptstyle g'_a}\downarrow & & \downarrow{\scriptstyle g'_{a'}} \\ Z_a & \xleftarrow[r_a^{a'}]{} & Z_{a'} \end{array}$$

By (4) and (10), we have

$$g'_a m_{\delta(a)}^{\delta(a')} p_{\delta(a')} = g'_a p_{\delta(a)} q_{\delta(a)}^{\delta(a')} = g_a q_{\varphi(a)}^{\delta(a)} q_{\delta(a)}^{\delta(a')} = g_a q_{\varphi(a)}^{\delta(a')} ;$$

on the other hand, by (10) and by the commutativity of the diagrams for $\boldsymbol{g}$,

$$r_a^{a'} g'_{a'} p_{\delta(a')} = (r_a^{a'} g_{a'}) q_{\varphi(a')}^{\delta(a')} = g_a q_{\varphi(a)}^{\delta(a')} ;$$

thus

$$(g'_a m_{\delta(a)}^{\delta(a')}) p_{\delta(a')} = (r_a^{a'} g'_{a'}) p_{\delta(a')} .$$

Hence, by (1),

$$g'_a m_{\delta(a)}^{\delta(a')} = r_a^{a'} g'_{a'} ,$$

i.e. the diagram commutes. So, we get

$$\boldsymbol{g}' = (\delta, g'_a)\colon \boldsymbol{M} \to \boldsymbol{Z} .$$

The condition (10) implies

(11) $$[\boldsymbol{g}] = [\boldsymbol{g}'][\boldsymbol{p}] .$$

By (7), (8) and (11) we obtain (ii). ■

As a consequence of Propositions 3.3 and 3.4, we obtain

3.5. Proposition. *Given an ordinary map* $\boldsymbol{f} = (f_a)\colon \underline{X} \to \underline{Y}$ *in* $\mathcal{K}^*$, *let* $\mathrm{Im} f_a = (V_a, s_a)$ *for any* $a \in A$. *Then there exists a collection of morphism in* $\mathcal{K}$, $(v_a^{a'})_{a' \geqslant a}$, *such that*

(i) $(V_a, v_a^{a'}, A) \in \mathrm{Ob}_{\mathcal{K}^*}$ *and* $(1_A, s_a) \in \mathrm{Mor}_{\mathcal{K}^*}$,

(ii) $\mathrm{Im}[\boldsymbol{f}] = (\boldsymbol{V}, [\boldsymbol{s}])$, *where* $\boldsymbol{V} = (V_a, v_a^{a'}, A)$ *and* $\boldsymbol{s} = (s_a)$.

Proof. Let $\mathrm{Coker} f_a = (M_a, p_a)$, then

$$\mathrm{Ker} p_a = \mathrm{Im} f_a = (V_a, s_a) .$$

By 3.3, there exists a collection $(v_a^{a'})_{a' \geqslant a}$ such that

$$(V_a, v_a^{a'}, A) \in \mathrm{Ob}_{\mathcal{K}^*} , \qquad (1_A, s_a) \in \mathrm{Mor}_{\mathcal{K}^*}$$

and

(1) $$\mathrm{Ker}[\boldsymbol{p}] = (\boldsymbol{V}, [\boldsymbol{s}]) , \quad \text{where} \quad \boldsymbol{V} = (V_a, v_a^{a'}, A) \text{ and } \boldsymbol{s} = (s_a) .$$

By 3.4, there exists a collection $(m_a^{a'})_{a' \geqslant a}$ such that

$$(M_a, m_a^{a'}, A) \in \mathrm{Ob}_{\mathcal{K}^*} , \qquad (1_A, p_a) \in \mathrm{Mor}_{\mathcal{K}^*}$$

and

(2) $$\mathrm{Coker}[\boldsymbol{f}] = (\boldsymbol{M}, [\boldsymbol{p}]) , \quad \text{where} \quad \boldsymbol{M} = (M_a, m_a^{a'}, A) \text{ and } \boldsymbol{p} = (p_a) .$$

By (1) and (2), we get

$$\mathrm{Im}[\boldsymbol{f}] = \mathrm{Ker}[\boldsymbol{p}] = (\boldsymbol{V}, [\boldsymbol{s}]) .$$

By Propositions 3.3 and 3.5, we obtain

3.6. Corollary. *Given a finite or infinite diagram*

$$\mathfrak{D}\colon \ \ldots \xrightarrow{f^{(0)}} X^{(1)} \xrightarrow{f^{(1)}} X^{(2)} \xrightarrow{f^{(2)}} \ldots \ \textit{in the category } \mathcal{K}^* ,$$

where $\boldsymbol{f}^{(k)} = (f_a^{(k)})_{a \in A}$ *for* $k = 0, \pm 1, \ldots$ *and* $\boldsymbol{X}^{(k)} = (X_a^{(k)}, p_a^{(k)a'}, A)$, *consider a system of diagrams* $(\mathfrak{D}_a)_{a \in A}$,

$$\mathfrak{D}_a\colon \ \ldots \xrightarrow{f_a^{(0)}} X_a^{(1)} \xrightarrow{f_a^{(1)}} X_a^{(2)} \xrightarrow{f_a^{(2)}} \ldots \ \textit{in } \mathcal{K} .$$

If all the diagrams $\mathfrak{D}_a$ *are exact in* $\mathcal{K}$ *then the diagram* $\mathfrak{D}$ *is exact in* $\hat{\mathcal{K}}^*$. ■

Consider now a weak additive category $\langle \mathcal{K}, \{+_{XY}\}, \{\omega_{XY}\} \rangle$. For any pair of objects $\boldsymbol{X}$, $\boldsymbol{Y}$ in $\mathcal{K}^*$, define an operation $+_{\boldsymbol{XY}}$ in $\mathrm{Mor}_{\hat{\mathcal{K}}^*}(\boldsymbol{X}, \boldsymbol{Y})$ as follows:

Take two maps in $\mathrm{Mor}_{\mathcal{K}^*}(\boldsymbol{X}, \boldsymbol{Y})$:

$$\boldsymbol{f} = (\varphi, f_a) \quad \text{and} \quad \boldsymbol{g} = (\psi, g_a) .$$

1° If $\varphi = \psi$, then

$$\boldsymbol{f} + \boldsymbol{g} \underset{\mathrm{Df}}{=} (\varphi, f_a + g_a) .$$

2° If $\varphi \neq \psi$, then, by Lemma 5 of [5], there exists $\delta\colon (A, \leqslant) \to (A, \leqslant)$ such that $\delta(a) \geqslant \varphi(a), \psi(a)$ for any a; let

$$\tilde{\boldsymbol{f}} = (\delta, f_a p_{\varphi(a)}^{\delta(a)}) \quad \text{and} \quad \tilde{\boldsymbol{g}} = (\delta, g_a p_{\psi(a)}^{\delta(a)}) ;$$

put

$$\boldsymbol{f} \underset{(\delta)}{+} \boldsymbol{g} \underset{\mathrm{Df}}{=} \tilde{\boldsymbol{f}} + \tilde{\boldsymbol{g}} .$$

Notice that $[f \underset{(\delta)}{+} g]$ is independent of the choice of δ. This enables us to define $[f]+[g]$ by the formula

$$[f]+[g] \underset{\mathrm{Df}}{=} [f \underset{(\delta)}{+} g]$$

for an arbitrary $\delta\colon (A, \leqslant)\to(A, \leqslant)$.

It is easy to show that

3.7. *The system* $\langle \hat{K}^*, \{+_{[X][Y]}\}, \{[\omega_{XY}]\}\rangle$ *is a weak additive category.* ■

Section 2

1. Exactness property for homotopy systems. The homotopy system $\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)$ has been defined in [12] and studied in [13]. It is understood as an inverse system of nth absolute homotopy groups; more precisely, if $(\boldsymbol{X}, \boldsymbol{x}_0) = \big((X_a, x_a), p_a^{a'}, A\big)$, then for every $n \geqslant 1$,

$$\pi_n(\boldsymbol{X}, \boldsymbol{x}_0) \underset{\mathrm{Df}}{=} \big(\pi_n(X_a, x_a), (p_a^{a'})_n, A\big),$$

where $\pi_n(X_a, x_a)$ is the nth homotopy group of X_a at x_a and $(p_a^{a'})_n\colon \pi_n(X_{a'}, x_{a'})\to\pi_n(X_a, x_a)$ is the homomorphism induced by the map $p_a^{a'}\colon (X_{a'}, x_{a'})\to(X_a, x_a)$.

In a similar way for every $n \geqslant 2$ the *relative n-th homotopy system* $\pi_n(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$ can be defined: if $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0) = \big((Z_a, X_a, x_a), r_a^{a'}, A\big)$, then

$$\pi_n(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0) \underset{\mathrm{Df}}{=} \big(\pi_n(Z_a, X_a, x_a), (r_a^{a'})_n, A\big),$$

where $\pi_n(Z_a, X_a, x_a)$ is the nth relative homotopy group of Z_a modulo X_a at x_a, and $(r_a^{a'})_n$ is the homomorphism induced by $r_a^{a'}$ (see [4]).

Remark. The category $\mathcal{T}$ of pointed Hausdorff spaces can be treated as a subcategory of the category $\mathcal{T}'$ of Hausdorff triplets, since any pair (X, x_0) can be identified with the triplet $\big(X, (x_0), x_0\big)$. Thus the functor of relative nth homotopy group is an extension of the functor of absolute nth homotopy group. Hence, in similar, the functor of relative nth homotopy system is an extension of the functor of absolute nth homotopy system. This justifies the using of the same symbol π_n for both the functors.

We use the following notation:

$I^n = \{(t_1, \ldots, t_n)\colon \bigwedge_{i=1}^{n} 0 \leqslant t_i \leqslant 1\}$ — the unit n-dimensional cube,

$I^{n-1} = \{(t_1, \ldots, t_n) \in I^n\colon\ t_n = 0\}$ — the initial $(n-1)$-face of I^n,

$J^{n-1} = \big\{(t_1, \ldots, t_n) \in I^n\colon\ \big(\bigvee_{i<n} t_i = 0\big) \vee \big(\bigvee_{i} t_i = 1\big)\big\}$ — the union of remaining $(n-1)$-faces,

$\dot{I}^n = I^{n-1} \cup J^{n-1}$ — the boundary of I^n.

Let $\mathcal{G}$ be the category of groups.

We are interested in the exactness property for homotopy systems. As was noticed in § 2 of Section 1, the exactness of a diagram can be defined in any category with zero-objects. Thus, for our purpose, the group structure of the absolute or relative homotopy groups can be neglected. In other words, it is sometimes convenient to treat π_n (for $n \geqslant 1$) as a functor from the category of triplets to the following category $\mathcal{C}$. The objects of $\mathcal{C}$ are pairs $\langle G, 0\rangle$ consisting of the set G and the fixed element 0 of G; the morphisms are functions preserving the fixed elements. Of course $\mathcal{C}$ has zero-objects. Then, $\pi_n(Z, X, x_0)$ is considered as the following pair:

$$\pi_n(Z, X, x_0) = \langle [Z, X, x_0]^{[I^n, \dot{I}^n, J^{n-1}]}, 0\rangle,$$

where 0 is the homotopy class of the constant map. Consequently,

$$\pi_n(X, x_0) = \pi_n(X, (x_0), x_0) = \langle [X, x_0]^{[I^n, \dot{I}^n]}, 0\rangle$$

(see Remark). Of course, such approach enables us to include the case $n = 1$ to our consideration.

Let us take an inverse-system of triplets,

$$(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0) = \big((Z_a, X_a, x_a), r_a^{a'}, A\big),$$

X_a being an arcwise connected subset of Z_a and x_a being any point of X_a. Let $\boldsymbol{i}\colon (\boldsymbol{X}, \boldsymbol{x}_0)\to(\boldsymbol{Z}, \boldsymbol{x}_0)$ be the ordinary inclusion (see [10]), i.e. $\boldsymbol{i} = (1_A, i_a)$, $i_a(x) = x$ for any $x \in X_a$. Consider the following diagram in the category $\mathcal{C}^*$:

$$\mathfrak{D}\colon\ \ldots\to\pi_n(\boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{i}_n}\pi_n(\boldsymbol{Z}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\xi}_n}\pi_n(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\partial}_n}\pi_{n-1}(\boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{i}_{n-1}}\pi_{n-1}(\boldsymbol{Z}, \boldsymbol{x}_0)\to\ldots$$
$$\ldots\to\pi_1(\boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{i}_1}\pi_1(\boldsymbol{Z}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\xi}_1}\pi_1(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\partial}_1}\boldsymbol{0},$$

where $\boldsymbol{0}$ is an inverse system consisting of zero-objects in $\mathcal{C}$ (thus all the more $\boldsymbol{0}$ is a zero-object in $\mathcal{C}^*$), $\boldsymbol{i}_n$ is the morphism induced by the inclusion $\boldsymbol{i}$ and $\boldsymbol{\xi}_n, \boldsymbol{\partial}_n$ are defined as follows:

$$\boldsymbol{\xi}_n = (1_A, \xi_{a,n}),\qquad \xi_{a,n}\colon \pi_n(Z_a, x_a)\to\pi_n(Z_a, X_a, x_a),$$
$$\xi_{a,n}[\varphi] = [\varphi]_{\mathrm{rel}}\quad \text{for any } \varphi\colon (I^n, \dot{I}^n)\to(Z_a, x_a),$$
$$\boldsymbol{\partial}_n = (1_A, \partial_{a,n}),\qquad \partial_{a,n}\colon \pi_n(Z_a, X_a, x_a)\to\pi_{n-1}(X_a, x_a),$$
$$\partial_{a,n}[\varphi]_{\mathrm{rel}} = [\varphi|I^{n-1}]\quad \text{for any } \varphi\colon (I^n, \dot{I}^n, J^{n-1})\to(Z_a, X_a, x_a).$$

The diagram $\mathfrak{D}$ will be refered to as the *homotopy diagram* of the inverse system $(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)$.

Consider the following finite subdiagrams of $\mathfrak{D}$;

$$\mathfrak{D}_n\colon\ \pi_n(\boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{i}_n}\pi_n(\boldsymbol{Z}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\xi}_n}\pi_n(\boldsymbol{Z}, \boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{\partial}_n}\pi_{n-1}(\boldsymbol{X}, \boldsymbol{x}_0)\xrightarrow{\boldsymbol{i}_{n-1}}\pi_{n-1}(\boldsymbol{Z}, \boldsymbol{x}_0)\quad \text{for } n \geqslant 2$$

and

$$\mathfrak{D}_1:\ \pi_1(X, x_0) \xrightarrow{i_1} \pi_1(Z, x_0) \xrightarrow{\xi_1} \pi_1(Z, X, x_0) \xrightarrow{\partial_1} 0 .$$

For $n \geqslant 2$, $\mathfrak{D}_n$ is a diagram in $\mathfrak{G}^*$; $\mathfrak{D}_1$ is a diagram in $\mathfrak{C}^*$.

By the exactness of the diagram

$$\ldots \to \pi_n(X_a, x_a) \xrightarrow{(i_a)_n} \pi_n(Z_a, x_a) \xrightarrow{\xi_{a,n}} \pi_n(Z_a, X_a, x_a) \xrightarrow{\partial_{a,n}} \pi_{n-1}(X_a, x_a) \to \ldots$$
$$\ldots \to \pi_1(Z_a, X_a, x_a) \xrightarrow{\partial_{a,1}} 0$$

for every $a \in A$, the statement 3.6 of Section 1 implies

1.1. *The homotopy diagram $\mathfrak{D}$ of any inverse system (Z, X, x_0) is exact.* ■

More precisely

1.2. *The diagram $\mathfrak{D}_n$ is exact in $\hat{\mathfrak{G}}^*$ for $n \geqslant 2$ and is exact in $\hat{\mathfrak{C}}^*$ for $n = 1$.* ■

Now, we apply 1.2 to prove

1.3. *Let (Z, X, x_0) be an inverse system of triplets of arcwise connected Hausdorff spaces and let $i\colon X \to Z$ be the inclusion. Take a natural number n_0. If $[i_n]$ is a bimorphism in $\hat{\mathfrak{C}}^*$ for $n \leqslant n_0$ and is an epimorphism in $\hat{\mathfrak{G}}^*$ for $n = n_0+1$, then $\pi_n(Z, X, x_0)$ is a zero-object in $\hat{\mathfrak{C}}^*$ for $n = 1, \ldots, n_0+1$.*

Proof. Since $[i_n]$ is a bimorphism in $\hat{\mathfrak{C}}^*$ for $n \leqslant n_0$, it is a bimorphism in $\hat{\mathfrak{G}}^*$ as well.

Let $2 \leqslant n \leqslant n_0+1$. Consider the subdiagram $\mathfrak{D}_n$ of the homotopy diagram for (Z, X, x_0),

$$\mathfrak{D}_n:\ \pi_n(X, x_0) \xrightarrow{i_n} \pi_n(Z, x_0) \xrightarrow{\xi_n} \pi_n(Z, X, x_0) \xrightarrow{\partial_n} \pi_{n-1}(X, x_0) \xrightarrow{i_{n-1}} \pi_{n-1}(Z, x_0) .$$

By 1.2, the diagram $\mathfrak{D}_n$ is exact in the category $\hat{\mathfrak{G}}^*$. Since $[i_n]$ is an epimorphism and $[i_{n-1}]$ is a monomorphism in $\hat{\mathfrak{G}}^*$, by the statements 2.3 and 3.7 of Section 1 it follows that

(1) $\qquad \pi_n(Z, X, x_0)$ is a zero-object in $\hat{\mathfrak{G}}^*$.

Let $n = 1$. Consider the diagram $\mathfrak{D}_1$ for (Z, X, x_0),

$$\mathfrak{D}_1:\ \pi_1(X, x_0) \xrightarrow{i_1} \pi_1(Z, x_0) \xrightarrow{\xi_1} \pi_1(Z, X, x_0) \xrightarrow{\partial_1} 0 .$$

By 1.2, the diagram $\mathfrak{D}_1$ is exact in the category $\hat{\mathfrak{C}}^*$. Since $[i_1]$ is an epimorphism in $\hat{\mathfrak{C}}^*$, by the statement 1.9 of Section 1 it follows that

(2) $\qquad \pi_1(Z, X, x_0)$ is a zero object in $\hat{\mathfrak{C}}^*$.

By (1) and (2), $\pi_n(Z, X, x_0)$ is a zero object in $\hat{\mathfrak{C}}^*$ for $n = 1, \ldots, n_0+1$. ■

Let us establish two consequences of the algebraic condition $\pi_n(Z, X, x_0) = 0$ in $\hat{\mathfrak{C}}^*$. First of them is the following

1.4. Proposition. *Let $(Z, X, x_0) = \big((Z_a, X_a, x_a), r_a^{a'}, A\big)$ be an inverse system of Hausdorff triplets over a closure-finite directed set $(A, \leqslant)$, n — a natural number. If $\pi_n(Z, X, x_0)$ is a zero-object in $\hat{\mathfrak{C}}^*$, then there exists an increasing function $\delta\colon A \to A$ such that*

(i) $\qquad \delta(a) \geqslant a \quad$ *for* $\quad a \in A$,

(ii) *for any map $\xi\colon (I^n, \dot{I}^n, J^{n-1}) \to (Z_{\delta(a)}, X_{\delta(a)}, x_{\delta(a)})$, $r_a^{\varphi(a)}\xi \simeq$ const in $(Z_a, X_a, x_a)^{(I^n, \dot{I}^n, J^{n-1})}$.*

Proof. Consider the identity map

$$1_Z\colon (Z, X, x_0) \to (Z, X, x_0)$$

and the induced identity morphism

$$(1_Z)_n\colon \pi_n(Z, X, x_0) \to \pi_n(Z, X, x_0) .$$

Since $\pi_n(Z, X, x_0)$ is a zero-object in $\hat{\mathfrak{C}}^*$, $[1_Z]_n$ is a zero-morphism in $\hat{\mathfrak{C}}^*$, i.e.

$$\bigwedge_a \bigvee_{a' \geqslant a} \operatorname{Im}(r_a^{a'})_n = 0 \in \pi_n(Z_a, X_a, x_a) .$$

Thus, for any map $\xi\colon (I^n, \dot{I}^n, J^{n-1}) \to (Z_{a'}, X_{a'}, x_{a'})$, $r_a^{a'}\xi \simeq$ const in (Z_a, X_a, x_a). Of course, a' can be replaced by any greater one, hence, by Lemma 5 of [5], there exists an increasing function $\delta\colon A \to A$ satisfying (i) and (ii). ■

Let us notice that

1.5. *If (Z, X) is an inverse system of pairs of arcwise connected Hausdorff spaces, then for any two threads x_0 and x_0' in X, the homotopy systems $\pi_n(Z, X, x_0)$ and $\pi_n(Z, X, x_0')$ are isomorphic (in $\hat{\mathfrak{C}}^*$ for $n = 1$ and in $\hat{\mathfrak{G}}^*$ for $n > 1$).* ■

The last statement enables us — in the case of arcwise connected spaces — to use the symbol $\pi_n(Z, X)$ to denote the nth homotopy system of (Z, X, x_0) for any x_0 in X.

As the second consequence of the condition $\pi_n(Z, X, x_0) = 0$ in $\hat{\mathfrak{C}}^*$, we get

1.6. Proposition. *Let $(Z, X) = \big((Z_a, X_a), r_a^{a'}, A\big)$ be an inverse system of pairs of arcwise connected Hausdorff spaces with $r_a^{a'}(X_{a'}) = X_a$ for $a' \geqslant a$. Let n be a natural number. If $\pi_n(Z, X)$ is a zero object in $\hat{\mathfrak{C}}^*$, then there exists an increasing function $\delta\colon A \to A$ satisfying the following two conditions*

(i) $\qquad \delta(a) \geqslant a \quad$ *for* $\quad a \in A$,

(ii′) *for every $\hat{\xi}\colon (I^n, \dot{I}^n) \to (Z_{\delta(a)}, X_{\delta(a)})$, $r_a^{\delta(a)}\hat{\xi} \simeq$ const in $(Z_a, X_a)^{(I^n, \dot{I}^n)}$.*

Proof. By 1.4, there exists an increasing function $\delta\colon A \to A$ satisfying (i) and (ii).

Take a map $\hat{\xi}\colon (I^n, \dot{I}^n) \to (Z_{\delta(a)}, X_{\delta(a)})$ and a point $a \in \hat{\xi}(\dot{I}^n)$. Since $r_a^{a'}(X_{a'}) = X_a$ for every $a' \geqslant a$, there exists a thread $(x_a)_{a \in A}$ in X, such that $a = x_{\delta(a)}$.

Notice that there exists a map

$$\xi\colon (I^n, \dot{I}^n) \to (Z_{\delta(a)}, X_{\delta(a)})$$

such that

$$\xi(J^{n-1}) = x_{\delta(a)} \tag{1}$$

and

$$\xi \simeq \hat{\xi} \quad \text{in } (Z_{\delta(a)}, X_{\delta(a)})^{(I^n, \dot{I}^n)}. \tag{2}$$

Indeed, take $t_a \in \hat{\xi}^{-1}(x_{\delta(a)}) \cap \dot{I}^n$; obviously, there exists a homotopy $h\colon J^{n-1} \times I \to \dot{I}^n$ such that

$$h(t, 0) = t \quad \text{and} \quad h(t, 1) = t_a;$$

thus the homotopy $h'\colon J^{n-1} \times I \to X_{\delta(a)}$ defined by the formula

$$h'(t, s) = \hat{\xi} h(t, s)$$

satisfies the conditions

$$h'(t, 0) = \hat{\xi}(t) \quad \text{and} \quad h'(t, 1) = x_{\delta(a)};$$

i.e. $\hat{\xi}|J^{n-1} \simeq \text{const}$ in $(X_{\delta(a)})^{J^{n-1}}$.

By theorem on extension of homotopy for polyhedra, (see [4], p. 14), there is a map $\xi'\colon \dot{I}^n \to X_{\delta(a)}$ such that

$$\xi'(J^{n-1}) = x_{\delta(a)} \quad \text{and} \quad \xi' \simeq \hat{\xi}|\dot{I}^n \quad \text{in } (X_{\delta(a)})^{\dot{I}^n};$$

by the same argument, there exists a map

$$\xi\colon (I^n, \dot{I}^n) \to (Z_{\delta(a)}, X_{\delta(a)})$$

such that

$$\xi(J^{n-1}) = x_{\delta(a)} \quad \text{and} \quad \xi \simeq \hat{\xi} \quad \text{in } (Z_{\delta(a)}, X_{\delta(a)})^{(I^n, \dot{I}^n)},$$

i.e. ξ satisfies (1) and (2). By 1.4, we have

$$r_a^{\delta(a)} \xi \simeq \text{const} \quad \text{in } (Z_a, X_a)^{(I^n, \dot{I}^n)}; \tag{3}$$

hence, by (2),

$$r_a^{\delta(a)} \hat{\xi} \simeq \text{const} \quad \text{in } (Z_a, X_a)^{(I^n, \dot{I}^n)}. \ \blacksquare$$

2. Inverse sequences of polyhedra. We shall use the following terminology. First, consider any map of inverse systems over the same directed set,

$$f = (\varphi, f_a)\colon X = (X_a, p_a^{a'}, A) \to (Y_a, q_a^{a'}, A) = Y.$$

By the definition, all the diagrams

$$\begin{array}{ccc} X_{\varphi(a)} & \xleftarrow{\;p_{\varphi(a)}^{\varphi(a')}\;} & X_{\varphi(a')} \\ \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_{a'}} \\ Y_a & \xleftarrow[\;q_a^{a'}\;]{} & Y_{a'} \end{array}$$

commute up to homotopy,

i.e. for $a' \geqslant a$ there exists a homotopy $h_a^{a'}\colon X_{\varphi(a')} \times I \to Y_a$ such that $h_a^{a'}(x, 0) = f_a p_{\varphi(a)}^{\varphi(a')}(x)$ and $h_a^{a'}(x, 1) = q_a^{a'} f_{a'}(x)$. These homotopies $h_a^{a'}$ will be called *connecting homotopies for the map f*.

Now, consider a system (φ, f_a) with $\varphi\colon A \to A$ being an increasing function and $f_a\colon X_{\varphi(a)} \to Y_a$ being any maps (with no assumption concerning the diagrams). Such a system (φ, f_a) will be referred to as a *pseudo-map* of inverse systems, in symbols $(\varphi, f_a)\colon X \dashrightarrow Y$.

Let $\mathfrak{T}$ be the category with polyhedral pairs as objects and with morphisms defined as follows: the map $g\colon (K, L) \to (K', L')$ is a morphism in $\mathfrak{T}$ whenever g is simplicial with respect to some subdivisions of K and K'. It is convenient to identify a polyhedron K with the pair $(K, \emptyset)$ and thus to consider polyhedra as some objects of $\mathfrak{T}$. The symbol K^n denotes the n-dimensional skeleton of K for every $n \geqslant 0$ and denotes the empty set for $n = -1$.

For any inverse system $K = (K_a, u_a^{a'}, A)$ in the category $\mathfrak{T}$ let

$$\dim K \underset{\mathrm{Df}}{=} \sup_{a \in A} \dim K_a.$$

We are going to establish two propositions concerning maps of inverse systems of polyhedral pairs (2.5 and 2.6). We start by the following

2.1. Lemma. *Given a homotopy $h\colon (I^n, \dot{I}^n, J^{n-1}) \times I \to (Z, X, x_0)$ satisfying the condition*

$$h(I^n \times (1)) \subset X,$$

there exists a homotopy $h'\colon (I^n, \dot{I}^n, J^{n-1}) \times I \to (Z, X, x_0)$ satisfying the conditions

$$h'(t, s) = h(t, 0) \quad \textit{for} \quad (t, s) \in I^n \times (0) \cup \dot{I}^n \times I$$

and

$$h'(I^n \times (1)) \subset X.$$

Proof ([5]). Take the homotopy h and construct h' as follows:

$$h'(t_1, \dots, t_n; s) \underset{\mathrm{Df}}{=} \begin{cases} h\left(t_1, \dots, t_{n-1}, \dfrac{t_n}{1 - \frac{1}{2}s}; s\right) & \text{if } 0 \leqslant t_n \leqslant 1 - \frac{1}{2}s, \\ h(t_1, \dots, t_{n-1}, 1; 2(1 - t_n)) & \text{if } 1 - \frac{1}{2}s \leqslant t_n \leqslant 1. \end{cases}$$

([5]) Due to R. Rubinstein.

Evidently, $h'(\dot{I}^n\times I)\subset X$ and $h'(J^{n-1}\times I)=x_0$. The function h' is continuous, since $t_n=1-\frac{1}{2}s$ implies

$$h\Big(t_1,\ldots,t_{n-1},\frac{t_n}{1-\frac{1}{2}s};s\Big)=h(t_1,\ldots,t_{n-1},1;s)=h\big(t_1,\ldots,t_{n-1},1;2(1-t_n)\big).$$

If $(t,s)\in I^n\times(0)$, then $h'(t;s)=h(t;0)$. Let $(t,s)\in\dot{I}^n\times I$; then either $t_n=1$ or $(t,s)\in J^{n-1}\times I$; in both cases $h'(t,s)=h(t,0)$.

At last, $h'(t;1)\in h(\dot{I}^n\times I)\cup I^n\times(1)\subset X$ for every $t\in I^n$. ■

Take a natural n. A function $\delta\colon A\to A$ will be referred to as *n-characteristic function of* (Z,X) whenever it satisfies the conditions (i) and (ii′) of Proposition 1.6. Then, Proposition 1.6 can be formulated as follows:

2.2. *Let* $(\boldsymbol{Z},\boldsymbol{X})=\big((Z_a,X_a),r_a^{a'},A\big)$ *be an inverse system of pairs of arcwise connected Hausdorff spaces with* $r_a^{a'}(X_{a'})=X_a$ *for* $a'\geqslant a$. *If* $\pi_n(\boldsymbol{Z},\boldsymbol{X})=0$ *in* $\hat{\mathfrak{C}}^*$, *then there exists n-characteristic function* $\delta\colon A\to A$ *of* $(\boldsymbol{Z},\boldsymbol{X})$.

Let us prove two Lemmas concerning n-characteristic function (2.3 and 2.4). In both of them $(\boldsymbol{Z},\boldsymbol{X})$ is assumed to be an inverse system of pairs of arcwise connected Hausdorff spaces, $(\boldsymbol{Z},\boldsymbol{X})=\big((Z_a,X_a),r_a^{a'},A\big)$, with $r_a^{a'}(X_{a'})=X_a$ for $a'\geqslant a$.

2.3. Lemma. *Let n be a natural number and let* $\delta\colon A\to A$ *be n-characteristic function of* $(\boldsymbol{Z},\boldsymbol{X})$. *Then for every polyhedral pair* (K,L) *and for every map*

$$p\colon (K^n,L^n)\times I\to(Z_{\delta(a)},X_{\delta(a)})$$

such that

$$p(x,s)=p(x,0)\quad\text{for}\quad(x,s)\in L^n\times I$$

and

$$p(x,1)\in X_{\delta(a)}\quad\text{for}\quad x\in K^{n-1},$$

there exists a map

$$\hat{p}\colon (K^n,L^n)\times I\to(Z_a,X_a)$$

such that

$$\hat{p}(x,s)=\begin{cases} r_a^{\delta(a)}p(x,s) & \text{for}\quad (x,s)\in K^n\times(0)\cup L^n\times I,\\ r_a^{\delta(a)}p(x,2s) & \text{for}\quad (x,s)\in K^{n-1}\times\langle 0,\tfrac{1}{2}\rangle,\\ r_a^{\delta(a)}p(x,1) & \text{for}\quad (x,s)\in K^{n-1}\times\langle\tfrac{1}{2},1\rangle,\end{cases}$$

and

$$\hat{p}(x,1)\in X_a\quad\text{for every}\quad x\in K^n.$$

Proof. Since $\delta\colon A\to A$ is n-characteristic function of $(\boldsymbol{Z},\boldsymbol{X})$, hence

(1) for every $\xi\colon(I^n,\dot{I}^n)\to(Z_{\delta(a)},X_{\delta(a)})$, $r_a^{\delta(a)}\xi\simeq\text{const}$ in $(Z_a,X_a)^{(I^n,\dot{I}^n)}$.

Notice that

(2) $K^n=K^{n-1}\cup L^n\cup D$, where $D=\emptyset$ or $D=\bigcup_{j=1}^{l}\Delta_j^n$, with $\Delta_1^n,\ldots,\Delta_l^n$ being n-dimensional simplexes which lie in K^n but not in L^n.

If $D=\emptyset$, then the existence of $\hat{p}$ is obvious. If $D\neq\emptyset$, then there exist topological imbeddings

$$\xi_j\colon(I^n,\dot{I}^n)\to(K^n,K^{n-1}),\quad j=1,\ldots,l,$$

such that

(3) $$\xi_j(I^n)=\Delta_j^n\quad\text{and}\quad\xi_j(\dot{I}^n)=\dot{\Delta}_j^n.$$

Then, let us define the maps

$$\hat{\xi}_j\colon(I^n,\dot{I}^n)\to(Z_{\delta(a)},X_{\delta(a)})$$

by the formula

$$\hat{\xi}_j(t)\underset{\mathrm{Df}}{=}p\big(\xi_j(t),1\big).$$

By (1),

$$r_a^{\delta(a)}\hat{\xi}_j\simeq\text{const}\quad\text{in}\ (Z_a,X_a)^{(I^n,\dot{I}^n)},$$

i.e. there exists a homotopy

$$h_j\colon(I^n,\dot{I}^n)\times I\to(Z_a,X_a)$$

such that

(4) $$h_j(t,0)=r_a^{\delta(a)}p\big(\xi_j(t),1\big)\quad\text{and}\quad h_j(t,1)\in X_a\quad\text{for } t\in I^n.$$

By Lemma 2.1, the homotopy h_j can be assumed to satisfy the following additional condition

(5) $$h_j(t,s)=h_j(t,0)\quad\text{for every }(t,s)\in\dot{I}^n\times I.$$

Now, let us define the function

$$\hat{p}\colon(K^n,L^n)\times I\to(Z_a,X_a)$$

by the formula

(6) $$\hat{p}(x,s)\underset{\mathrm{Df}}{=}\begin{cases} r_a^{\delta(a)}p(x,2s) & \text{for}\quad (x,s)\in K^n\times\langle 0,\tfrac{1}{2}\rangle,\\ r_a^{\delta(a)}p(x,1) & \text{for}\quad (x,s)\in(K^{n-1}\cup L^n)\times\langle\tfrac{1}{2},1\rangle,\\ h_j(\xi_j^{-1}(x),2s-1) & \text{for}\quad (x,s)\in\Delta_j^n\times\langle\tfrac{1}{2},1\rangle,\ j=1,\ldots,l.\end{cases}$$

By (4) and (5), $\hat{p}$ is continuous. By (6), since $p(x, s) = p(x, 1)$ for $(x, s) \in L^n \times I$, we get

$$\hat{p}(x, s) = r_a^{\delta(a)} p(x, s) \quad \text{for} \quad (x, s) \in K^n \times (0) \cup L^n \times I .$$

By (6),

$$\hat{p}(x, s) = \begin{cases} r_a^{\delta(a)} p(x, 2s) & \text{for} \quad (x, s) \in K^{n-1} \times \langle 0, \tfrac{1}{2} \rangle, \\ r_a^{\delta(a)} p(x, 1) & \text{for} \quad (x, s) \in K^{n-1} \times \langle \tfrac{1}{2}, 1 \rangle. \end{cases}$$

At last, by (4) and (6),

$$\hat{p}(x, 1) \in X_a \quad \text{for every } x \in K^n. \blacksquare$$

2.4. Lemma. *Let $n \geqslant 0$ and let $\delta\colon A \to A$ be $(n+1)$-characteristic function of (Z, X). Let (K, L) be a polyhedral pair and let*

$$M_n = (K^n \times I) \times (0) \cup [(K^{n-1} \cup L^n) \times I \cup K^n \times \{0, 1\}] \times I .$$

Then, for every map

$$p\colon \big(M_n, (L_n \times I) \times I\big) \to (Z_{\delta(a)}, X_{\delta(a)})$$

such that

$$p\big((x, t), 1\big) \in X_{\delta(a)} \quad \textit{for} \quad \big((x, t), 1\big) \in M_n ,$$

there exists a map

$$\hat{p}\colon (K^n \times I, L^n \times I) \times I \to (Z_a, X_a)$$

such that

$$\hat{p}\big((x, t), s\big) = r_a^{\delta(a)} p\big((x, t), s\big) \quad \textit{for} \quad \big((x, t), s\big) \in M_n$$

and

$$\hat{p}\big((x, t), 1\big) \in X_a \quad \textit{for} \quad (x, t) \in K^n \times I .$$

Proof. If $K^n = K^{n-1} \cup L^n$, then we put $\hat{p} \underset{\mathrm{Df}}{=} r_a^{\delta(a)} p$. Let us assume that

$$K^n = K^{n-1} \cup L^n \cup \bigcup_{j=1}^{l} \Delta_j^n,$$

$\Delta_1^n, \ldots, \Delta_l^n$ being n-dimensional simplexes which lie in K^n but not in L^n.

Let B_j be a union of all $(n+1)$-dimensional faces of $(\Delta_j^n \times I) \times I$ with the exception of the face $(\Delta_j^n \times I) \times (1)$; i.e.

$$B_j = (\Delta_j^n \times I) \times (0) \cup (\Delta_j^n \times \{0, 1\} \cup \dot{\Delta}_j^n \times I) \times I .$$

There exist homeomorphisms

$$h_j\colon I^{n+1} \times I \to (\Delta_j^n \times I) \times I$$

such that

(1) $$h_j\big(I^{n+1} \times (0)\big) = B_j$$

and

(2) $$h_{j_1}^{-1}\big((x, t), s\big) = h_{j_2}^{-1}\big((x, t), s\big) \quad \text{for} \quad \big((x, t), s\big) \in \big((\Delta_{j_1}^n \cap \Delta_{j_2}^n) \times I\big) \times I .$$

Then, of course,

(3) $$h_j\big(\dot{I}^{n+1} \times (0)\big) = \dot{B}_j = (\Delta_j^n \times \{0, 1\} \cup \dot{\Delta}_j^n \times I) \times (1) .$$

Take the map $p|B_j\colon B_j \to Z_{\delta(a)}$; by the assumption, we have

(4) $$p(\dot{B}_j) \subset X_{\delta(a)} .$$

The conditions (1), (3) and (4) enable us to define maps

$$\overline{p}_j\colon (I^{n+1}, \dot{I}^{n+1}) \to (Z_{\delta(a)}, X_{\delta(a)})$$

by the formula

(5) $$\overline{p}_j(x) \underset{\mathrm{Df}}{=} p h_j(x, 0) \quad \text{for} \quad x \in I^{n+1},\ j = 1, \ldots, l .$$

Since δ was assumed to be $(n+1)$-characteristic function, we get

$$r_a^{\delta(a)} \overline{p}_j \simeq \text{const} \quad \text{in } (Z_a, X_a)^{(I^{n+1}, \dot{I}^{n+1})},$$

i.e. there exist homotopies

$$v_j\colon (I^{n+1}, I^{n+1}) \times I \to (Z_a, X_a)$$

such that

(6) $$v_j(x, 0) = r_a^{\delta(a)} \overline{p}_j(x)$$

and

(7) $$v_j(x, 1) \in X_a .$$

Let us notice that

(8) $$(K^n \times I) \times I = M_n \cup \bigcup_{j=1}^{l} (\Delta_j^n \times I) \times I$$

and

(9) $$M_n \cap [(\Delta_j^n \times I) \times I] = B_j \quad \text{for} \quad j = 1, \ldots, l .$$

Let us define the map

$$\hat{p}\colon (K^n \times I, L^n \times I) \times I \to (Z_a, X_a)$$

by the formula

$$(10)\qquad \hat{p}\big((x,t),s\big) \underset{\mathrm{Df}}{=} \begin{cases} r_\alpha^{\delta(\alpha)}p\big((x,t),s\big) & \text{for } \big((x,t),s\big)\in M_n\,,\\ v_j h_j^{-1}\big((x,t),s\big) & \text{for } \big((x,t),s\big)\in(\Delta_j^n\times I)\times I\,,\\ & \qquad j=1,\dots,l\,. \end{cases}$$

By (1) and (6), if $\big((x,t),s\big)\in B_j$, then $v_jh_j^{-1}\big((x,t),s\big)=v_j(y,0)$ $=r_\alpha^{\delta(\alpha)}\overline{p}_j(y)$, where $h_j(y,0)=\big((x,t),s\big)$; thus, by (5), $v_jh_j^{-1}\big((x,t),s\big)$ $=r_\alpha^{\delta(\alpha)}p\big((x,t),s\big)$; therefore, by (2), (8) and (9), $\hat{p}$ is continuous.

By (10) and (3), we infer that $\hat{p}$ satisfies the required conditions. ∎

Now, let us prove

2.5. Proposition. *Let $(\boldsymbol{Z},\boldsymbol{X})=\big((Z_\alpha,X_\alpha),r_\alpha^{\alpha'},N\big)$ be an inverse sequence of pairs of arcwise connected Hausdorff spaces with $r_\alpha^{\alpha'}(X_{\alpha'})=X_\alpha$ for $\alpha'\geqslant\alpha$, and let n_0 be a natural number. If $\pi_n(\boldsymbol{Z},\boldsymbol{X})$ is a zero-object in $\hat{\mathcal{C}}^*$ for $n=1,\dots,n_0+1$, then there exists an increasing function $\varphi\colon N\to N$ with the following property: for any inverse sequence in $\mathcal{T}$, $(\boldsymbol{K},\boldsymbol{L})$ $=\big((K_\alpha,L_\alpha),u_\alpha^{\alpha'},N\big)$ with $\dim\boldsymbol{K}=n_0$, and for every map $\boldsymbol{g}=(\psi,g_\alpha)$: $(\boldsymbol{K},\boldsymbol{L})\to(\boldsymbol{Z},\boldsymbol{X})$ with connecting homotopies $h_\alpha^{\alpha'}$, there exists a map $\hat{\boldsymbol{g}}=(\psi\varphi,\hat{g}_\alpha)$: $\boldsymbol{K}\to\boldsymbol{X}$ with connecting homotopies $\hat{h}_\alpha^{\alpha'}$, a map $\boldsymbol{k}=(\psi\varphi,k_\alpha)$: $(\boldsymbol{K},\boldsymbol{L})\times I\to(\boldsymbol{Z},\boldsymbol{X})$ and double homotopies*

$$H_\alpha^{\alpha+1}\colon (K_{\psi\varphi(\alpha+1)}\times I,\ L_{\psi\varphi(\alpha+1)}\times I)\times I\to(Z_\alpha,X_\alpha)$$

such that

$$(\mathrm{i})_\alpha\qquad k_\alpha(x,s)=r_\alpha^{\varphi(\alpha)}g_{\varphi(\alpha)}(x)\qquad \textit{for}\quad (x,s)\in K_{\psi\varphi(\alpha)}\times(0)\cup L_{\psi\varphi(\alpha)}\times I\,,$$

$$(\mathrm{ii})_\alpha\qquad k_\alpha(x,1)=\hat{g}_\alpha(x)\qquad \textit{for}\quad x\in K_{\psi\varphi(\alpha)}\,,$$

$$(\mathrm{iii})_\alpha^{\alpha+1}\qquad \hat{h}_\alpha^{\alpha+1}(x,t)=r_\alpha^{\varphi(\alpha)}h_{\varphi(\alpha)}^{\varphi(\alpha+1)}(x,t)\qquad \textit{for}\quad (x,t)\in L_{\psi\varphi(\alpha+1)}\times I\,,$$

$$(\mathrm{iv})_\alpha^{\alpha+1}\qquad H_\alpha^{\alpha+1}\big((x,0),s\big)=k_\alpha\big(u_{\psi\varphi(\alpha)}^{\psi\varphi(\alpha+1)}(x),s\big)\,,\qquad H_\alpha^{\alpha+1}\big((x,1),s\big)=r_\alpha^{\alpha+1}k_{\alpha+1}(x,s)$$
$$\textit{for}\quad x\in K_{\psi\varphi(\alpha+1)},\ s\in I\,,$$

$$(\mathrm{v})_\alpha^{\alpha+1}\qquad H_\alpha^{\alpha+1}\big((x,t),s\big)=r_\alpha^{\varphi(\alpha)}h_{\varphi(\alpha)}^{\varphi(\alpha+1)}(x,t)$$
$$\textit{for}\quad \big((x,t),s\big)\in(K_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup(L_{\psi\varphi(\alpha+1)}\times I)\times I$$

and

$$(\mathrm{vi})_\alpha^{\alpha+1}\qquad H_\alpha^{\alpha+1}\big((x,t),1\big)=\hat{h}_\alpha^{\alpha+1}(x,t)\qquad \textit{for}\quad (x,t)\in K_{\psi\varphi(\alpha+1)}\times I\,.$$

or every $\alpha\in N$.

Proof. By 2.2, there exists $\delta\colon N\to N$ being n-characteristic function of $(\boldsymbol{Z},\boldsymbol{X})$ for $n=1,\dots,n_0+1$. Let us define the function $\varphi\colon N\to N$ as follows. Let

$$(1)\qquad \varphi_0\underset{\mathrm{Df}}{=}1_N\,,\qquad \varphi_n\underset{\mathrm{Df}}{=}\delta\varphi_{n-1}\qquad \text{for}\quad n\geqslant 1$$

and let

$$(2)\qquad \varphi\underset{\mathrm{Df}}{=}\varphi_{2n_0+1}\,.$$

Let us take an $\alpha\in N$. We are going to define six maps,

$$\hat{g}_\beta\colon K_{\psi\varphi(\beta)}\to X_\beta,\qquad k_\beta\colon (K_{\psi\varphi(\beta)},L_{\psi\varphi(\beta)})\times I\to(Z_\beta,X_\beta)\qquad \text{for}\quad \beta=\alpha,\ \alpha+1\,,$$
$$\hat{h}_\alpha^{\alpha+1}\colon K_{\psi\varphi(\alpha+1)}\times I\to X_\alpha$$

and

$$H_\alpha^{\alpha+1}\colon (K_{\psi\varphi(\alpha+1)}\times I,L_{\psi\varphi(\alpha+1)}\times I)\times I\to(Z_\alpha,X_\alpha)\,,$$

satisfying $(\mathrm{i})_\beta$, $(\mathrm{ii})_\beta$ for $\beta=\alpha,\alpha+1$, and $(\mathrm{iv})_\alpha^{\alpha+1}$-$(\mathrm{vi})_\alpha^{\alpha+1}$.

A. Take an arbitrary $\alpha\in N$ and arbitrary simplicial division of the polyhedron $K_{\psi\varphi(\alpha)}$. We construct two finite sequences of maps

$$\hat{g}_\alpha^n\colon K_{\psi\varphi(\alpha)}^n\to X_{\varphi_{2n_0+1-n}(\alpha)}$$

and

$$k_\alpha^n\colon (K_{\psi\varphi(\alpha)}^n,L_{\psi\varphi(\alpha)}^n)\times I\to(Z_{\varphi_{2n_0+1-n}(\alpha)},X_{\varphi_{2n_0+1-n}(\alpha)})$$

for $n=0,\dots,n_0$, satisfying for $n\geqslant 0$ the following two conditions:

$$(\mathrm{i})_\alpha^n\qquad k_\alpha^n(x,s)=r_{\varphi_{2n_0+1-n}(\alpha)}^{\varphi(\alpha)}g_{\varphi(\alpha)}(x)\qquad \text{for}\quad (x,s)\in K_{\psi\varphi(\alpha)}^n\times(0)\cup L_{\psi\varphi(\alpha)}^n\times I\,,$$

$$(\mathrm{ii})_\alpha^n\qquad k_\alpha^n(x,1)=\hat{g}_\alpha^n(x)\qquad \text{for}\quad x\in K_{\psi\varphi(\alpha)}^n,$$

and, for $n\geqslant 1$, the condition

$$(\mathrm{iii})_\alpha^n\qquad k_\alpha^n(x,s)=\begin{cases} r_{\varphi_{2n_0+1-n}(\alpha)}^{\delta\varphi_{2n_0+1-n}(\alpha)}k_\alpha^{n-1}(x,2s) & \text{for}\quad (x,s)\in K_{\psi\varphi(\alpha+1)}^{n-1}\times\langle 0,\tfrac12\rangle\,,\\ r_{\varphi_{2n_0+1-n}(\alpha)}^{\delta\varphi_{2n_0+1-n}(\alpha)}k_\alpha^{n-1}(x,1) & \text{for}\quad (x,s)\in K_{\psi\varphi(\alpha+1)}^{n-1}\times\langle\tfrac12,1\rangle\,. \end{cases}$$

The existence of $\hat{g}_\alpha^0$ and k_α^0 satisfying $(\mathrm{i})_\alpha^0$ and $(\mathrm{ii})_\alpha^0$ follows immediately by the arcwise connectedness of $Z_{\varphi(\alpha)}$.

Take $n\geqslant 1$ and assume that there exist $\hat{g}_\alpha^{n-1}$ and k_α^{n-1} satisfying $(\mathrm{i})_\alpha^{n-1}$ and $(\mathrm{ii})_\alpha^{n-1}$. Let us define

$$\bar{k}_\alpha^n\colon (K_{\psi\varphi(\alpha)}^{n-1}\cup L_{\psi\varphi(\alpha)}^n,L_{\psi\varphi(\alpha)}^n)\times I\to(Z_{\delta\varphi_{2n_0+1-n}(\alpha)},X_{\delta\varphi_{2n_0+1-n}(\alpha)})$$

by the formula

$$(3)\qquad \bar{k}_\alpha^n(x,s)\underset{\mathrm{Df}}{=}\begin{cases} k_\alpha^{n-1}(x,s) & \text{for}\quad (x,s)\in K_{\psi\varphi(\alpha)}^{n-1}\times I\,,\\ r_{\delta\varphi_{2n_0+1-n}(\alpha)}^{\varphi(\alpha)}g_{\varphi(\alpha)}(x) & \text{for}\quad (x,s)\in L_{\psi\varphi(\alpha)}^n\times I\,. \end{cases}$$

By $(\mathrm{i})_\alpha^{n-1}$, the function $\bar{k}_\alpha^n$ is continuous. By $(\mathrm{ii})_\alpha^{n-1}$, it satisfies the condition

$$(4)\qquad \bar{k}_\alpha^n(x,1)\in X_{\delta\varphi_{2n_0+1-n}(\alpha)}\qquad \text{for}\quad x\in K_{\psi\varphi(\alpha)}^{n-1}\cup L_{\psi\varphi(\alpha)}^n\,.$$

By the theorem on extension of homotopy for polyhedra (see e.g. [4], p. 14), there exists a homotopy

$$\bar{\bar{k}}{}^n_\alpha\colon (K^n_{\psi\varphi(\alpha)}, L^n_{\psi\varphi(\alpha)})\times I\to (Z_{\delta\varphi_{2n_0+1-n}(\alpha)}, X_{\delta\varphi_{2n_0+1-n}(\alpha)})$$

such that

(5) $$\bar{k}^n_\alpha\subset \bar{\bar{k}}{}^n_\alpha$$

and

(6) $$\bar{\bar{k}}{}^n_\alpha(x,0)=r^{\varphi(\alpha)}_{\delta\varphi_{2n_0+1-n}(\alpha)}g_{\varphi(\alpha)}(x)\quad\text{for}\quad x\in K^n_{\psi\varphi(\alpha)}.$$

By (3) and (5), we get

(7) $$\bar{\bar{k}}{}^n_\alpha(x,s)=\bar{\bar{k}}{}^n_\alpha(x,0)\quad\text{for}\quad (x,s)\in L^n_{\psi\varphi(\alpha)}\times I.$$

By (4) and (5), we get

(8) $$\bar{\bar{k}}{}^n_\alpha(x,1)\in X_{\delta\varphi_{2n_0+1-n}(\alpha)}\quad\text{for}\quad x\in K^{n-1}_{\psi\varphi(\alpha)}\cup L^n_{\psi\varphi(\alpha)}.$$

Thus, applying Lemma 2.3, we infer that there exists a map

$$k^n_\alpha\colon (K^n_{\psi\varphi(\alpha)}, L^n_{\psi\varphi(\alpha)})\times I\to (Z_{\varphi_{2n_0+1-n}(\alpha)}, X_{\varphi_{2n_0+1-n}(\alpha)})$$

such that

(9) $$k^n_\alpha(x,s)=r^{\delta\varphi_{2n_0+1-n}(\alpha)}_{\varphi_{2n_0+1-n}(\alpha)}\bar{\bar{k}}{}^n_\alpha(x,s)\quad\text{for}\quad (x,s)\in K^n_{\psi\varphi(\alpha)}\times(0)\cup L^n_{\psi\varphi(\alpha)}\times I,$$

(10) $$k^n_\alpha(x,s)=\begin{cases} r^{\delta\varphi_{2n_0+1-n}(\alpha)}_{\varphi_{2n_0+1-n}(\alpha)}\bar{\bar{k}}{}^n_\alpha(x,2s) & \text{for}\quad (x,s)\in K^{n-1}_{\psi\varphi(\alpha)}\times\langle 0,\tfrac12\rangle,\\ r^{\delta\varphi_{2n_0+1-n}(\alpha)}_{\varphi_{2n_0+1-n}(\alpha)}\bar{\bar{k}}{}^n_\alpha(x,1) & \text{for}\quad (x,s)\in K^{n-1}_{\psi\varphi(\alpha)}\times\langle\tfrac12,1\rangle\end{cases}$$

and

(11) $$k^n_\alpha(x,1)\in X_{\varphi_{2n_0+1-n}(\alpha)}\quad\text{for every}\quad x\in K^n_{\psi\varphi(\alpha)}.$$

The conditions (6), (7) and (9) imply (i)$^n_\alpha$. The condition (11) enables us to define $\hat{g}^n_\alpha$ by the formula

(12) $$g^n_\alpha(x)\underset{\mathrm{Df}}{=}k^n_\alpha(x,1)\quad\text{for}\quad x\in K^n_{\psi\varphi(\alpha)}.$$

By (12), we get (ii)$^n_\alpha$. By (3), (5) and (10), we get (iii)$^n_\alpha$.

B. Fix an $\alpha\in N$ and consider the map

$$u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}\colon K_{\psi\varphi(\alpha+1)}\to K_{\psi\varphi(\alpha)}.$$

Since $u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}$ is a morphism in the category $\mathcal{S}$, there exist simplicial subdivisions of $K_{\psi\varphi(\alpha+1)}$ and $K_{\psi\varphi(\alpha)}$ such that

$$u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}(K^n_{\psi\varphi(\alpha+1)})\subset K^n_{\psi\varphi(\alpha)}\quad\text{for}\quad n=0,\dots,n_0.$$

For these divisions we already constructed four sequences of maps,

$$\hat{g}^n_\alpha,\ \hat{g}^n_{\alpha+1},\ k^n_\alpha\ \text{and}\ k^n_{\alpha+1},$$

satisfying the conditions (i)$^n_\alpha$-(iii)$^n_\alpha$ and (i)$^n_{\alpha+1}$-(iii)$^n_{\alpha+1}$. Now, let us define two sequences,

$$\hat{h}^{\alpha+1,n}_\alpha\colon K^n_{\psi\varphi(\alpha+1)}\times I\to X_{\varphi_{n_0-n}(\alpha)}$$

and

$$H^{\alpha+1,n}_\alpha\colon (K^n_{\psi\varphi(\alpha+1)}\times I, L^n_{\psi\varphi(\alpha+1)}\times I)\times I\to (Z_{\varphi_{n_0-n}(\alpha)}, X_{\varphi_{n_0-n}(\alpha)})$$

for $n=0,\dots,n_0$, satisfying the following three conditions:

(iv)$^{\alpha+1,n}_\alpha$ $$H^{\alpha+1,n}_\alpha((x,0),s)=r^{\varphi_{2n_0+1-n}(\alpha)}_{\varphi_{n_0-n}(\alpha)}k^n_\alpha(u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}(x),s),$$
$$H^{\alpha+1,n}_\alpha((x,1),s)=r^{\varphi_{2n_0+1-n}(\alpha+1)}_{\varphi_{n_0-n}(\alpha)}k^n_{\alpha+1}(x,s)$$
$$\text{for } x\in K^n_{\psi\varphi(\alpha+1)},\ s\in I;$$

(v)$^{\alpha+1,n}_\alpha$ $$H^{\alpha+1,n}_\alpha((x,t),s)=r^{\varphi(\alpha)}_{\varphi_{n_0-n}(\alpha)}h^{\varphi(\alpha+1)}_{\varphi(\alpha)}(x,t)$$
$$\text{for } ((x,t),s)\in(K^n_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup(L^n_{\psi\varphi(\alpha+1)}\times I)\times I$$

and

(vi)$^{\alpha+1,n}_\alpha$ $$H^{\alpha+1,n}_\alpha((x,t),1)=\hat{h}^{\alpha+1,n}_\alpha(x,t)\quad\text{for}\quad (x,t)\in K^n_{\psi\varphi(\alpha+1)}\times I.$$

Let $n=0$. Consider the set

$$M_0\underset{\mathrm{Df}}{=}(K^0_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup[K^0_{\psi\varphi(\alpha+1)}\times\{0,1\}\cup L^0_{\psi\varphi(\alpha+1)}\times I]\times I$$

and define the map

$$\bar{H}^{\alpha+1,0}_\alpha\colon (M_0, (L^0_{\psi\varphi(\alpha+1)}\times I)\times I)\to (Z_{\delta\varphi_{n_0}(\alpha)}, X_{\delta\varphi_{n_0}(\alpha)})$$

by the formula

(13) $$\bar{H}^{\alpha+1,0}_\alpha((x,t),s)$$
$$\underset{\mathrm{Df}}{=}\begin{cases} r^{\varphi(\alpha)}_{\delta\varphi_{n_0}(\alpha)}h^{\varphi(\alpha+1)}_{\varphi(\alpha)}(x,t) & \text{for } ((x,t),s)\in(K^0_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup(L^0_{\psi\varphi(\alpha+1)}\times I)\times I,\\ r^{\varphi(\alpha)}_{\delta\varphi_{n_0}(\alpha)}k^0_\alpha(u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}(x),s) & \text{for } ((x,t),s)\in(K^0_{\psi\varphi(\alpha+1)}\times(0))\times I,\\ r^{\varphi(\alpha+1)}_{\delta\varphi_{n_0}(\alpha)}k^0_{\alpha+1}(x,s) & \text{for } ((x,t),s)\in(K^0_{\psi\varphi(\alpha+1)}\times(1))\times I.\end{cases}$$

Applying Lemma 2.4 we infer that there exists a map

$$H^{\alpha+1,0}_\alpha\colon (K^0_{\psi\varphi(\alpha+1)}\times I, L^0_{\psi\varphi(\alpha+1)}\times I)\times I\to (Z_{\varphi_{n_0}(\alpha)}, X_{\varphi_{n_0}(\alpha)})$$

such that

(14) $$H^{\alpha+1,0}_\alpha((x,0),s)=r^{\varphi_{2n_0+1}(\alpha)}_{\varphi_{n_0}(\alpha)}k^0_\alpha(u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}(x),s)\quad\text{for}\quad x\in K^0_{\psi\varphi(\alpha+1)},\ s\in I,$$

(15) $$H^{\alpha+1,0}_\alpha((x,1),s)=r^{\varphi_{2n_0+1}(\alpha)}_{\varphi_{n_0}(\alpha)}k^n_{\alpha+1}(x,s)\quad\text{for}\quad x\in K^0_{\psi\varphi(\alpha+1)},\ s\in I$$

and

$$(16)\qquad H_\alpha^{\alpha+1}\big((x,t),s\big)=r_{\varphi_{n_0}(\alpha)}^{\varphi(\alpha)}h_{\varphi(\alpha)}^{\varphi(\alpha+1)}(x,t)$$
$$\text{for}\ \big((x,t),s\big)\in(K^0_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup L^n_{\psi\varphi(\alpha+1)}\times I)\times I$$

i.e. the conditions $(\mathrm{iv})_\alpha^{\alpha+1,0}$ and $(\mathrm{v})_\alpha^{\alpha+1,0}$ are satisfied. Moreover, we have

$$(17)\qquad H_\alpha^{\alpha+1,0}\big((x,t),1\big)\in X_\alpha\quad\text{for}\quad (x,t)\in K^0_{\psi\varphi(\alpha+1)}\times I;$$

hence, the map $\hat{h}_\alpha^{\alpha+1,0}$ can be defined by the formula

$$\hat{h}_\alpha^{\alpha+1,0}(x,t)\underset{\mathrm{Df}}{=}H_\alpha^{\alpha+1,0}\big((x,t),1\big);$$

then, $(\mathrm{vi})_\alpha^{\alpha+1,0}$ is also satisfied.

Let $n\geqslant 1$. Assume that there exist $\hat{h}_\alpha^{\alpha+1,n-1}$ and $\hat{H}_\alpha^{\alpha+1,n-1}$ satisfying $(\mathrm{iv})_\alpha^{\alpha+1,n-1}$-$(\mathrm{vi})_\alpha^{\alpha+1,n-1}$. Consider the set

$$M_n\underset{\mathrm{Df}}{=}(K^n_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup[(K^{n-1}_{\psi\varphi(\alpha+1)}\cup L^n_{\psi\varphi(\alpha+1)})\times I\cup K^n\times\{0,1\}]\times I$$

and define the map

$$\overline{H}_\alpha^{\alpha+1,n}\colon\ \big(M_n,(L^n_{\psi\varphi(\alpha+1)}\times I)\times I\big)\to Z_{\delta\varphi_{n_0-n}(\alpha)},X_{\delta\varphi_{n_0-n}(\alpha)})$$

by the formula

$$(18)\qquad \overline{H}_\alpha^{\alpha+1,n}\big((x,t),s\big)$$
$$\underset{\mathrm{Df}}{=}\begin{cases} r^{\varphi(\alpha)}_{\delta\varphi_{n_0-n}(\alpha)}h^{\varphi(\alpha+1)}_{\varphi(\alpha)}(x,t) & \text{for}\ \big((x,t),s\big)\in(K^n_{\psi\varphi(\alpha+1)}\times I)\times(0)\cup\\ & \qquad\cup(L^n_{\psi\varphi(\alpha+1)}\times I)\times I,\\ H_\alpha^{\alpha+1,n-1}\big((x,t),2s\big) & \text{for}\ \big((x,t),s\big)\in(K^{n-1}_{\psi\varphi(\alpha+1)}\times I)\times\langle 0,\tfrac12\rangle,\\ H_\alpha^{\alpha+1,n-1}\big((x,t),1\big) & \text{for}\ \big((x,t),s\big)\in(K^{n-1}_{\psi\varphi(\alpha+1)}\times I)\times\langle \tfrac12,1\rangle,\\ r^{\varphi_{2n_0+1-n}(\alpha)}_{\delta\varphi_{n_0-n}(\alpha)}k^n_\alpha\big(u^{\psi\varphi(\alpha+1)}_{\psi\varphi(\alpha)}(x),s\big) & \text{for}\ \big((x,t),s\big)\in\big(K^n_{\psi\varphi(\alpha+1)}\times(0)\big)\times I,\\ r^{\varphi_{2n_0+1-n}(\alpha+1)}_{\delta\varphi_{n_0-n}(\alpha)}k^n_{\alpha+1}(x,s) & \text{for}\ \big((x,t),s\big)\in\big(K^n_{\psi\varphi(\alpha+1)}\times(1)\big)\times I.\end{cases}$$

By $(\mathrm{v})_\alpha^{\alpha+1,n-1}$, $(\mathrm{i})_\alpha^n$, $(\mathrm{i})_{\alpha+1}^n$, $(\mathrm{iii})_\alpha^n$, $(\mathrm{iii})_{\alpha+1}^n$ and $(\mathrm{iv})_\alpha^{\alpha+1,n-1}$, $\overline{H}_\alpha^{\alpha+1,n}$ is continuous. By $(\mathrm{vi})_\alpha^{\alpha+1,n-1}$ and (18) we get

$$(19)\qquad \overline{H}_\alpha^{\alpha+1,n}\big((x,t),1\big)\in X_{\delta\varphi_{n_0-n}(\alpha)}\quad\text{for}\quad\big((x,t),1\big)\in M_n.$$

Thus, by Lemma 2.4, there exists a map

$$H_\alpha^{\alpha+1,n}\colon\ (K^n_{\psi\varphi(\alpha+1)}\times I,L^n_{\psi\varphi(\alpha+1)}\times I)\times I\to(Z_{\varphi_{n_0-n}(\alpha)},X_{\varphi_{n_0-n}(\alpha)})$$

such that

$$(20)\qquad H_\alpha^{\alpha+1,n}\big((x,t),s\big)=r^{\delta\varphi_{n_0-n}(\alpha)}_{\varphi_{n_0-n}(\alpha)}\overline{H}_\alpha^{\alpha+1,n}\big((x,t),s\big)\quad\text{for}\quad\big((x,t),s\big)\in M_n$$

and

$$(21)\qquad H_\alpha^{\alpha+1,n}\big((x,t),1\big)\in X_{\varphi_{n_0-n}(\alpha)}\quad\text{for}\quad(x,t)\in K^n_{\psi\varphi(\alpha+1)}\times I.$$

By (18) and (20), we obtain $(\mathrm{iv})_\alpha^{\alpha+1,n}$ and $(\mathrm{v})_\alpha^{\alpha+1,n}$. The condition (21) enables us to define $\hat{h}_\alpha^{\alpha+1,n}$ by the formula

$$(22)\qquad \hat{h}_\alpha^{\alpha+1,n}(x,t)\underset{\mathrm{Df}}{=}H_\alpha^{\alpha+1,n}\big((x,t),1\big).$$

Thus $(\mathrm{vi})_\alpha^{\alpha+1,n}$ is also satisfied.

C. Now, let us define the desired maps

$$\hat{g}_\beta,\ k_\beta,\ \hat{h}_\alpha^{\alpha+1}\ \text{and}\ H_\alpha^{\alpha+1}$$

by the formulae:

$$(23)\qquad \hat{g}_\beta(x)\underset{\mathrm{Df}}{=}r_\beta^{\varphi_{n_0+1}(\beta)}\hat{g}_\beta^{n_0}(x)\qquad(\beta=\alpha,\alpha+1),$$

$$(24)\qquad k_\beta(x,s)\underset{\mathrm{Df}}{=}r_\beta^{\varphi_{n_0+1}(\beta)}k_\beta^{n_0}(x,s)\qquad(\beta=\alpha,\alpha+1),$$

$$(25)\qquad \hat{h}_\alpha^{\alpha+1}(x,t)\underset{\mathrm{Df}}{=}\hat{h}_\alpha^{\alpha+1,n_0}(x,t)$$

and

$$(26)\qquad H_\alpha^{\alpha+1}\big((x,t),s\big)\underset{\mathrm{Df}}{=}H_\alpha^{\alpha+1,n_0}\big((x,t),s\big).$$

The following implications hold:

$$(\mathrm{i})_\beta^{n_0}\wedge(23)\Rightarrow(\mathrm{i})_\beta;\qquad(\mathrm{ii})_\beta^{n_0}\wedge(23)\Rightarrow(\mathrm{ii})_\beta,\ \text{for}\ \beta=\alpha,\alpha+1;$$
$$(\mathrm{v})_\alpha^{\alpha+1,n_0}\wedge(26)\Rightarrow(\mathrm{v})_\alpha^{\alpha+1};\qquad(\mathrm{vi})_\alpha^{\alpha+1,n_0}\wedge(25)\wedge(26)\Rightarrow(\mathrm{vi})_\alpha^{\alpha+1};$$
$$(\mathrm{v})_\alpha^{\alpha+1,n_0}\wedge(\mathrm{vi})_\alpha^{\alpha+1}\wedge(26)\Rightarrow(\mathrm{iii})_\alpha^{\alpha+1}.$$

At last

$$(\mathrm{iv})_\alpha^{\alpha+1,n_0}\wedge(26)\wedge(24)\Rightarrow(\mathrm{iv})_\alpha^{\alpha+1},$$

and

$$(\mathrm{ii})_\alpha\wedge(\mathrm{ii})_{\alpha+1}\wedge(\mathrm{iv})_\alpha^{\alpha+1}\wedge(\mathrm{vi})_\alpha^{\alpha+1}\Rightarrow(\hat{h}_\alpha^{\alpha+1}\ \text{are connecting homotopies for}\ \hat{g}).$$

Thus the proof is complete. ∎

Now, we shall prove a kind of theorem on extension of homotopy for inverse sequences of polyhedra. It concerns with extensions of maps of inverse systems understood as follows. Given two inverse systems, $(\boldsymbol{K},\boldsymbol{L})=\big((K_\alpha,L_\alpha),u_\alpha^{\alpha'},A\big)$ and $\boldsymbol{Z}=(Z_\alpha,r_\alpha^{\alpha'},A)$, and two pseudo-maps, $(\varphi,f_\alpha)\colon\boldsymbol{L}\dashrightarrow\boldsymbol{Z}$ and $(\varphi,\bar{f}_\alpha)\colon\boldsymbol{K}\dashrightarrow\boldsymbol{Z}$, the pseudo-map $(\varphi,\bar{f}_\alpha)$ is an extension of (φ,f_α) whenever $\bar{f}_\alpha$ is an extension of f_α for every $\alpha\in A$. If, in particular,

$f = (\varphi, f_a)$ is a map, then the extension $f = (\varphi, \bar{f}_a)$ is also required to be a map [6].

2.6. PROPOSITION. *Let* $(K, L) = \big((K_a, L_a), u_a^{a'}, N\big)$ *be an inverse sequence of polyhedral pairs,* $(Z, X) = \big((Z_a, X_a), r_a^{a'}, N\big)$ *— an inverse sequence of pairs of Hausdorff spaces. Let two maps*

$$f = (1_N, f_a)\colon L \to Z \quad \text{and} \quad g = (\varphi, g_a)\colon L \to X$$

and a pseudo-map (φ, k_a) *satisfy the condition*

$$(1) \qquad k_a(x, 0) = r_a^{\varphi(a)} f_{\varphi(a)}(x), \quad k_a(x, 1) = g_a(x).$$

If the map f *has an extension* $\bar{f} = (1_N, \bar{f}_a)\colon K \to Z$, *then the map* g *has an extension* $\bar{g} = (\varphi, \bar{g}_a)\colon (K, L) \to (Z, X)$ *and the pseudo-map* (φ, k_a) *has an extension* $(\varphi, \bar{k}_a)$ *such that*

$$(\mathrm{j})_a \quad \bar{k}_a(x, 0) = r_a^{\varphi(a)} \bar{f}_{\varphi(a)}(x) \quad \text{and} \quad \bar{k}_a(x, 1) = \bar{g}_a(x) \quad \text{for} \quad x \in K_{\varphi(a)}.$$

Let, moreover, a_a^{a+1}, $\bar{a}_a^{a+1}$ *and* b_a^{a+1} *be connecting homotopies for* $f, \bar{f}$ *and* g *respectively and let* $a_a^{a+1} \subset \bar{a}_a^{a+1}$. *If there are double homotopies*

$$H_a^{a+1}\colon (L_{\varphi(a+1)} \times I) \times I \to Z_a$$

satisfying the conditions

$$(2) \quad H_a^{a+1}\big((x, 0), s\big) = k_a\big(u_{\varphi(a)}^{\varphi(a+1)}(x), s\big) \quad \text{and} \quad H_a^{a+1}\big((x, 1), s\big) = r_a^{a+1} k_{a+1}(x, s)$$

and

$$(3) \quad H_a^{a+1}\big((x, t), 0\big) = r_a^{\varphi(a)} a_{\varphi(a)}^{\varphi(a+1)}(x, t) \quad \text{and} \quad H_a^{a+1}\big((x, t), 1\big) = b_a^{a+1}(x, t),$$

then (φ, k_a) *is a map, and* b_a^{a+1} *have extensions* $\bar{b}_a^{a+1}$ *such that*

$$(\mathrm{jj})_a^{a+1} \quad \bar{b}_a^{a+1}(x, 0) = \bar{g}_a u_{\varphi(a)}^{\varphi(a+1)}(x) \quad \text{and} \quad \bar{b}_a^{a+1}(x, 1) = r_a^{a+1} \bar{g}_{a+1}(x),$$

i.e. $\bar{b}_a^{a+1}$ *are connecting homotopies for* $\bar{g}$.

Proof. Let f, g and (φ, k_a) satisfy the condition (1) and let $f \subset \bar{f}\colon K \to Z$.

Take an $a \in N$. Since $r_a^{\varphi(a)} f_{\varphi(a)} \subset r_a^{\varphi(a)} \bar{f}_{\varphi(a)}$, by theorem on extension of homotopy for polyhedra (see [4], p. 14) together with (1), it follows that k_a has an extension

$$\bar{k}_a\colon K_{\varphi(a)} \to Z_a$$

such that

$$(4) \qquad \bar{k}_a(x, 0) = r_a^{\varphi(a)} \bar{f}_{\varphi(a)}(x) \quad \text{for} \quad x \in K_{\varphi(a)}.$$

[6] This notion of extension of maps is a particular case of the notion of extension which was introduced by the author in [10].

Define the map

$$\bar{g}_a\colon (K_{\varphi(a)}, L_{\varphi(a)}) \to (Z_a, X_a)$$

by the formula

$$(5) \qquad \bar{g}_a(x) \underset{\mathrm{Df}}{=} \bar{k}_a(x, 1).$$

We get two pseudo-maps $(\varphi, \bar{g}_a)$ and $(\varphi, \bar{k}_a)$ satisfying $(\mathrm{j})_a$ for every a.

Now, given an $a \in N$, take $\bar{g}_a, \bar{g}_{a+1}, \bar{k}_a$ and $\bar{k}_{a+1}$. Let a_a^{a+1}, $\bar{a}_a^{a+1}$ and b_a^{a+1} be connecting homotopies for $f, \bar{f}$ and g, let H_a^{a+1} satisfy (2) and (3) and let $a_a^{a+1} \subset \bar{a}_a^{a+1}$. We are going to extend b_a^{a+1} and H_a^{a+1} to $\bar{b}_a^{a+1}$ and $\bar{H}_a^{a+1}$ satisfying $(\mathrm{jj})_a^{a+1}$ and the following two conditions:

$$(\mathrm{jjj})_a^{a+1} \quad \bar{H}_a^{a+1}\big((x, 0), s\big) = \bar{k}_a\big(u_{\varphi(a)}^{\varphi(a+1)}(x), s\big) \quad \text{and} \quad \bar{H}_a^{a+1}\big((x, 1), s\big) = r_a^{a+1} \bar{k}_{a+1}(x, s)$$

and

$$(\mathrm{jv})_a^{a+1} \quad \bar{H}_a^{a+1}\big((x, t), 0\big) = r_a^{\varphi(a)} \bar{a}_{\varphi(a)}^{\varphi(a+1)}(x, t) \quad \text{and} \quad \bar{H}_a^{a+1}\big((x, t), 1\big) = \bar{b}_a^{a+1}(x, t)$$

for every $x \in K_{\varphi(a+1)}$. For this purpose consider a subpolyhedron

$$P = (K_{\varphi(a+1)} \times I) \times (0) \cup (K_{\varphi(a+1)} \times \{0, 1\}) \times I \cup (L_{\varphi(a+1)} \times I) \times I$$

of the polyhedron $(K_{\varphi(a+1)} \times I) \times I$. The conditions (2), (3) and (4) enable us to define a map

$$\hat{H}_a^{a+1}\colon P \to Z_a$$

by the formula

$$(6) \quad \hat{H}_a^{a+1}\big((x, t), s\big) \underset{\mathrm{Df}}{=} \begin{cases} r_a^{\varphi(a)} \bar{a}_{\varphi(a)}^{\varphi(a+1)}(x, t) & \text{for} \quad \big((x, t), s\big) \in (K_{\varphi(a+1)} \times I) \times (0), \\ \bar{k}_a\big(u_{\varphi(a)}^{\varphi(a+1)}(x), s\big) & \text{for} \quad \big((x, t), s\big) \in \big(K_{\varphi(a+1)} \times (0)\big) \times I, \\ r_a^{a+1} \bar{k}_{a+1}(x, s) & \text{for} \quad \big((x, t), s\big) \in \big(K_{\varphi(a+1)} \times (1)\big) \times I, \\ H_a^{a+1}\big((x, t), s\big) & \text{for} \quad \big((x, t), s\big) \in (L_{\varphi(a+1)} \times I) \times I. \end{cases}$$

Since P is a retract of $(K_{\varphi(a+1)} \times I) \times I$, the map $\hat{H}_a^{a+1}$ can be extended to a map

$$\bar{H}_a^{a+1}\colon (K_{\varphi(a+1)} \times I) \times I \to Z_a.$$

Define the map

$$\bar{b}_a^{a+1}\colon (K_{\varphi(a+1)}, L_{\varphi(a+1)}) \times I \to (Z_a, X_a)$$

by the formula

$$(7) \qquad \bar{b}_a^{a+1}(x, t) \underset{\mathrm{Df}}{=} \bar{H}_a^{a+1}\big((x, t), 1\big).$$

By (5), (6) and (7) we get $(\mathrm{jj})_\alpha^{\alpha+1}$-$(\mathrm{jv})_\alpha^{\alpha+1}$. By $(\mathrm{jj})_\alpha^{\alpha+1}$, $(\varphi, \bar{g}_\alpha)$ is a map with connecting homotopies $\bar{b}_\alpha^{\alpha+1}$, by $(\mathrm{jjj})_\alpha^{\alpha+1}$, (φ, k_α) is a map; by $(\mathrm{jv})_\alpha^{\alpha+1}$, $\bar{b}_\alpha^{\alpha+1}$ is an extension of $b_\alpha^{\alpha+1}$. ■

3. The Whitehead Theorem for inverse sequences of polyhedra. We are going now to prove an analogue of the Whitehead Theorem for inverse sequences of polyhedra (3.7). Consider the category $\mathcal{T}$ as defined in § 2 and let

$$(\boldsymbol{X}, \boldsymbol{x}_0) = \big((X_\alpha, x_\alpha), p_\alpha^{\alpha'}, N\big) \quad \text{and} \quad (\boldsymbol{Y}, \boldsymbol{y}_0) = \big((Y_\alpha, y_\alpha), q_\alpha^{\alpha'}, N\big)$$

be two inverse sequences in the category $\mathcal{T}$. Take an ordinary map

$$\boldsymbol{f}^\bullet = (1_N, f_\alpha)\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$$

and let $\boldsymbol{f} = (1_N, f_\alpha)\colon \boldsymbol{X} \to \boldsymbol{Y}$. By the definition of map of inverse systems, the diagram

$$\begin{array}{ccc} X_\alpha & \xleftarrow{p_\alpha^{\alpha'}} & X_{\alpha'} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\alpha'}} \\ Y_\alpha & \xleftarrow[q_\alpha^{\alpha'}]{} & Y_{\alpha'} \end{array} \quad \text{commutes up to homotopy for every } \alpha' \geqslant \alpha .$$

Let us fix the sequence of homotopies $k_\alpha^{\alpha'}$ between $f_\alpha p_\alpha^{\alpha'}$ and $q_\alpha^{\alpha'} f_{\alpha'}$. The mapping cylinder $\boldsymbol{C}_f$ (rel. $(k_\alpha^{\alpha'})$) was defined in [11] as follows:

$$\boldsymbol{C}_f = (C_{f_\alpha}, r_\alpha^{\alpha'}, N),$$

where

$$(*) \qquad r_\alpha^{\alpha+1}[z] \underset{\mathrm{Df}}{=} \begin{cases} [p_\alpha^{\alpha+1}(x), 2t] & \text{for} \quad z = (x, t) \in X_{\alpha+1} \times \langle 0, \tfrac{1}{2} \rangle, \\ [k_\alpha^{\alpha+1}(x, 2t-1)] & \text{for} \quad z = (x, t) \in X_{\alpha+1} \times \langle \tfrac{1}{2}, 1 \rangle, \\ [q_\alpha^{\alpha+1}(y)] & \text{for} \quad z = y \in Y_{\alpha+1}. \end{cases}$$

Let $\boldsymbol{i}\colon \boldsymbol{X} \to \boldsymbol{C}_f$ and $\boldsymbol{j}\colon \boldsymbol{Y} \to \boldsymbol{C}_f$ be the inclusions (see [11]). By the statement 1.2 of [11], the map $\boldsymbol{j}$ is a homotopy equivalence; by 1.3 of [11], $\boldsymbol{i} \simeq \boldsymbol{j}\boldsymbol{f}$. Hence

3.1. *For any natural n, the morphism* $[f_n]$ *is a monomorphism (epimorphism) if and only if* $[i_n]$ *is a monomorphism (epimorphism).* ■

3.2. $\boldsymbol{f}$ *is a homotopy equivalence if and only if* $\boldsymbol{i}$ *is a homotopy equivalence.* ■

First, let us consider the particular case of $\boldsymbol{f}$ being usual. Then, all the diagrams

$$\begin{array}{ccc} X_\alpha & \xleftarrow{p_\alpha^{\alpha'}} & X_{\alpha'} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\alpha'}} \\ Y & \xleftarrow[q_\alpha^{\alpha'}]{} & Y_{\alpha'} \end{array} \quad \text{are commutative}, \quad \text{i.e. } f_\alpha p_\alpha^{\alpha'} = q_\alpha^{\alpha'} f_{\alpha'} \text{ for } \alpha' \geqslant \alpha,$$

and the bonding maps

$$r_\alpha^{\alpha'}\colon C_{f_{\alpha'}} \to C_{f_\alpha}$$

can be defined as follows (see [10]):

$$(**) \qquad r_\alpha^{\alpha'}[z] \underset{\mathrm{Df}}{=} \begin{cases} [p_\alpha^{\alpha'}(x), t] & \text{for} \quad z = (x, t) \in X_{\alpha'} \times I, \\ [q_\alpha^{\alpha'}(y)] & \text{for} \quad z = y \in Y_{\alpha'}. \end{cases}$$

The mapping cylinder $\boldsymbol{C}_f$ with $r_\alpha^{\alpha'}$ determined by $(**)$ will be called *usual* (7).

Let us define the *basic inverse system* $\boldsymbol{K}$ for the usual mapping cylinder $\boldsymbol{C}_f$:

$$\boldsymbol{K} = (K_\alpha, u_\alpha^{\alpha'}, N), \quad \text{where} \quad K_\alpha \underset{\mathrm{Df}}{=} X_\alpha \times I \cup Y_\alpha$$

and

$$(***) \qquad u_\alpha^{\alpha'}(z) \underset{\mathrm{Df}}{=} \begin{cases} (p_\alpha^{\alpha'}(x), t) & \text{for} \quad z = (x, t) \in X_{\alpha'} \times I, \\ q_\alpha^{\alpha'}(y) & \text{for} \quad z = y \in Y_{\alpha'}. \end{cases}$$

Consider the sequence of natural projections

$$e_\alpha\colon K_\alpha \to C_{f_\alpha}, \quad e_\alpha(z) \underset{\mathrm{Df}}{=} [z] \quad \text{for} \quad z \in K_\alpha, \ \alpha \in N.$$

It is evident that

3.3. *The maps* e_α *form a usual map of the basic inverse sequence* $\boldsymbol{K}$ *into the usual mapping cylinder* $\boldsymbol{C}_f$,

$$\boldsymbol{e} = (1_N, e_\alpha)\colon \boldsymbol{K} \to \boldsymbol{C}_f. \ ■$$

We shall refer to the map $\boldsymbol{e}$ as the *natural map*.

Consider a map $g_\alpha\colon K_{\alpha'} \times I \to Z_\alpha$. The map g_α is said to be *compatible with* $e_{\alpha'}$ whenever

$$g_\alpha\big((x, 1), s\big) = g_\alpha\big(f_{\alpha'}(x), s\big) \quad \text{for} \quad x \in X_{\alpha'}, \ s \in I.$$

A pseudo-map $(\varphi, g_\alpha)\colon \boldsymbol{K} \times I \dashrightarrow \boldsymbol{Z}$ is said to be *compatible with* $\boldsymbol{e}$ whenever g_α is compatible with $e_{\varphi(\alpha)}$ for every $\alpha \in N$.

A map $\boldsymbol{g} = (\varphi, g_\alpha)\colon \boldsymbol{K} \to \boldsymbol{Z}$ is said to be *compatible with* $\boldsymbol{e}$ whenever g_α is compatible with $e_{\varphi(\alpha)}$ for every α and there exist connecting homotopies $h_\alpha^{\alpha'}$ compatible with $e_{\varphi(\alpha')}$ for every $\alpha, \alpha' \in N$, $\alpha' \geqslant \alpha$.

Notice that

(7) Setting in $(*)$: $k_\alpha^{\alpha+1}(x, t) = f_\alpha p_\alpha^{\alpha+1}(x)$ for every $t \in I$, one obtains bonding maps $r_\alpha^{\alpha+1}$ different from those defined by $(**)$ (but homotopic to). However, the statements 3.1 and 3.2 are valid for the usual mapping cylinder as well (compare 4.2 of [10]).

3.4. *If a pseudo-map* (ψ, g_a): $K\times I \dashrightarrow Z$ *is compatible with* e, *then it generates the pseudo-map* $(\psi, \hat{g}_a)$: $C_f\times I \dashrightarrow Z$ *defined by the formula*

$$\hat{g}_a(z) = g_a e^{-1}_{\psi(a)}(z)\,. \blacksquare$$

3.5. *If a map* $g = (\psi, g_a)$: $K\to Z$ *is compatible with* e *then it generates the map* $\hat{g} = (\psi, \hat{g}_a)$: $C_f \to Z$ *defined by the formula*

$$\hat{g}_a(z) = g_a e^{-1}_{\psi(a)}(z)\,. \blacksquare$$

Let us prove the following

3.6. PROPOSITION. *Let* $X = (X_a, p_a^{a'}, N)$ *and* $Y = (Y_a, q_a^{a'}, N)$ *be two inverse sequences in the category* $\mathfrak{T}$ *with all* X_a *and* Y_a *being connected and all* $p_a^{a'}$ *being "onto". Let* $n_0 = \max(1+\dim X, \dim Y)$ *and let* C_f *be the usual mapping cylinder of a usual map* $f = (1_N, f_a)$: $X\to Y$. *If* $\pi_n(C_f, X)$ *is a zero object in* $\hat{\mathfrak{C}}^*$ *for* $n = 1, \ldots, n_0+1$, *then the inclusion* i: $X\to C_f$ *is a homotopy equivalence.*

Proof. Take X and Y and let $f = (1_N, f_a)$ be a usual map. Take the usual mapping cylinder $C_f = (C_f, r_a^{a'}, N)$ and its basic inverse sequence $K = (K_a, u_a^{a'}, N)$. By (**), since $p_a^{a'}$ are "onto", we get $r_a^{a'}(X_{a'}) = X_a$ for $a' \geqslant a$. Obviously

$$\dim K_a = \max(1+\dim X_a, \dim Y_a) \quad \text{for} \quad a \in N;$$

thus

(1) $$\dim K = n_0\,.$$

Consider a subpolyhedron L_a of K_a,

$$L_a \underset{\mathrm{Df}}{=} X_a\times\{0, 1\} \cup Y_a\,.$$

Since $u_a^{a+1}(L_{a+1}) \subset L_a$ for every a, we obtain an inverse sequence (K, L) in the category $\mathfrak{T}$.

We are going to prove that i is a homotopy equivalence, i.e. there exists a map

$$\hat{c} = (\psi, \hat{c}_a): C_f \to X$$

such that

(2) $$i\hat{c} \simeq 1_{C_f} \qquad \begin{array}{ccc} C_{f_a} & \xleftarrow{r_a^{\psi(a)}} & C_{f_{\psi(a)}} \\ {\scriptstyle 1_{C_{f_a}}}\downarrow & \swarrow{\scriptstyle i_a\hat{c}_a} & \\ C_{f_a} & & \end{array}$$

and

(3) $$\hat{c}i \simeq 1_X \qquad \begin{array}{ccc} X_a & \xleftarrow{p_a^{\psi(a)}} & X_{\psi(a)} \\ {\scriptstyle 1_{X_a}}\downarrow & \swarrow{\scriptstyle \hat{c}_a i_{\psi(a)}} & \\ X_a & & \end{array}$$

Thus, we shall define a pseudo-map

$$(\psi, \hat{\theta}_a): (C_f, X)\times I \dashrightarrow (C_f, X)$$

such that

(4) $$\hat{\theta}_a([z], 0) = r_a^{\psi(a)}[z] \quad\text{and}\quad \hat{\theta}_a([z], 1) = i_a\hat{c}_a[z] \quad\text{for}\quad [z] \in C_{f_{\psi(a)}}$$

and

(5) $$\hat{\theta}_a(x, 0) = p_a^{\psi(a)}(x) \quad\text{and}\quad \hat{\theta}_a(x, 1) = \hat{c}_a i_{\psi(a)}(x)\ (^8)\,.$$

By 3.4 and 3.5 it suffices to find a map

$$c = (\psi, c_a): K\to X$$

with connecting homotopies $w_a^{a'}$, and a pseudo-map

$$(\psi, \theta_a): (K, X)\times I \dashrightarrow (C_f, X)$$

such that

(6) $$\theta_a((x, 1), s) = \theta_a(f_{\psi(a)}(x), s) \quad\text{for}\quad x \in X_{\psi(a)},\ s \in I\,,$$

(7) $$w_a^{a'}((x, 1), s) = w_a^{a'}(f_{\psi(a')}(x), s) \quad\text{for}\quad x \in X_{\psi(a')},\ s \in I\,,$$

and

(8) $$\theta_a(z, 0) = r_a^{\psi(a)}[z] \quad\text{and}\quad \theta_a(z, 1) = c_a(z) \quad\text{for}\quad z \in K_{\psi(a)}\,.$$

By (1), $\pi_n(C_f, X)$ is a zero object in $\hat{\mathfrak{C}}^*$ for $n = 1, \ldots, \dim K+1$.

A. Take the natural map $e = (1_N, e_a)$: $K\to C_f$ and its restriction $e|Y$: $(Y, \emptyset)\to(C_f, X)$. By Proposition 2.5, there exist an increasing function φ: $N\to N$, a map

$$b = (\varphi, b_a): Y\to X$$

with connecting homotopies $\hat{h}_a^{a+1}$, a map

$$h = (\varphi, h_a): Y\times I \to C_f\,,$$

and double homotopies

$$H_a^{a+1}: (Y_{\varphi(a+1)}\times I)\times I \to C_{f_a}\,,$$

(8) We identify $[x, 0]$ with x for every $x \in X_a$, $a \in N$.

such that, for $y \in Y_{\varphi(a)}$, $a \in N$,

(9) $$h_a(y,0) = r_a^{\varphi(a)} e_{\varphi(a)}(y) \quad \text{and} \quad h_a(y,1) = b_a(y)\,,$$

(10) $$H_a^{a+1}\big((y,0),s\big) = h_a\big(u_{\varphi(a)}^{\varphi(a+1)}(x),s\big) \quad \text{and} \quad H_a^{a+1}\big((y,1),s\big) = r_a^{a+1} h_{a+1}(x,s)$$

and

(11) $$H_a^{a+1}\big((y,t),0\big) = r_a^{\varphi(a+1)} e_{\varphi(a+1)}(y) \quad \text{and} \quad H_a^{a+1}\big((y,t),1\big) = \hat{h}_a^{a+1}(y,t)\,.$$

These maps $\boldsymbol{b}$ and $\boldsymbol{h}$ and homotopies $\hat{h}_a^{a+1}$ and H_a^{a+1} can be respectively extended to the maps

$$\boldsymbol{b}' = (\varphi, b'_a)\colon \boldsymbol{L} \to \boldsymbol{X} \quad \text{and} \quad \boldsymbol{h}' = (\varphi, h'_a)\colon \boldsymbol{L}\times I \to \boldsymbol{C}_f$$

and homotopies

$$h_a'^{a+1}\colon L_{\varphi(a+1)} \times I \to X_a \quad \text{and} \quad H_1'^{a+1}\colon (L_{\varphi(a+1)}\times I)\times I \to C_{f_a}\,,$$

defined by the formulae

(12) $$H_a'^{a+1}\big((z,t),s\big) \underset{\mathrm{Df}}{=} \begin{cases} r_a^{\varphi(a+1)} e_{\varphi(a+1)}(x,0) & \text{for} \quad z = (x,0)\,,\\ H_a^{a+1}\big(\big(f_{\varphi(a+1)}(x),t\big),s\big) & \text{for} \quad z=(x,1)\,,\\ H_a^{a+1}\big((y,t),s\big) & \text{for} \quad z = y\,,\end{cases}$$

(13) $$h_a'^{a+1}(z,t) \underset{\mathrm{Df}}{=} H_a'^{a+1}\big((z,t),1\big)\,,$$

(14) $$h'_a(z,s) \underset{\mathrm{Df}}{=} \begin{cases} r_a^{\varphi(a)} e_{\varphi(a)}(x,0) & \text{for} \quad z=(x,0)\,,\\ h_a\big(f_{\varphi(a)}(x),s\big) & \text{for} \quad z=(x,1)\,,\\ h_a(y,s) & \text{for} \quad z=y\,,\end{cases}$$

and

(15) $$b'_a(z) \underset{\mathrm{Df}}{=} h'_a(z,1)\,.$$

It is easy to check, that $h_a'^{a+1}$ are connecting homotopies for $\boldsymbol{b}'$ and

(16) $$h'_a(z,0) = r_a^{\varphi(a)} e_{\varphi(a)}(z)\,.$$

Now, let us take in Proposition 2.6 $\boldsymbol{C}_f$ for $\boldsymbol{Z}$, $\boldsymbol{e}|\boldsymbol{L}$ for $\boldsymbol{f}$, $\boldsymbol{e}$ for $\bar{\boldsymbol{f}}$, $\boldsymbol{b}'$ for $\boldsymbol{g}$, $\boldsymbol{h}'$ for $\boldsymbol{k}$, $r_a^{a+1}e_{a+1}$ for $\bar{a}_a^{a+1}$, $h_a'^{a+1}$ for b_a^{a+1} and $H_a'^{a+1}$ for H_a^{a+1}. Since, by (11)-(16), the conditions (1), (2) and (3) of Proposition 2.6 are satisfied, it follows that $\boldsymbol{b}'$, $\boldsymbol{h}'$ and $h_a'^{a+1}$ have extensions

$$\boldsymbol{b}'' = (\varphi, b''_a)\colon (\boldsymbol{K},\boldsymbol{L}) \to (\boldsymbol{C}_f, \boldsymbol{X})\,, \qquad \boldsymbol{h}'' = (\varphi, h''_a)\colon \boldsymbol{K} \to \boldsymbol{C}_f$$

and

$$v_a^{a+1}\colon (K_{\varphi(a+1)}, L_{\varphi(a+1)})\times I \to (C_{f_a}, X_a)$$

respectively, such that

(17) $$h''_a(z,0) = r_a^{\varphi(a)} e_{\varphi(a)}(z) \quad \text{and} \quad h''_a(z,1) = b''_a(z) \quad (\text{by (j)}_a)$$

and v_a^{a+1} are connecting homotopies for $\boldsymbol{b}''$.

By (12) and (13) we get

(18) $$v_a^{a+1}\big((x,1),s\big) = v_a^{a+1}\big(f_{\varphi(a+1)}(x),s\big) \quad \text{for} \quad x \in X_{\varphi(a+1)},\ s \in I\,,$$

i.e. v_a^{a+1} is compatible with $e_{\varphi(a+1)}$.

B. By Proposition 2.5, there exist an increasing function $\eta\colon N\to N$, a map

$$\boldsymbol{c} = (\varphi\eta, c_a)\colon \boldsymbol{K} \to \boldsymbol{X}$$

with connecting homotopies w_a^{a+1}, and a map

$$\boldsymbol{k} = (\varphi\eta, k_a)\colon (\boldsymbol{K},\boldsymbol{L})\times I \to (\boldsymbol{C}_f, \boldsymbol{X})$$

such that

(19) $$k_a(z,s) = r_a^{\eta(a)} b''_{\eta(a)}(z) \quad \text{for} \quad (z,s) \in K_{\varphi\eta(a)}\times(0) \cup L_{\varphi\eta(a)}\times I,\ a \in N\,,$$

(20) $$k_a(z,1) = c_a(z) \quad \text{for} \quad z \in K_{\varphi\eta(a)},\ a \in N$$

and

(21) $$w_a^{a+1}(z,s) = r_a^{\eta(a)} v_{\eta(a)}^{\eta(a+1)}(x,s) \quad \text{for} \quad (x,s) \in L_{\varphi\eta(a+1)}\times I\,.$$

By (18) and (21), we get

(22) $$w_a^{a+1}\big((x,1),s\big) = w_a^{a+1}\big(f_{\varphi\eta(a+1)}(x),s\big) \quad \text{for} \quad (x,s) \in X_{\varphi\eta(a+1)}\times I\,,$$

i.e. w_a^{a+1} is compatible with $e_{\varphi\eta(a+1)}$.

Now, let us put

$$\psi \underset{\mathrm{Df}}{=} \varphi\eta$$

and let the pseudo-map

$$(\varphi, \theta_a)\colon (\boldsymbol{K},\boldsymbol{X})\times I \dashrightarrow (\boldsymbol{C}_f, \boldsymbol{X})$$

be defined by the formula

(23) $$\theta_a(z,s) \underset{\mathrm{Df}}{=} \begin{cases} r_a^{\eta(a)} h''_{\eta(a)}(z,2s) & \text{for} \quad s \in \langle 0,\tfrac{1}{2}\rangle\,,\\ k_a(z,2s-1) & \text{for} \quad s \in \langle \tfrac{1}{2},1\rangle\,.\end{cases}$$

If $s=\frac{1}{2}$, then, by (17) and (19),

$$r_a^{\eta(a)}h''_{\eta(a)}(z,2s)=r_a^{\eta(a)}b''_{\eta(a)}(z)=k_a(z,0)=k_a(z,2s-1)\,,$$

thus θ_a is continuous.

By (23), (19) and (14), we get

$$\theta_a\big((x,1),s\big)=\left\{\begin{array}{ll} r_a^{\eta(a)}h_{\eta(a)}\big(f_{\varphi\eta(a)}(x),2s\big) & \text{for } s\in\langle 0,\frac{1}{2}\rangle\\ r_a^{\eta(a)}b'_{\eta(a)}(x,1) & \text{for } s\in\langle \frac{1}{2},1\rangle\end{array}\right\}=\theta_a\big(f_{\varphi(a)}(x),s\big)\,,$$

i.e. the condition (6) is satisfied.

By (22), the condition (7) is satisfied.

At last, by (23) together with (17) we get

$$\theta_a(z,0)=r_a^{\varphi(a)}[z]\,,$$

and by (23) together with (20) we get

$$\theta_a(z,1)=c_a(z)\,,$$

thus the condition (8) is also satisfied. ■

As a consequence of 3.1, 1.3, 3.6 and 3.2 we obtain the following

3.7. THEOREM. *Let* (X,x_0) *and* (Y,y_0) *be two inverse sequences in the category* $\mathfrak{S}$, *with all the spaces being connected and all the bonding maps being "onto". Let* $n_0=\max(1+\dim X,\dim Y)$. *Let* $f_n\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$ *be the morphism in* $\mathfrak{G}^*$ *induced by a usual ordinary map* $f^{\bullet}\colon (X,x_0)\to(Y,y_0)$. *If* $[f_n]$ *is a bimorphism in* $\hat{\mathfrak{C}}^*$ *for* $n\leqslant n_0$ *and is an epimorphism in* $\hat{\mathfrak{G}}^*$ *for* $n=n_0+1$, *then* $f\colon X\to Y$ *is a homotopy equivalence.* ■

THEOREM 3.7 is an analogue of the Whitehead Theorem for inverse sequences of polyhedra in the particular case of f being a usual map.

Now, we pass to the general case of an arbitrary map f. Let us prove

3.8. THEOREM. *Let* (X,x_0) *and* (Y,y_0) *be two inverse sequences of finite dimension in the category* $\mathfrak{S}$, *with all the spaces being connected and all the bonding maps being "onto". Let* $n_0=\max(1+\dim X,\dim Y)$. *Let* $f_n\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$ *be the morphism in* $\mathfrak{G}^*$ *induced by a map* $f^{\bullet}=(1_N,f_a)\colon (X,x_0)\to(Y,y_0)$. *If* $[f_n]$ *is a bimorphism in* $\hat{\mathfrak{C}}^*$ *for* $n\leqslant n_0$ *and is an epimorphism in* $\hat{\mathfrak{G}}^*$ *for* $n=n_0+1$, *then* $f\colon X=Y$ *is a homotopy equivalence.*

Proof. Take $f^{\bullet}=(1_N,f_a)\colon (X,x_0)\to(Y,y_0)$. Since all the diagrams

$$\begin{array}{ccc} X_a & \xleftarrow{\;p_a^{a'}\;} & X_{a'}\\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_{a'}}\\ Y_a & \xleftarrow[\;q_a^{a'}\;]{} & Y_{a'}\end{array}$$

are assumed commutative up to homotopy,

hence all the maps f_a can be replaced by simplicial mappings and then C_{f_a} are polyhedra for all $a\in N$ (see [17]). By (*), all the maps $r_a^{a'}$ can be made simplicial. Thus C_f is an inverse sequence in $\mathfrak{S}$. As it was noticed in [11], the map i is usual. At last,

$$\max(1+\dim X,\dim C_f)=\max\big(1+\dim X,\max(1+\dim X,\dim Y)\big)$$
$$=\max(1+\dim X,\dim Y)=n_0\,.$$

Thus, Theorem 3.7 can be applied to the map $i\colon X\to C_f$; we get the implication

(1) if $[i_n]$ is a bimorphism in $\hat{\mathfrak{C}}^*$ for $n\leqslant n_0$ and is an epimorphism in $\hat{\mathfrak{G}}^*$ for $n=n_0+1$, then i is a homotopy equivalence.

The statements 3.1 and 3.2 together with (1) complete the proof. ■

4. The Whitehead Theorem in the shape theory. The notion of fundamental dimension introduced by K. Borsuk and W. Holsztyński in [3] for compacta, can be analogically defined for pointed compacta:

$$\mathrm{Fd}(X,x_0)\underset{\mathrm{Df}}{=}\min\{\dim Y\colon \bigvee_{y_0\in Y}\mathrm{Sh}(Y,y_0)\geqslant\mathrm{Sh}(X,x_0)\}\,.$$

As proved by W. Holsztyński (non published), for any compactum X

$$\mathrm{Fd}\,X=\min\{\dim Y\colon \mathrm{Sh}\,Y=\mathrm{Sh}\,X\}\,.$$

By a slight modification of his proof one obtains the following statement

4.1. *For any pointed compactum* (X,x_0),

$$\mathrm{Fd}(X,x_0)=\min\{\dim Y\colon \bigvee_{y_0\in Y}\mathrm{Sh}(Y,y_0)=\mathrm{Sh}(X,x_0)\}\,.$$

Let us prove a statement concerning the notion of shape map (see [8]).

4.2. *Let* (X,x_0) *and* (Y,y_0) *be two ANR-systems over the same closure finite directed set* $(A,\leqslant)$. *For every cofinal map* $f=(\varphi,f_a)\colon (X,x_0)\to(Y,y_0)$ *there exist an ANR-system* $(\tilde{X},\tilde{x}_0)$ *over* $(A,\leqslant)$ *and a map* $\tilde{f}=(1_A,\tilde{f}_a)\colon (\tilde{X},\tilde{x}_0)\to(Y,y_0)$ *such that* $\tilde{X}$ *is cofinal to* X *and both* f *and* $\tilde{f}$ *represent the same shape map.*

Proof. Let $(X,x_0)=\big((X_a,x_a),p_a^{a'},A\big)$, $(Y,y_0)=\big((Y_a,y_a),q_a^{a'},A\big)$ and $f=(\varphi,f_a)$. Define $(\tilde{X},\tilde{x}_0)$ and $\tilde{f}$ as follows:

$$(\tilde{X},\tilde{x}_0)=\big((\tilde{X}_a,\tilde{x}_a),\tilde{p}_a^{a'},A\big)\,,\quad\text{where}\quad \tilde{X}_a=X_{\varphi(a)},\ \tilde{x}_a=x_{\varphi(a)},\ \tilde{p}_a^{a'}=p_{\varphi(a)}^{\varphi(a')}$$

and

$$\tilde{f}=(1_A,\tilde{f}_a)\,,\quad\text{where}\quad \tilde{f}_a=f_a\,.$$

Let $(X, x_0) = \varprojlim(\boldsymbol{X}, \boldsymbol{x}_0)$. By the assumption f is cofinal, i.e. $\varphi(A)$ is cofinal with A; so $(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{x}}_0)$ is cofinal with $(\boldsymbol{X}, \boldsymbol{x}_0)$. Thus

$$\varprojlim(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{x}}_0) = (X, x_0) = \varprojlim(\boldsymbol{X}, \boldsymbol{x}_0) .$$

Let $p_a\colon (X, x_0) \to (X_a, x_a)$, $\tilde{p}_a\colon (X, x_0) \to (\tilde{X}_a, \tilde{x}_a)$ be the projections. Let us define the map $\boldsymbol{e}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{x}}_0)$ by the formulae

$$\boldsymbol{e} = (\varphi, e_a)\,, \qquad e_a = 1_{X_{\varphi(a)}}\colon X_{\varphi(a)} \to \tilde{X}_a\,,$$

and notice that $\boldsymbol{e}$ is associated with $\mathbf{1}_X$. Indeed, $e_a p_{\varphi(a)} = \tilde{p}_a$, so $e_a p_{\varphi(a)} \simeq \tilde{p}_a$ for $a \in A$. The only to be proved more is that the diagram

$$\begin{array}{ccc} (\boldsymbol{X}, \boldsymbol{x}_0) & \xrightarrow{\ \boldsymbol{f}\ } & (\boldsymbol{Y}, \boldsymbol{y}_0) \\ {\scriptstyle \boldsymbol{e}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_Y} \\ (\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{x}}_0) & \xrightarrow[\ \tilde{\boldsymbol{f}}\ ]{} & (\boldsymbol{Y}, \boldsymbol{y}_0) \end{array} \quad \text{commutes up to homotopy .}$$

We have

$$\tilde{\boldsymbol{f}}\boldsymbol{e} = (\varphi, \tilde{f}_a e_a) = (\varphi, f_a) = \boldsymbol{f} = \mathbf{1}_Y \boldsymbol{f}\,. \ ■$$

Now, let us establish

4.3. MAIN THEOREM. *Let (X, x_0) and (Y, y_0) be two movable pointed continua of finite fundamental dimension and let $n_0 = \max\big(1 + \mathrm{Fd}(X, x_0), \mathrm{Fd}(Y, y_0)\big)$. Let $\underline{f}_n\colon \underline{\pi}_n(X, x)_0 \to \underline{\pi}_n(Y, y_0)$ be the homomorphism of n-th fundamental groups induced by a fundamental sequence $\underline{f}^{\bullet} = \{f^k, (X, x_0), (Y, y_0)\}$. If $\underline{f}_n$ is an isomorphism for $n \leqslant n_0$ and is an epimorphism for $n = n_0 + 1$, then $\underline{f} = \{f^k, X, Y\}$, is a fundamental equivalence.*

Proof. By the assumption, $\mathrm{Fd}(X, x_0) \leqslant n_0 - 1$ and $\mathrm{Fd}(Y, y_0) \leqslant n_0$. Since each of two pointed spaces (X, x_0) and (Y, y_0) can be replaced by any pointed compactum of the same shape, Theorem 4.1 enables us to assume that

$$\dim X \leqslant n_0 - 1 \quad \text{and} \quad \dim Y \leqslant n_0\,. \tag{1}$$

By the Freudental Theorem (see [14], p. 158), it follows that there exist two inverse sequences

$$(\boldsymbol{X}, \boldsymbol{x}_0) = \big((X_a, x_a), p_a^{a'}, N\big) \quad \text{and} \quad (\boldsymbol{Y}, \boldsymbol{y}_0) = \big((Y_a, y_a), q_a^{a'}, N\big)$$

associated with (X, x_0) and (Y, y_0), such that

(2) X_a and Y_a are polyhedra with $\dim X_a \leqslant n_0 - 1$ and $\dim Y_a \leqslant n_0$,

(3) $p_a^{a'}$ and $q_a^{a'}$ are "onto" for $a, a' \in A$, $a' \geqslant a$.

Since X and Y are both connected, all the spaces X_a and Y_a can be assumed connected. Take any map $\boldsymbol{f}^{\bullet}\colon (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ related to $\underline{f}^{\bullet}$.

By 4.2, one can assume that

$$\boldsymbol{f}^{\bullet} = (1_N, f_a)\,. \tag{4}$$

By the assumption, $\underline{f}_n$ is an isomorphism in $\mathcal{G}$ (so all the more in $\mathcal{C}$) for $n \leqslant n_0$ and is an epimorphism in $\mathcal{G}$ for $n = n_0 + 1$. Thus, by 1.1 of Section 1, applying the statement 6.5 of [13], we infer that

(5) $\varprojlim f_n$ is a bimorphism in $\mathcal{C}$ for $n \leqslant n_0$ and is an epimorphism in $\mathcal{G}$ for $n = n_0 + 1$.

By Remark 6.7 of [12], since (X, x_0) and (Y, y_0) are movable, they are uniformly movable. Thus, by Corollary 6.6 of [12] it follows that

(6) $[f_n]$ is a bimorphism in $\hat{\mathcal{C}}^*$ for $n \leqslant n_0$ and is an epimorphism in $\hat{\mathcal{G}}^*$ for $n = n_0 + 1$.

Thus, by Theorem 3.8, $\boldsymbol{f}$ is a homotopy equivalence. Hence, by the results of [6], $\underline{f}$ is a fundamental equivalence. ■

5. Remarks. 1. The hypothesis of Theorem 4.3 refers to a pointed fundamental sequence, though the result is obtained for non pointed fundamental sequence. However that theorem can be formulated without use of pointed spaces. For this purpose, consider an inverse system of arcwise connected spaces, $\boldsymbol{X} = (X_a, p_a^{a'}, A)$. Since for every natural n the nth homotopy group of X_a does not depend on the choice of a basic point, one can define the functor $\pi_n\colon \mathcal{T} \to \mathcal{G}$ from the category of arcwise connected spaces with continuous maps as morphisms into the category of groups (or, more precisely, into the quotient category of isomorphic types of groups). This functor generates the functor $\pi_n\colon \mathcal{T}^* \to \mathcal{G}^*$ defined by the formulae

$$\pi_n(\boldsymbol{X}) = \big(\pi_n(X_a), \pi_n(p_a^{a'}), A\big) \quad \text{for} \quad \boldsymbol{X} = (X_a, p_a^{a'}, A)$$

and

$$\pi_n(\boldsymbol{f}) = \big(\varphi, \pi_n(f_a)\big) \quad \text{for} \quad \boldsymbol{f} = (\varphi, f_a)\colon \boldsymbol{X} \to \boldsymbol{Y} = (Y_\beta, q_\beta^{\beta'}, B)\,.$$

In a similar way we define the homotopy systems of $(\boldsymbol{Z}, \boldsymbol{X}) = \big((Z_a, X_a), r_a^{a'}, A\big)$, for arcwise connected Z_a, X_a and consequently we get the functor of relative homotopy systems. It enables us to replace in Theorems 3.7 and 3.8 the pointed spaces and pointed maps by non pointed ones. Then Theorem 3.8 admits the following form:

5.1. THEOREM. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two inverse sequences of finite dimension in the category $\mathcal{T}$, with all the spaces being connected and all the bonding* *maps being* "onto". *Let* $n_0 = \max(1+\dim X, \dim Y)$. *Let* $f_n\colon \pi_n(X) \to \pi_n(Y)$ *be the morphism in* $\mathcal{G}^*$ *induced by a map* $f = (1_N, f_a)\colon X \to Y$. *If* $[f_n]$ *is a bimorphism in* $\hat{\mathcal{C}}^*$ *for* $n \leqslant n_0$ *and is an epimorphism in* $\hat{\mathcal{G}}^*$ *for* $n = n_0+1$, *then* f *is a homotopy equivalence.* ■

Consequently, the main result, Theorem 4.5, can be reformulated as follows:

5.2. THEOREM. *Let* X *and* Y *be two movable continua of finite fundamental dimension and let* $n_0 = \max(1+\mathrm{Fd}\,X, \mathrm{Fd}\,Y)$. *Let* $[\![f]\!]_n^*\colon \pi_n^*(X) \to \pi_n^*(Y)$ *be the homomorphism of* n*-th limit homotopy groups induced by the shape map* $[\![f]\!]\colon X \to Y$. *If* $[\![f]\!]_n^*$ *is an isomorphism for* $n \leqslant n_0$ *and is an epimorphism for* $n = n_0+1$, *then* f *is a shape equivalence.* ■

2. The assumption of movability in the statement 5.2 (as well as in 4.5) is essential. Indeed, let X be a solenoid and let $Y = \{y_0\}$. Then $[\![f]\!]$ is a constant map, which fails to be a shape equivalence though all $[\![f]\!]_n$ are isomorphisms.

3. Theorems 5.1 and 5.2 solve one of the problems raised recently by D. A. Edwards and R. Geoghegan ([18], Question 1). On the other hand, their Example 2 shows that the assumption of finite fundamental dimension in Theorems 5.1 and 5.2 (as well as in 3.8 and 4.5) is essential.

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] — *On movable compacta*, Fund. Math. 66 (1969), pp. 137–146.

[3] — and W. Holsztyński, *Hopf classification theorem in the shape theory*, Bull. Acad. Polon. Sci. 19 (1971), pp. 387–390.

[4] Sze-Tsen-Hu, *Homotopy Theory*, 1959.

[5] S. Mardešić and J. Segal, *Shapes of compacta and* ANR*-systems*, Fund. Math. 72 (1971), pp. 41–59.

[6] — — *Equivalence of the Borsuk and the* ANR*-system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.

[7] — — *Movable compacta and* ANR*-systems*, Bull. Acad. Pol. Sci. 18 (1970), pp. 649–654.

[8] — *Retracts in shape theory*, Glasnik Mat. 6 (26) (1971), p. 153–163.

[9] B. Mitchell, *Theory of Categories*, 1965.

[10] M. Moszyńska, *On shape and fundamental deformation retracts I*, Fund. Math. 75 (1972), pp. 145–167.

[11] — *On shape and fundamental deformation retracts II*, Fund. Math. 77 (1973), pp. 235–240.

[12] — *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.

[13] — *Various approaches to the fundamental groups*, Fund. Math. 78 (1973), pp. 107–118.

[14] K. Nagami, *Dimension Theory*, 1970.

[15] J. H. C. Whitehead, *On the homotopy type of* ANR's, BAMS 54 (1948), pp. 1125–1132.

[16] — *Combinatorial Homotopy I*, BAMS 55 (1949), pp. 213–245.

[17] — *Simplicial spaces, nuclei and m-groups*, Proc. Lond. Math. Soc. 45 (1939), p. 243–327.

[18] D. A. Edwards and R. Geoghegan, *Compacta weak shape equivalent to* ANR's, to appear in Fund. Math.

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