Strongly cellular subsets of $E^3$

by

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Abstract. This paper investigates properties of strongly cellular subsets of $E^3$ and $E^2$. A subset $Z$ of $E^3$ is strongly cellular if there is an $n$-cell $C$ in $E^3$ and a homeomorphism $H$ of $C$ in $C$ such that $H_a = 1$, $H_1 = 1$ for all $t$, $H((Bd C) \times [0, 1])$ is a homeomorphism into $E^3 \setminus Z$, and $H_i$ is $Z$. The main results are the following.

Theorem. A subset of $E^3$ is strongly cellular if and only if it is a locally connected continuum not separating $E^3$.

Theorem. A one-dimensional subset of $E^3$ is strongly cellular if and only if it is a tame dendrite.

1. Introduction. Bing and Kirko [4] introduced the notion of strong cellularity and proved that a strongly cellular arc in $E^3$ is tame. This paper generalizes this result to the following satisfying theorem, which not only provides a characterization of tame dendrites but also a new characterization of tame trees.

Theorem 1. A one dimensional subset of $E^3$ is strongly cellular if and only if it is a tame dendrite.

There are a number of results and questions involving strong cellularity. Bing and Kirko’s paper [4] has already been mentioned. Griffith and Howell [5] have shown that strongly cellular two-cells and three-cells in $E^3$ are tame. We use a number of results and techniques from both these papers. Bean has shown in [3] that every cactoid has an embedding $C$ in $E^3$ so that $C$ is strongly cellular, and in [2] asked whether a monotone decomposition of $E^3$ into points and countably many strongly cellular sets has $E^3$ as decomposition space. Our theorem shows a relationship between this question and that of Armentrout [1, Question 3] asking the same question for tame dendrites.

2. Preliminaries and statement of results. We first recall the original definition of Bing and Kirko [4] as modified in [5]. I denotes the real interval $[0, 1]$. A homotopy of $S$ in $T$ is a continuous function $H$ from $S \times I$ into $T$, and $H_t$ denotes the function given by $H_t(s) = H(s, t)$. Also, if $C$ is a cell, then $Bd C$, $Int C$ denote its combinatorial boundary and interior respectively.
3. Proof of Theorem 3. Any strongly cellular set is the continuous image of a cell, and so is a locally connected continuum. The complement of a strongly cellular set is the increasing union of complements of cells and so is connected. This proves one-half of Theorem 3.

Let M be a locally connected continuum in \( E^2 \) with connected complement. We show that M is strongly cellular by constructing a coocooming map H for M and applying Theorem 2. We use freely results concerning the topology of the plane and of locally connected continua, all of which can be found in [7] or [8].

Let B be the point-set boundary of M. We know that B is a regular curve. For each point x of B there is an open neighborhood \( U_x \) of x in B such that \( U_x \) is connected, \( \text{diam} U_x \) is less than \( \frac{1}{2} \), and the boundary of \( U_x \) in B is finite. Let \( U_1, U_2, \ldots, U_n \) be a collection of such \( U_x \)'s covering B, and let \( K_i \) be the closure of \( U_i \). B \( \supseteq U_i \) has only finitely many components, each lying except for a single point in the unbounded component of \( E^2 - K_i \). Hence there is a closed connected set \( L_i \) such that \( B \cap K_i \cap L_i \) and \( K_i \cap L_i \) is finite.

An application of the Plane Separation Theorem [8, page 108] yields simple closed curves \( C_1, C_2, \ldots, C_n \), bounding disks \( D_1, D_2, \ldots, D_n \) such that \( C_i \subset B(K_i, 1) \), \( \text{diam} D_i < \frac{1}{2} \), and \( B \cap D_i = K_i \). (For any set A and positive number \( \epsilon \) we let \( B(A, \epsilon) \) denote the set of points with \( \epsilon \) of a point of A.) Also, if J is a simple closed curve in \( E^2 \), then \( \text{Int} J \) denotes the bounded complementary domain of J. By adjusting the \( C_i \)'s if necessary, we may assume that each component of \( B \cap D_i \) contains a point of B, that each \( C_i \) is locally polygonal off B, and that the \( C_i \)'s are in general position off B, i.e., at each point of intersection of two \( C_i \)'s not on B the \( C_i \)'s look locally like the letter \( "x" \).

Let \( J_1 \) be the boundary of the unbounded component of \( E^2 - \bigcup C_i \). \( J_1 \) is a polygonal simple closed curve whose interior contains M and which lies in \( B(M, 1) \). By starting with sufficiently small neighborhoods \( U_x \) and following the above procedure, we can obtain continua \( K_{21}, K_{22}, \ldots, K_{2n} \) such that \( \text{diam} K_{2i} < \frac{1}{4} \), each \( K_{2i} \) is contained in some \( K_{1i} \), the interiors of the \( K_{2i} \)'s in B cover B, and there are continua \( L_{21}, L_{22}, \ldots, L_{2n} \) such that \( B \cap K_{2i} \cap L_{2i} \) and \( K_{2i} \cap L_{2i} \) is finite.

An application of the Plane Separation Theorem and adjustment of the resulting curves now yield simple closed curves \( C_{21}, C_{22}, \ldots, C_{2n} \), bounding disks \( D_{21}, D_{22}, \ldots, D_{2n} \) such that

1. \( \text{diam} D_{2i} < \frac{1}{4} \),
2. \( D_{2i} \cap B(K_{2i}, \frac{1}{4}) \cap \text{Int} C_{1i} \) whenever \( K_{2i} \subset K_{1i} \),
3. \( B \cap D_{2i} = K_{2i} \).
(4) each component of \( D_{j_1} \cap D_{i_1} \) contains a point of \( B_i \).
(5) each component of \( D_{j_1} \cap D_{i_1} \) contains a point of \( B_i \).
(6) each \( C_{i_1} \) is locally polygonal off \( B_i \) and the \( C_{i_1} \)'s and \( C_{i_1} \)'s are in general position off \( B \).

Let \( J_i \) be the boundary of the unbounded component of \( \mathbb{E}^2 \setminus \bigcup_{i=1}^n C_{i_1} \). \( J_i \) is a polygonal simple closed curve such that \( M \subseteq \text{Int } J_i \), \( J_i \subseteq \text{Int } J_i \), and \( J_i \subseteq B(M, \epsilon) \). Let \( J \) be the annulus \( (J_1 \cup \text{Int } J_1) \setminus \text{Int } J_1 \). We show that there is a homeomorphism \( h \) from \( J \times \mathbb{I} \) onto \( A \) such that \( h(z, 0) = z \) and \( h\left( (J \cap C_{i_1} ) \times \mathbb{I} \right) \subseteq D_{i_1} \) for \( i = 1, 2, \ldots, \).

Consider \( D_{j_1} \cap (J_1 \cup \text{Int } J_1) \). By (5) each component of this set contains points of \( B_i \) and hence \( D_{j_1} \cap B_i \) is connected, there is only one component. By the general position requirements \( D_{j_1} \cap J_1 \) consists of either (I) one arc \( ab \) spanning \( D_{j_1} \) or (II) two such arcs, say \( ay \) and \( ew \). Also, \( B \cap D_{j_1} \) lies in the disk bounded by \( (I) \) \( ab \) and one component of \( C_{i_1} \), or (II) \( ay \), \( ev \), and two components of \( C_{i_1} \), \( y, y, \pm \).

Since \( J_1 \subseteq \text{Int } C_{i_1} \), the \( C_{i_1} \) are in general position off \( B \), there is a homeomorphism \( h_{j_1} \) of \( C_{i_1} \cap J_1 \) into \( J_1 \) such that if \( z \) is a point of \( C_{i_1} \cap J_1 \cap C_{i_1} \), then \( h_{j_1}(z) \times \mathbb{I} \) lies in \( D_{j_1} \). The desired homeomorphism \( h \) is obtained by pasting together such \( h_{j_1}, j = 1, 2, \ldots \).

The entire procedure given above can clearly be iterated, yielding a sequence of simple closed curves \( J_1, J_2, \ldots \) such that for \( i = 1, 2, \ldots \)

\[
M \subseteq \text{Int } J_{i+1} \setminus \text{Int } J_{i+1} \subseteq \text{Int } J_{i+1} \setminus B(M, i, 1) + 1.
\]

Further, if we let \( A_i \) be the annulus bounded by \( J_{i+1} \cup J_{i+1} \), then the curves \( J_1 \) may be so constructed that there is a homeomorphism \( h_1 \) from \( J_{i+1} \times \mathbb{I} \) onto \( A_i \) such that \( \text{diam } h_1(\{x\} \times \mathbb{I}) \) is less than \( 1/2^i \) and \( h_1(x, 0) = x \) for each \( x \) in \( J_1 \) and \( i = 1, 2, \ldots \).

The sequence \( h_1, h_{j_1}, h_{j_1}^2, \ldots \) converges to a continuous function \( g \). It is now easy to show that the function \( h \) from \( J_{i+1} \times \mathbb{I} \) into \( \mathbb{E}^2 \) defined by

\[
H(x, t) = \left[ h^m h^m_{j_1} \circ \ldots \circ h^m_{j_1} \circ m \times (m+1) t - m^2 - 1, t \right. \\
\left. m - 1 \right] m \times (m+1) + 1, \quad m = 1, 2, \ldots , \quad t = 1, 2, \ldots, \quad g(x),
\]

is a coocooning map for \( M \). This concludes the proof of Theorem 3.

4. Proof of Theorem 4. Let \( W \) be a one-dimensional subset of \( \mathbb{E}^2 \) with coocooning map \( h \) defined on \( S \times I \). We first note that \( W \) is a dendrite if and only if \( h \) is a monotone map. For if \( h \) is monotone, then \( W \) is the monotone image of a 2-sphere, hence a cactoid [6], and a one-dimensional cactoid is a dendrite. Conversely, if \( W \) is a dendrite, the \( W - \{x\} \) is simply connected for each point \( x \) of \( W \) and so by Proposition 1 \( h^{-1}(x) \) is connected.

The rest of the proof is by contradiction. Suppose \( x \) is a point of \( W \) such that \( h^{-1}(x) \) is not connected. There is a simple closed curve \( J \) meeting \( h^{-1}(x) \) such that each complementary domain of \( J \) meets \( h^{-1}(x) \). Now \( h(S - J) \) is a neighborhood \( U \) of \( x \) in \( W \). Since \( W \) is one-dimensional, there is a compact 0-dimensional set \( K \) contained in \( U \) which separates \( x \) from \( h(J) \) in \( W \). Then \( h^{-1}(K) \) separates \( h^{-1}(x) \) from \( J \) on \( S \).

Let \( D \) be either of the disks bounded by \( J \) on \( S \) and let \( A \) be a component of \( h^{-1}(x) \) in \( D \). Some component \( C \) of \( h^{-1}(x) \) separates \( A \) from \( J \) (T. page 123). Since \( K \) is 0-dimensional, \( h(C) \) is a single point. Let \( J_1, J_{i+1}, \ldots \) be simple closed curves in \( \text{Int } D \) such that \( C \cap J_{i+1} \subseteq \text{Int } D \) bounded by \( J_i \) for \( i = 1, 2, \ldots \). Define a function \( g \) from \( J \times \mathbb{I} \) into \( \mathbb{E}^2 \) by

\[
g(x, t) = \begin{cases} h(f(x, t)), & t < 1, \\ h(C), & t = 1. \end{cases}
\]

It is easy to see that \( g \) is continuous and hence a homotopy shrinking \( J \) to a point.

Now let \( y \) be a point of \( A \) and \( y_1 \) a point of \( h^{-1}(x) \) in \( S \). Let \( J' \) be a simple closed curve formed by \( y \) and \( y_1 \) together with the arc \( h(y, x) \times \mathbb{I} \) and \( h(y_1, x) \times \mathbb{I} \). Then \( J \) and \( J' \) are linked, but \( g \) shrinks \( J \) to a point in the complement of \( J' \). (See [5, Lemma 4.1] for a similar argument.) This contradiction establishes Theorem 4.

5. Proof of Theorem 5. The proof of this theorem follows [4] so closely that we omit details and give only a general outline. Let \( K \) and \( K' \) be any two strongly cellular homeomorphic dendrites in \( \mathbb{E}^2 \). In particular, by Theorem 3 \( K \) can be planar. Let \( h \) be a homeomorphism from \( K \) onto \( K' \), and assume that \( K \cup K' C = \{x \in \mathbb{E}^2 : |x| < 1\} \) and that \( H', H'' \) are homotopies of \( C \times I \) into \( C \) as in the definition of strongly cellular for \( K, K' \), respectively. Let \( S_1, S_2 \) be \( H'(C_1, C_2) \), \( H''(B_1, B_2) \) respectively, and \( C_1, C_2 \) be \( H(I, C) \), \( H''(I, C) \) respectively.

Let \( p_1, p_2, \ldots \) be a dense set of points of order 2 in \( K \) such that for each positive \( \epsilon \) there is an integer \( N \) for which each component of \( K - \{p_1, p_2, ..., p_N\} \) has diameter less than \( \epsilon \). Let \( p_i = h(y_i) \). We now construct topological disks \( D_1, D_2, \ldots \) and \( D'_1, D'_2, \ldots \) in \( C \) satisfying the following conditions.
(1) $D_t \cap S_t, D'_t \cap S'_t$ are simple closed curves for $t < 1$ and $p_t, p'_t$ respectively for $t = 1$.

(2) Each component of $C \setminus D_t$, $C \setminus D'_t$ contains exactly one component of $K \setminus p_t$, $K \setminus p'_t$ respectively.

(3) $D_t, D'_t$ are locally tame except possibly at $p_t, p'_t$ respectively.

(4) For $i \neq j$ there is a number $t$ such that $0 < t < 1$ and $C_i \cap D_t = O = C_j \cap D'_t$.

(5) If $i, j,$ and $t$ are as in (4), $t < s < 1$ and $W_s, W'_s$ are the closures of $C_s \setminus C_t$, $C'_s \setminus C'_t$ respectively, then the closures of the components of $W_s \setminus (D_t \cup D'_t)$ and $W'_s \setminus (D'_t \cup D_t)$ are two tame 3-cells and a tame solid torus.

Let $t_1, t_2, \ldots$ be a monotone increasing sequence of positive numbers with limit 1 such that for $m \neq n$ and $m, n \leq i + 1$,

$$C_i \cap D_m \cap D_n = O = C'_i \cap D'_m \cap D'_n.$$ 

The $S_{t_1}, S_{t_2}, S'_{t_1}$ and $D_{t_1}, S'_{t_2}$ form isomorphic decompositions of $C$ into tame 3-cells, tame solid tori, and points of $K$ and $K'$. Now it is not difficult to extend the homeomorphism $h$ inductively over elements of the decompositions to obtain a homeomorphism of $C$ onto itself. This completes the outline of the proof of Theorem 5.

REFERENCES


MONOTONE DECOMPOSITIONS OF CONTINUUMS INTO GENERALIZED ARCS AND SIMPLE CLOSED CURVES

by

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Abstract. For compact, Hausdorff continua that are irreducible between two points, and those that are separated by no subcontinuum, sufficient conditions are given in order for the continuum to have a monotone, upper semi-continuous decomposition onto a generalized arc and generalized simple closed curve, respectively. The conditions involve the use of saturated and bi-saturated collections of continua. For metric continua the conditions are both necessary and sufficient. It is also shown that the elements $Q$ in the decomposition with void interiors are of the form $T(T(x))$, $x \in Q$, where $T$ is the asymmetric set function. The structure of the elements with non-void interiors is described and two open questions relating to the paper are discussed.

1. Introduction. A compact, Hausdorff continuum $M$ that is irreducible between a pair of its points is type $A$ [4] if $M$ has a monotone, upper semi-continuous decomposition whose quotient space is a generalized arc (a continuum with exactly two non-separating points). If $M$ is separated by no subcontinuum let us say $M$ is also type $A$ if there exists a monotone, upper semi-continuous decomposition whose quotient space is a generalized simple closed curve (a continuum in which every set of two distinct points separates). The primary theorems of this paper establish sufficient conditions for these two kinds of continua to be type $A$. If the continuum is metric the conditions are both necessary and sufficient where, of course, the quotient space is now a simple arc or a simple closed curve. Whether in the Hausdorff setting these conditions characterize type $A$ continua is not known.

Prior work for non-metric continua has been done by Gordin [4], and FitzGerald and Swingle [5] in the case where the continuum is irreducible. Gordin generalizes the work of Thomas on metric continua [7] to prove that a compact, Hausdorff continuum $M$ is type $A$ with the elements of the decomposition nowhere dense if and only if $M$ contains no region-containing indecomposable continuum. FitzGerald and Swingle give sufficient conditions for the continuum to be type $A$ and give results concerning the nature of the elements of the minimal decomposition.

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Reçu par la Réduction le 18. 1. 1972