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Strongly cellular subsets of E^3

by

Arlo W. Schurle (Charlotte, N. C.)

Abstract. This paper investigates properties of strongly cellular subsets of E^2 and E^3. A subset Z of E^n is strongly cellular if there is an n-cell C in E^n and a homotopy H of C in C such that $H_0 = \mathrm{Id}$, $H_t|Z = \mathrm{Id}$ for all t, $H|(\mathrm{Bd}\,C)\times[0,1]$ is a homeomorphism into $E^n\setminus Z$, and $H_1(C) = Z$. The main results are the following.

Theorem. *A subset of E^2 is strongly cellular if and only if it is a locally connected continuum not separating E^2.*

Theorem. *A one-dimensional subset of E^3 is strongly cellular if and only if it is a tame dendrite.*

1. Introduction. Bing and Kirkor [4] introduced the notion of strong cellularity and proved that a strongly cellular arc in E^3 is tame. This paper generalizes this result to the following satisfying theorem, which not only provides a characterization of tame dendrites but also a new characterization of tame trees.

THEOREM 1. *A one dimensional subset of E^3 is strongly cellular if and only if it is a tame dendrite.*

There are a number of results and questions involving strong cellularity. Bing and Kirkor's paper [4] has already been mentioned. Griffith and Howell [5] have shown that strongly cellular two-cells and three-cells in E^3 are tame. We use a number of results and techniques from both these papers. Bean has shown in [3] that every cactoid has an embedding C in E^3 so that C is strongly cellular, and in [2] asked whether a monotone decomposition of E^3 into points and countably many strongly cellular sets has E^3 as decomposition space. Our theorem shows a relationship between this question and that of Armentrout [1, Question 3] asking the same question for tame dendrites.

2. Preliminaries and statement of results. We first recall the original definition of Bing and Kirkor [4] as modified in [5]. I denotes the real interval $[0, 1]$. A homotopy of S in T is a continuous function H from $S\times I$ into T, and H_t denotes the function given by $H_t(x) = H(x, t)$. Also, if C is a cell, then $\mathrm{Bd}\,C$, $\mathrm{Int}\,C$ denote its combinatorial boundary and interior respectively.

A set Z in E^n is strongly cellular if there is an n-cell C in E^n and a homotopy of C in C such that, if $S = \mathrm{Bd}\, C$, then

(1) H_0 is the identity map, and $H_t|Z$ is the identity for all t,

(2) $H_t|S$ is a homeomorphism and $H_t(S) \cap Z = \emptyset$ for $t < 1$,

(3) $H_t(S) \cap H_u(S) = \emptyset$ for $t \neq u$, and

(4) $H_1(C) = Z$.

Following Griffith and Howell [5] we say that a set Z in E^n *has a cocoon* if there is an $(n-1)$-sphere S in $E^n - Z$ and a homotopy h of S in E^n such that

(1) h_0 is the identity,

(2) h_t is an embedding for $t < 1$,

(3) $h_t(S) \cap h_u(S) = \emptyset$ for $t \neq u$, and

(4) $h_1(S) = \mathrm{Bdry}\, Z$, where Bdry denotes point-set boundary.

The homotopy h will be called a *cocooning map for* Z.

We will need the following results from [5]. It should be noted that we do not require that a simply connected set be connected.

THEOREM 2. ([Theorem 2.1, 5]) *If Z is a compact subset of E^n with a connected complement, then Z is strongly cellular if and only if Z has a cocoon.*

PROPOSITION 1. ([Lemma 4.1, 5]) *If h is a cocooning map for a subset Z of E^3 and A is an arcwise connected closed subset of* $\mathrm{Bdry}\, Z$ *with* $\mathrm{Bdry}\, Z \setminus A$ *simply connected, then $h_1^{-1}(A)$ is connected.*

Our first result concerns strongly cellular subsets of the plane.

THEOREM 3. *A subset of E^2 is strongly cellular if and only if it is a locally connected continuum with connected complement.*

Recall that a dendrite is a locally connected continuum containing no simple closed curve. It is well known that all dendrites can be embedded in E^2, that all dendrites lying in E^2 have connected complements, and that dendrites are one-dimensional. (See, for example, [8, page 77], [8, page 107], and [8, page 99].) A dendrite D in E^3 is said to be *tame* if there is a homeomorphism h of E^3 onto itself such that $h(D)$ is contained in $E^2 = \{(x, y, z) \in E^3\colon z = 0\}$. Since a strongly cellular subset of E^2 is easily seen to be strongly cellular in E^3, Theorem 3 gives us half of Theorem 1. The next two theorems give us the other half.

THEOREM 4. *A one-dimensional strongly cellular subset of E^3 is a dendrite.*

THEOREM 5. *A strongly cellular dendrite in E^3 is tame.*

The remaining sections of this paper consist of proofs of Theorems 3, 4, and 5.

3. Proof of Theorem 3. Any strongly cellular set is the continuous image of a cell, and so is a locally connected continuum. The complement of a strongly cellular set is the increasing union of complements of cells and so is connected. This proves one-half of Theorem 3.

Let M be a locally connected continuum in E^2 with connected complement. We show that M is strongly cellular by constructing a cocooning map H for M and applying Theorem 2. We use freely results concerning the topology of the plane and of locally connected continua, all of which can be found in [7] or [8].

Let B be the point-set boundary of M. We know that B is a regular curve. For each point x of B there is an open neighborhood U_x of x in B such that U_x is connected, $\mathrm{diam}\, U_x$ is less than $\frac{1}{2}$, and the boundary of U_x in B is finite. Let $U_1, U_2, \ldots, U_{s_1}$ be a collection of such U_x's covering B, and let K_{1i} be the closure of U_i. $B \setminus U_i$ has only finitely many components, each lying except for a single point in the unbounded component of $E^2 - K_{1i}$. Hence there is a closed connected set L_{1i} such that $B \subset K_{1i} \cup L_{1i}$ and $K_{1i} \cap L_{1i}$ is finite.

An application of the Plane Separation Theorem [8, page 108] yields simple closed curves $C_{11}, C_{12}, \ldots, C_{1s_1}$ bounding disks $D_{11}, D_{12}, \ldots, D_{1s_1}$ such that $C_{1i} \subset B(K_{1i}, 1)$, $\mathrm{diam}\, D_{1i} < \frac{1}{2}$, and $B \cap D_{1i} = K_{1i}$. (For any set A and positive number ε we let $B(A, \varepsilon)$ denote the set of points within ε of a point of A. Also, if J is a simple closed curve in E^2, then $\mathrm{Int}\, J$ denotes the bounded complementary domain of J.) By adjusting the C_{1i}'s if necessary, we may assume that each component of $D_{1i} \cap D_{1j}$ contains a point of B, that each C_{1i} is locally polygonal off B, and that the C_{1i}'s are in general position off B, i.e., at each point of intersection of two C_{1i}'s not on B the C_{1i}'s look locally like the letter "X."

Let J_1 be the boundary of the unbounded component of $E^2 \setminus \bigcup_{i=1}^{s_1} C_{1i}$. J_1 is a polygonal simple closed curve whose interior contains M and which lies in $B(M, 1)$.

By starting with sufficiently small neighborhoods U_x and following the above procedure, we can obtain continua $K_{21}, K_{22}, \ldots, K_{2s_2}$ such that $\mathrm{diam}\, K_{2i} < \frac{1}{4}$, each K_{2i} is contained in some K_{1j}, the interiors of the K_{2i}'s in B cover B, and there are continua $L_{21}, L_{22}, \ldots, L_{2s_2}$ such that $B \subset K_{2i} \cup L_{2i}$ and $K_{2i} \cap L_{2i}$ is finite.

An application of the Plane Separation Theorem and adjustment of the resulting curves now yield simple closed curves $C_{21}, C_{22}, \ldots, C_{2s_2}$ bounding disks $D_{21}, D_{22}, \ldots, D_{2s_2}$ such that

(1) $\mathrm{diam}\, D_{2j} < \frac{1}{4}$,

(2) $D_{2j} \subset B(K_{2j}, \frac{1}{2}) \cap \mathrm{Int}\, C_{1k}$ whenever $K_{2j} \subset K_{1k}$,

(3) $B \cap D_{2j} = K_{2j}$,

(4) each component of $D_{2j} \cap D_{2i}$ contains a point of B,

(5) each component of $D_{2j} \cap D_{1i}$ contains a point of B,

(6) each C_{2j} is locally polygonal off B, and the C_{1i}'s and C_{2j}'s are in general position off B.

Let J_2 be the boundary of the unbounded component of $E^2 \setminus \bigcup_{i=1}^{s_2} C_{2i}$. J_2 is a polygonal simple closed curve such that $M \subset \operatorname{Int} J_2$, $J_2 \subset \operatorname{Int} J_1$, and $J_2 \subset B(M, \frac{1}{2})$. Let A be the annulus $(J_1 \cup \operatorname{Int} J_1) \setminus \operatorname{Int} J_2$. We show that there is a homeomorphism h from $J_1 \times I$ onto A such that $h(x, 0) = x$ and $h\big((J_1 \cap C_{1i}) \times I\big) \subset D_{1i}$ for $i = 1, 2, \dots, s_1$.

Consider $D_{1j} \cap (J_2 \cup \operatorname{Int} J_2)$. By (5) each component of this set contains points of B, and since $D_{1j} \cap B$ is connected, there is only one component. By the general position requirements $D_{1j} \cap J_2$ consists of either (I) one arc ab spanning D_{1j} or (II) two such arcs, say xy and zw. Also, $B \cap D_{1j}$ lies in the disk bounded by (I) ab and one component of $C_{1j} \setminus \{a, b\}$ or (II) xy, zw, and two components of $C_{1j} \setminus \{x, y, z, w\}$.

Since $J_2 \subset \bigcup_{i=1}^{s_1} \operatorname{Int} C_{1i}$ and the C_{1i}'s, C_{2i}'s are in general position off B, there is a homeomorphism h_{1j} of $C_{1j} \cap J_1$ into D_{1j} such that if x is a point of $C_{1j} \cap J_1 \cap C_{1i}$, then $h_{1j}(\{x\} \times I)$ lies in $D_{1j} \cap D_{1i}$. The desired homeomorphism h is obtained by pasting together such h_{1j}'s, $j = 1, 2, \dots, s_1$.

The entire procedure given above can clearly be iterated, yielding a sequence of simple closed curves $J_1, J_2, \dots$ such that for $i = 1, 2, \dots$

$$M \subset \operatorname{Int} J_{i+1} \subset J_{i+1} \cup \operatorname{Int} J_{i+1} \subset (\operatorname{Int} J_i) \cap B(M, 1/i+1) .$$

Further, if we let A_i be the annulus bounded by $J_i \cup J_{i+1}$, then the curves J_i may be so constructed that there is a homeomorphism h^i from $J_i \times I$ onto A_i such that diam $h^i(\{x\} \times I)$ is less than $1/2^i$ and $h^i(x, 0) = x$ for each x in J_i and $i = 1, 2, \dots$

The sequence $h_1^1, h_1^2 \circ h_1^1, \dots$ converges to a continuous function g. It is now easy to show that the function H from $J_i \times I$ into E^2 defined by

$$H(x, t) = \begin{cases} h^m\big(h_1^{m-1} \circ \dots \circ h_1^1(x), m(m+1)t - m^2 + 1\big), & t \in \left[\dfrac{m-1}{m}, \dfrac{m}{m+1}\right], \ m = 1, 2, \dots, \\ g(x), & t = 1 \end{cases}$$

is a cocooning map for M. This concludes the proof of Theorem 3.

4. Proof of Theorem 4. Let W be a one-dimensional subset of E^3 with cocooning map h defined on $S \times I$. We first note that W is a dendrite if and only if h_1 is a monotone map. For if h_1 is monotone, then W is the monotone image of a 2-sphere, hence a cactoid [6], and a one-dimensional cactoid is a dendrite. Conversely, if W is a dendrite, the $W - \{x\}$ is simply connected for each point x of W, and so by Proposition 1 $h_1^{-1}(x)$ is connected.

The rest of the proof is by contradiction. Suppose x is a point of W such that $h_1^{-1}(x)$ is not connected. There is a simple closed curve J on S missing $h_1^{-1}(x)$ such that each complementary domain of J on S meets $h_1^{-1}(x)$. Now $h_1(S - J)$ is a neighborhood U of x in W. Since W is one-dimensional, there is a compact 0-dimensional set K contained in U which separates x from $h_1(J)$ in W. Then $h_1^{-1}(K)$ separates $h_1^{-1}(x)$ from J on S.

Let D be either of the disks bounded by J on S, and let A be a component of $h_1^{-1}(x)$ in D. Some component C of $h_1^{-1}(K)$ separates A from J ([7, page 123]). Since K is 0-dimensional, $h_1(C)$ is a single point. Let $J_1, J_2, \dots$ be simple closed curves in $\operatorname{Int} D$ such that $C \cup J_{i+1}$ is contained in the interior of the subdisk of D bounded by J_i for $i = 1, 2, \dots$, and $J_i \subset B(C, 1/i)$ for $i = 1, 2, \dots$

Let f be a homeomorphism of $J \times [0, 1)$ into D such that $f(x, 0) = x$ for all x in J and $f(J \times \{i/i+1\}) = J_i$ for $i = 1, 2, \dots$ Define a function g from $J \times I$ into E^3 by

$$g(x, t) = \begin{cases} h_1(f(x, t)), & t < 1, \\ h_1(C), & t = 1. \end{cases}$$

It is easy to see that g is continuous and hence a homotopy shrinking J to a point.

Now let y_1 be a point of A and y_2 a point of $h_1^{-1}(x)$ in $S \setminus D$. Let J' be a simple closed curve formed by an arc from y_1 to y_2 meeting $h(S \times I)$ only at y_1 and y_2 together with the arcs $h(y_1 \times I)$ and $h(y_2 \times I)$. Then J and J' are linked, but g shrinks J to a point in the complement of J'. (See [5, Lemma 4.1] for a similar argument.) This contradiction establishes Theorem 4.

5. Proof of Theorem 5. The proof of this theorem follows [4] so closely that we omit details and give only a general outline. Let K and K' be any two strongly cellular homeomorphic dendrites in E^3. In particular, by Theorem 3 K can be planar. Let h be a homeomorphism from K onto K', and assume that $K \cup K' \subset C = \{x \in E^3\colon |x| \leqslant 1\}$ and that H, H' are homotopies of $C \times I$ into C as in the definition of strongly cellular for K, K' respectively. Let S_t, S'_t be $H_t(\operatorname{Bd} C), H'_t(\operatorname{Bd} C)$ respectively, and C_t, C'_t be $H_t(C), H'_t(C)$ respectively.

Let $p_1, p_2, \dots$ be a dense set of points of order 2 in K such that for each positive ε there is an integer N for which each component of $K - \{p_1, p_2, \dots, p_N\}$ has diameter less than ε. Let $p'_i = h(p_i)$. We now construct topological disks $D_1, D_2, \dots$ and $D'_1, D'_2 \dots$ in C satisfying the following conditions.

(1) $D_i \cap S_t$, $D'_i \cap S'_t$ are simple closed curves for $t < 1$ and p_i, p'_i respectively for $t = 1$.

(2) Each component of $C \setminus D_i$, $C \setminus D'_i$ contains exactly one component of $K \setminus p_i$, $K' \setminus p'_i$ respectively.

(3) D_i, D'_i are locally tame except possibly at p_i, p'_i respectively.

(4) For $i \neq j$ there is a number t such that $0 \leqslant t < 1$ and

$$C_t \cap D_i \cap D_j = \emptyset = C'_t \cap D'_i \cap D'_j .$$

(5) If i, j, and t are as in (4), $t < s < 1$ and W, W' are the closures of $C_t \setminus C_s$, $C'_t \setminus C'_s$ respectively, then the closures of the components of $W \setminus (D_i \cup D_j)$ and $W' \setminus (D'_i \cup D'_j)$ are two tame 3-cells and a tame solid torus.

Let $t_1, t_2, \ldots$ be a monotone increasing sequence of positive numbers with limit 1 such that for $m \neq n$ and $m, n \leqslant i+1$,

$$C_{t_i} \cap D_m \cap D_n = \emptyset = C'_{t_i} \cap D'_m \cap D'_n .$$

The S_{t_i}'s, D_i's, S'_{t_i}'s and D'_i's form isomorphic decompositions of C into tame 3-cells, tame solid tori, and points of K and K'. Now it is not difficult to extend the homeomorphism h inductively over elements of the decompositions to obtain a homeomorphism of C onto itself. This completes the outline of the proof of Theorem 5.

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UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE

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Monotone decompositions of continua into generalized arcs and simple closed curves

by

Eldon J. Vought (Chico, Calif.)

Abstract. For compact, Hausdorff continua that are irreducible between two points, and those that are separated by no subcontinuum, sufficient conditions are given in order for the continuum to have a monotone, upper semi-continuous decomposition onto a generalized arc and generalized simple closed curve, respectively. The conditions involve the use of saturated and bi-saturated collections of continua. For metric continua the conditions are both necessary and sufficient. It is also shown that the elements Q in the decomposition with void interiors are of the form $T(T(x)), x \in Q$, where T is the aposyndetic set function. The structure of the elements with non-void interiors is described and two open questions relating to the paper are discussed.

1. Introduction. A compact, Hausdorff continuum M that is irreducible between a pair of its points is *type A* [4] if M has a monotone, upper semi-continuous decomposition whose quotient space is a generalized arc (a continuum with exactly two non-separating points). If M is separated by no subcontinuum let us say M is also type A if there exists a monotone, upper semi-continuous decomposition whose quotient space is a generalized simple closed curve (a continuum in which every set of two distinct points separates). The primary theorems of this paper establish sufficient conditions for these two kinds of continua to be type A. If the continuum is metric the conditions are both necessary and sufficient where, of course, the quotient space is now a simple arc or a simple closed curve. Whether in the Hausdorff setting these conditions characterize type A continua is not known.

Prior work for non-metric continua has been done by Gordh [4], and FitzGerald and Swingle [3] in the case where the continuum is irreducible. Gordh generalizes the work of Thomas on metric continua [7] to prove that a compact, Hausdorff continuum M is type A with the elements of the decomposition nowhere dense if and only if M contains no region-containing indecomposable continuum. FitzGerald and Swingle give sufficient conditions for the continuum to be type A and give results concerning the nature of the elements of the minimal decomposition.