The notion of an elementary subsystem for a Boolean-valued relational system

by

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Abstract. The paper is devoted to investigate the possibility of generalization of the notion of an elementary subsystem for a Boolean-valued relational system. Several possible generalizations are proposed and their mutual relations are studied. The Skolem-Löwenheim-Tarski theorem is studied in connection with this generalizations.

By a relational system one may mean the sequence

$$\mathfrak{S} = (S, R_0, \ldots, R_a, \ldots)$$

where R_a is a function from ${}^{n_a}S$ into $\{0, 1\}$. The relational system $\mathfrak{S}$ may be described in the first order language $\mathfrak{L}_{\mathfrak{S}}$. For any formula $\boldsymbol{F} \in \mathfrak{L}_{\mathfrak{S}}$ and any $\mathfrak{h} \in {}^{\omega}S$ we have $\mathrm{val}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}) \in 1$ if $\mathfrak{h}$ satisfies $\boldsymbol{F}$ in $\mathfrak{S}$ and $\mathrm{val}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}) \in 0$ otherwise. We may generalize the notion of relational system such that the functor $\mathrm{val}_{\mathfrak{S}}$ will take values in some, not necessarily two-valued, Boolean algebra (see e.g. [3], [4] and [5]). For a classical relational system (over a two-valued Boolean algebra) the notion of elementary subsystem has been introduced, and the well known Löwenheim-Skolem-Tarski theorem has been proved (see [7]). Below, the notion of elementary subsystem will be generalized to Boolean-valued relational systems in such a way that the Löwenheim-Skolem-Tarski theorem will be valid.

1. Boolean-valued relational systems. Usually by a Boolean algebra one means the algebra $\mathfrak{B} = (B, -, \cup, \cap)$ satisfying suitable conditions (see e.g. [6]). Since the operation $\cap$ may be defined in terms of $-$ and $\cup$, we shall mean by a Boolean algebra the reduct $(B, -, \cup)$ of $\mathfrak{B}$.

Let $\mathfrak{B} = (B, -, \cup)$ be a Boolean algebra. As is well known, we may define in $\mathfrak{B}$ a partial order on B:

$$a \leqslant b \leftrightarrow a \cup b = b\,.$$

This relation may be extended to subsets of B:

$$A \leqslant a \leftrightarrow \forall a'(a' \in A \rightarrow a' \leqslant a)\,.$$

If $A \leqslant a$ we say that a is an *upper bound of* A. The least upper bound of A (if it exists) is called the *supremum of* A (in symbols: $\sup_{\mathfrak{B}} A$). If the operation of supremum is defined for all subsets of B, the algebra $\mathfrak{B}$ is called *complete* (see [6], § 20). The two-valued Boolean algebra $\mathfrak{B}_2$, i.e. the Boolean algebra with the universe $\{0, 1\}$ is complete.

By the *Boolean-valued relational system over the algebra* $\mathfrak{B}$ we shall mean the (finite or transfinite) sequence

$$\mathfrak{S} = (S, R_0, \ldots, R_a, \ldots)$$

where S is a set called the *universe of* $\mathfrak{S}$, and R_a is an n-ary function defined on S with the values in B:

$$R_a\colon {}^{n_a}S \to B .$$

Any function R_a is called a *relation of* $\mathfrak{S}$.

To the Boolean-valued relational system $\mathfrak{S}$ there corresponds a *language* $\mathfrak{L}_{\mathfrak{S}}$, i.e. the set of all well-formed formulas, built up from the symbols $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n, \ldots$ (variables), $=$ (equality), $\sim$, $\vee$, $\exists$ (logical connectives) and $($, $)$ (parentheses). Let $\mathfrak{h}$ be any infinite sequence of elements of S. We put:

$$\text{(i)}\quad \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{x}_i = \boldsymbol{x}_j, \mathfrak{h}) = \begin{cases} 0 & \text{if } \mathfrak{h}_i \neq \mathfrak{h}_j , \\ 1 & \text{if } \mathfrak{h}_i = \mathfrak{h}_j , \end{cases}$$

$$\text{(ii)}\quad \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{R}_a(\boldsymbol{x}_{i_0}, \ldots, \boldsymbol{x}_{i_n}), \mathfrak{h}) = R_a(\mathfrak{h}_{i_0}, \ldots, \mathfrak{h}_{i_n}) ,$$

$$\text{(iii)}\quad \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\sim \boldsymbol{F}, \mathfrak{h}) = -\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}) ,$$

$$\text{(iv)}\quad \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F} \vee \boldsymbol{G}, \mathfrak{h}) = \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}) \cup \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{G}, \mathfrak{h}) ,$$

$$\text{(v)}\quad \mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\exists \boldsymbol{x}_i \boldsymbol{F}, \mathfrak{h}) = \sup_{\mathfrak{B}} \{\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}')\colon \forall j (i \neq j \to \mathfrak{h}_j = \mathfrak{h}'_j)\} .$$

The condition (v) is well defined in any complete Boolean algebra, in particular in $\mathfrak{B}_2$. In general the supremum of the set

$$\text{(1)}\qquad \{\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h})\colon \forall j (i \neq j \to \mathfrak{h}_j = \mathfrak{h}'_j)\}$$

does not necessarily exist. Thus we need to assume that the supremum of any set of the form (1) does exist in the Boolean algebra in question. This assumption is much weaker than the assumption of completeness

From the condition (i) it follows that the interpretation of equality, $=$, is standard. Sometimes (see e.g. [3], p. 53) the equality is treated just like any other predicate, i.e. there is a function eq: ${}^2S \to B$ such that $\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{x}_i = \boldsymbol{x}_j, \mathfrak{h}) = \mathrm{eq}(\mathfrak{h}_i, h_j)$. Since in such a case we may enrich the language by a new predicate, say $='$, which will denote standard equality, this nonstandard notion may be treated as just a particular case of that defined above.

The Boolean algebra underlying the Boolean relational system $\mathfrak{S}$ is not uniquely determined. If the Boolean relation system $\mathfrak{S}$ is over the Boolean algebra $\mathfrak{B}$, then it is over any extension of $\mathfrak{B}$ being a Boolean algebra. The least Boolean algebra underlying $\mathfrak{S}$ is said to be *supporting* $\mathfrak{S}$. Let $\mathfrak{S}$ be a Boolean-valued relational system over the Boolean algebra $\mathfrak{B}$. The algebra $\mathfrak{B}$ restricted to the set

$$\text{(2)}\qquad \{\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h})\colon \boldsymbol{F} \in \mathfrak{L}_{\mathfrak{S}} \wedge \mathfrak{h} \in {}^{\omega}S\}$$

is a Boolean algebra supporting $\mathfrak{S}$.

By the power of $\mathfrak{S} = (S, R_0, \ldots, R_a, \ldots)$ we shall mean the power of the set S. Let the power of S be σ. Let λ be the power of the language $\mathfrak{L}_{\mathfrak{S}}$. Then the power of the set (2) is not greater than the maximum of σ and λ (since at least λ is infinite). Thus we have

THEOREM 1. *The power of the Boolean algebra supporting the Boolean-valued relational system* $\mathfrak{S}$ *is not greater than the maximum of the powers of* $\mathfrak{S}$ *and* $\mathfrak{L}_{\mathfrak{S}}$.

Let $\mathfrak{S}$ be a Boolean-valued relational system over a Boolean algebra $\mathfrak{B}$, and $\mathfrak{S}'$ — over $\mathfrak{B}'$. We say that $\mathfrak{S}$ is a *subsystem* of $\mathfrak{S}'$ if the universe of $\mathfrak{S}$ is a subset of the universe of the universe of $\mathfrak{S}'$ and relations of $\mathfrak{S}$ are corresponding relations of $\mathfrak{S}'$ restricted to the universe of $\mathfrak{S}$. We say that $\mathfrak{S}$ is a *Boolean elementary subsystem of* $\mathfrak{S}'$ ($\mathfrak{S}'$ is a *Boolean elementary extension of* $\mathfrak{S}$) if there are Boolean algebras $\mathfrak{B}$ and $\mathfrak{B}'$, underlying $\mathfrak{S}$ and $\mathfrak{S}'$, respectively and such that

(I) $\mathfrak{S}$ is a subsystem of $\mathfrak{S}'$,

(II) $\mathfrak{B}$ is an elementary subalgebra [1] of $\mathfrak{B}'$,

(III) for any $\boldsymbol{F} \in \mathfrak{L}_{\mathfrak{S}}$ and any $\mathfrak{h} \in {}^{\omega}S$

$$\mathrm{val}^{\mathfrak{B}}_{\mathfrak{S}}(\boldsymbol{F}, \mathfrak{h}) = \mathrm{val}^{\mathfrak{B}'}_{\mathfrak{S}'}(\boldsymbol{F}, \mathfrak{h}) .$$

Since the only Boolean subalgebra (and therefore the only elementary subalgebra) of the two-valued Boolean algebra $\mathfrak{B}_2$ is the algebra $\mathfrak{B}_2$ itself, the notion of a Boolean elementary subsystem in case of classical systems (i.e. the systems over the algebra $\mathfrak{B}_2$) coincides with the classical notion of elementary subsystem (see [7]). Therefore the notion of a Boolean elementary subsystem is a generalization of the notion of an elementary subsystem.

2. Löwenheim-Skolem-Tarski theorem. Let $\mathfrak{S} = (S, R_0, \ldots, R_a, \ldots)$ be a Boolean-valued relational system supported by the Boolean algebra $\mathfrak{B} = (B, -, \cup)$. Let $\mathfrak{L}_p$ be the set of all formulas in which only the variables with indices less than p occur. We say that the relational system

$$\mathfrak{M} = (M, S, R_0, \ldots, R_a, \ldots, B, -, \cup, \boldsymbol{F}_0, \ldots, \boldsymbol{F}_q, \ldots, \mathrm{val}_0, \ldots, \mathrm{val}_r, \ldots)$$

(1) In the classical sense.

describes the Boolean-valued relational system $\mathfrak{S}$ if

(i) $S \cup B \cup \mathfrak{L}_{\mathfrak{S}} \subseteq M$,

(ii) $F_0, \ldots, F_q, \ldots$ is a sequence of all formulas of $\mathfrak{L}_{\mathfrak{S}}$,

(iii) val_r is a mapping of $\mathfrak{L}_r \times {}^r S$ into B such that for any $F \in \mathfrak{L}_r$ and $\mathfrak{h} \in {}^\omega S$

$$\mathrm{val}_r(F, \mathfrak{h}_0, \ldots, \mathfrak{h}_{r-1}) = \mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) .$$

It should be noticed that $\mathfrak{M}$ is a relational system in the classical sense: R_a is not a Boolean relation on S but an ordinary function from ${}^{n_a}S$ into B.

LEMMA 2. *If $\mathfrak{M}'$ is an elementary subsystem (elementary extension) of the system $\mathfrak{M}$ describing a Boolean-valued relational system $\mathfrak{S}$, then the suitable reduct of $\mathfrak{M}'$ is a Boolean elementary subsystem (Boolean elementary extension) of $\mathfrak{S}$.*

Proof. Let

$$\mathfrak{M}' = (M', S', R_0', \ldots, R_a', \ldots, B', -', \cup', F_0', \ldots, F_q', \ldots, \mathrm{val}_0', \ldots, \mathrm{val}_r', \ldots)$$

be an elementary subsystem of $\mathfrak{M}$. Then the reduct $\mathfrak{S}' = (S', R_0', \ldots, R_a', \ldots)$ is the Boolean-valued relational system over the Boolean algebra $\mathfrak{B}' = (B', -', \cup')$. Moreover, $\mathfrak{B}'$ is an elementary subalgebra of $\mathfrak{B}$. Therefore, to prove the lemma it will suffice to show that for any formula F and any $\mathfrak{h} \in {}^\omega S'$ we have

$$\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) = \mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) .$$

We shall prove this equality by induction according to the complexity of the formula F. Let F be an atomic formula $R_a(x_{i_0}, \ldots, x_{i_{n_a}})$; then

$$\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) = R_a(\mathfrak{h}_{i_0}, \ldots, \mathfrak{h}_{i_{n_a}}) = R_a'(\mathfrak{h}_{i_0}, \ldots, \mathfrak{h}_{i_{n_a}}) = \mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) .$$

Similarly for $F = (x_i = x_j)$. Let $F = \sim F$. Then

$$\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) = -\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F', \mathfrak{h}) = -'\mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F', \mathfrak{h}) = \mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) .$$

Similarly we may prove the equality in the case of $F = F' \vee F''$. Slightly more involved is the proof in the case of $F = \exists x_i F'$. Here we have to use the fact that $\mathfrak{M}'$ is an elementary subsystem of $\mathfrak{M}$ and the element

$$\sup_{\mathfrak{B}}\{\mathrm{val}_p(F', \mathfrak{h}_0, \ldots, \mathfrak{h}_{i-1}, x, \mathfrak{h}_{i+1}, \ldots, \mathfrak{h}_{p-1})\colon x \in S\}$$

is definable in $\mathfrak{L}_{\mathfrak{M}}$. Let $F \in \mathfrak{L}_p$, then

$$\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) = g$$

if and only if

$$g = \sup_{\mathfrak{B}}\{\mathrm{val}_p(F', \mathfrak{h}_0, \ldots, \mathfrak{h}_{i-1}, x, \mathfrak{h}_{i+1}, \ldots, \mathfrak{h}_{p-1})\colon x \in S\} .$$

Since $\mathfrak{M}'$ is an elementary subsystem of $\mathfrak{M}$, this is equivalent to

$$g = \sup_{\mathfrak{B}}\{\mathrm{val}_p'(F', \mathfrak{h}_0, \ldots, \mathfrak{h}_{i-1}, x, \mathfrak{h}_{i+1}, \ldots, \mathfrak{h}_{p-1})\colon x \in S'\} .$$

Therefore by the induction hypothesis

$$\mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) = g = \mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) .$$

THEOREM 3 (Lower Löwenheim-Skolem-Tarski theorem). *If $\mathfrak{S}$ is a Boolean-valued relational system of the power σ, $\mathfrak{L}_{\mathfrak{S}}$ is of the power λ and $\sigma \geqslant \mu \geqslant \lambda$, then there is an elementary Boolean subsystem $\mathfrak{S}'$ of $\mathfrak{S}$ of the power μ.*

Proof. Let $\mathfrak{S} = (S, R_0, \ldots, R_a, \ldots)$. We shall consider the following relational system:

$$\mathfrak{M} = (M, S, R_0, \ldots, R_a, \ldots, B, -, \cup, F_0, \ldots, F_q, \ldots, \mathrm{val}_0, \ldots, \mathrm{val}_r, \ldots, f)$$

(comp. the definition of a system describing the Boolean-valued relational system) where f is a function mapping S onto M. Such a function exists since by Theorem 1 the power of $\mathfrak{B}$ is less than or equal to σ. The power of $\mathfrak{L}_{\mathfrak{M}}$ is equal to λ. Therefore there is a relational system

$$\mathfrak{M}' = (M', S', R_0', \ldots, R_a', \ldots, B', -', \cup', F_0, \ldots, F_q, \ldots, \mathrm{val}_0', \ldots, \mathrm{val}_r', \ldots, f') ,$$

an elementary subsystem of $\mathfrak{M}$ of the power μ. Thus by Lemma 2 the Boolean-valued relational system

$$\mathfrak{S}' = (S', R_0', \ldots, R_a', \ldots)$$

is an elementary Boolean subsystem of $\mathfrak{S}$. Since $S' \subseteq M'$ and f' is a function mapping S' onto M', the power of $\mathfrak{S}'$ is μ.

More involved is the proof of

THEOREM 4 (Upper Löwenheim-Skolem-Tarski theorem). *If $\mathfrak{S}$ is a Boolean-valued relational system of the infinite power σ, $\mathfrak{L}_{\mathfrak{S}}$ is of the power λ and $\mu \geqslant \lambda + \sigma$, then there is an elementary Boolean extension $\mathfrak{S}'$ od $\mathfrak{S}$ of the power μ.*

Proof. Let us consider the relational system

$$\mathfrak{M} = (M, S, R_0, \ldots, R_a, \ldots, B, -, \cup, F_0, \ldots, F_q, \ldots \\ \ldots, \mathrm{val}_0, \ldots, \mathrm{val}_r, \ldots, s_0, \ldots, s_\beta, \ldots) ,$$

where $s_0, \ldots, s_\beta, \ldots$ is a sequence of all elements of M. The other notations are as in the proof of Theorem 3. There are an index set $\mathfrak{I}$ and an ultrafilter $\mathcal{F}$ such that the power of the ultrapower $S^{\mathfrak{I}}/_{\mathcal{F}}$ is greater than μ. Let $\mathfrak{M}''$ be the expansion of $\mathfrak{M}^{\mathfrak{I}}/_{\mathcal{F}}$ by the sequence $t_0, \ldots, t_\gamma, \ldots$ of μ elements of $S^{\mathfrak{I}}/_{\mathcal{F}}$. Since the power of $\mathfrak{L}_{\mathfrak{M}''}$ is equal to μ, there is an elementary subsystem of $\mathfrak{M}''$ of the power μ (see [7]). Let

$$\mathfrak{M}' = (M', S', R_0', \ldots, R_a', \ldots, B', -', \cup', F_0, \ldots, F_q, \ldots, \mathrm{val}_0', \ldots, \mathrm{val}_r', \ldots)$$

be the reduct of this subsystem. Since M' contains all the elements of the sequence $s_0, \ldots, s_\beta, \ldots$, $\mathfrak{M}'$ is an extension of the relational system describing $\mathfrak{S}$. Both are elementary subsystems of the reduct of $\mathfrak{M}''$. Thus $\mathfrak{M}'$ is an elementary extension of the relational system describing $\mathfrak{S}$. Therefore by Lemma 2 the Boolean-valued relational system

$$\mathfrak{S}' = (S', R_0', \ldots, R_a', \ldots)$$

is a Boolean elementary extension of $\mathfrak{S}$. Since S' is contained in the set M' of the power μ and contains μ elements $t_0, \ldots, t_\nu, \ldots$, therefore $\mathfrak{S}'$ is of the power μ. This completes the proof.

3. Modifications of the notion of an elementary Boolean subsystem. We have defined the notion of an elementary Boolean subsystem in the following way:

The Boolean-valued relational system $\mathfrak{S}$ is an elementary Boolean subsystem of $\mathfrak{S}'$ (in symbols $\mathfrak{S} \preccurlyeq \mathfrak{S}'$) if there are Boolean algebras $\mathfrak{B}$ and $\mathfrak{B}'$ underlying, respectively, $\mathfrak{S}$ and $\mathfrak{S}'$ and such that

(I) $\mathfrak{S}$ is a subsystem of $\mathfrak{S}'$,

(II) $\mathfrak{B}$ is an elementary subalgebra of $\mathfrak{B}'$,

(III) for any $F \in \mathfrak{L}_{\mathfrak{S}}$ and $\mathfrak{h} \in {}^{\omega}S$ (S is the universe of $\mathfrak{S}$) we have

$$\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) = \mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) .$$

We may introduce the modified notions of an elementary Boolean subsystem. We shall write

$$\mathfrak{S} \preccurlyeq_s \mathfrak{S}'$$

if $\mathfrak{S}$ and $\mathfrak{S}'$ satisfy the above definition with condition (II) replaced by a stronger one:

(II′) $\mathfrak{B} = \mathfrak{B}'$.

We shall write

$$\mathfrak{S} \preccurlyeq_w \mathfrak{S}'$$

if $\mathfrak{S}$ and $\mathfrak{S}'$ satisfy this definition with condition (II) replaced by a weaker one:

(II″) $\mathfrak{B}$ is Boolean subalgebra of $\mathfrak{B}'$.

Moreover, we may modify the notion of an elementary Boolean subsystem by the elimination of the existential quantifier: We shall write

$$\mathfrak{S} \preccurlyeq^s \mathfrak{S}'$$

if for Boolean algebras $\mathfrak{B}$ and $\mathfrak{B}'$ supporting, respectively, $\mathfrak{S}$ and $\mathfrak{S}'$ the conditions (I), (II) and (III) are satisfied. Similarly we shall write

$$\mathfrak{S} \preccurlyeq_s^s \mathfrak{S}' \quad \text{and} \quad \mathfrak{S} \preccurlyeq_w^s \mathfrak{S}'$$

if condition (II) is replaced by (II′) and (II″), respectively.

THEOREM 5.

$$\mathfrak{S} \preccurlyeq_s^s \mathfrak{S}' \to \mathfrak{S} \preccurlyeq_s \mathfrak{S}' \qquad \mathfrak{S} \preccurlyeq_s^s \mathfrak{S}' \to \mathfrak{S} \preccurlyeq^s \mathfrak{S}' \qquad \mathfrak{S} \preccurlyeq^s \mathfrak{S}' \to \mathfrak{S} \preccurlyeq_w^s \mathfrak{S}'$$

$$\mathfrak{S} \preccurlyeq^s \mathfrak{S}' \to \mathfrak{S} \preccurlyeq \mathfrak{S}' \qquad \mathfrak{S} \preccurlyeq_s \mathfrak{S}' \to \mathfrak{S} \preccurlyeq \mathfrak{S}' \qquad \mathfrak{S} \preccurlyeq \mathfrak{S}' \to \mathfrak{S} \preccurlyeq_w \mathfrak{S}'$$

$$\mathfrak{S} \preccurlyeq_w^s \mathfrak{S}' \leftrightarrow \mathfrak{S} \preccurlyeq_w \mathfrak{S}'$$

Proof. Almost all implications follow immediately from the definitions. Only the implication

$$\mathfrak{S} \preccurlyeq_w \mathfrak{S}' \to \mathfrak{S} \preccurlyeq_w^s \mathfrak{S}'$$

must be proved. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be such Boolean algebras underlying, respectively, $\mathfrak{S}$ and $\mathfrak{S}'$ that conditions (I), (II″) and (III) are satisfied. In particular $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}'$ and $\mathfrak{S}$ is a Boolean subsystem of $\mathfrak{S}'$. Thus by condition (III)

$$\{\mathrm{val}_{\mathfrak{S}}^{\mathfrak{B}}(F, \mathfrak{h}) \colon F \in \mathfrak{L}_{\mathfrak{S}} \wedge \mathfrak{h} \in {}^{\omega}S\} \subseteq \{\mathrm{val}_{\mathfrak{S}'}^{\mathfrak{B}'}(F, \mathfrak{h}) \colon F \in \mathfrak{L}_{\mathfrak{S}} \wedge \mathfrak{h} \in {}^{\omega}S'\}$$

and therefore the Boolean algebra supporting $\mathfrak{S}$ is a subalgebra of that supporting $\mathfrak{S}'$. This completes the proof.

On the other hand, $\mathfrak{S} \preccurlyeq \mathfrak{S}'$ does not imply $\mathfrak{S} \preccurlyeq_s \mathfrak{S}'$. In fact, for $\preccurlyeq$ the lower Löwenheim-Skolem-Tarski theorem holds (see section 2), but for $\preccurlyeq_s$ as has been proven by W. Guzicki (see [2]), it fails. Guzicki's proof may in fact be reduced to the following:

EXAMPLE. Let $\mathfrak{B}$ be the Boolean algebra of all subsets of the set of real numbers R. Further, let $P(r) = \{r\}$, where $r \in R$ and $\mathfrak{R} = (R, P)$. $\mathfrak{R}$ is a Boolean-valued relational system over the Boolean algebra $\mathfrak{B}$. Let us consider the following sentence:

$$\exists x_0 P(x_0) .$$

The value of this sentence, for any $\mathfrak{h} \in {}^{\omega}R$, is

$$\mathrm{val}_{\mathfrak{R}}^{\mathfrak{B}}(\exists x_0 P(x_0), \mathfrak{h}) = \sup_{\mathfrak{B}}\{\mathrm{val}_{\mathfrak{R}}^{\mathfrak{B}}(P(x_0), \mathfrak{h}') \colon \forall i \;\; i \neq 0 \to \mathfrak{h}_i = \mathfrak{h}_i'\} = R .$$

Now, let $\mathfrak{R}' = (R', P')$ be any proper Boolean subsystem of R; then

$$\mathrm{val}_{\mathfrak{R}'}^{\mathfrak{B}}(\exists x_0 P(x_0), \mathfrak{h}) = R' \subsetneqq R .$$

Thus

$$\mathrm{val}_{\mathfrak{R}'}^{\mathfrak{B}}(\exists x_0 P(x_0), \mathfrak{h}) \neq \mathrm{val}_{\mathfrak{R}}^{\mathfrak{B}}(\exists x_0 P(x_0), \mathfrak{h}) .$$

Moreover, we shall have the same inequality if we replace $\mathfrak{B}$ by any of its extensions. Therefore it is not true that $\mathfrak{R}' \preccurlyeq_s \mathfrak{R}$. There is no proper Boolean subsystem $\mathfrak{R}'$ of $\mathfrak{R}$ such that $\mathfrak{R}' \preccurlyeq_s \mathfrak{R}$.

The connections between different notions of a Boolean elementary subsystem may be visualised by means of the following diagram.

$$\begin{array}{ccc} \mathfrak{S} \precsim^{s}_{s} \mathfrak{S}' & \rightarrow & \mathfrak{S} \precsim_{s} \mathfrak{S}' \\ \downarrow & & \downarrow \not\uparrow \\ \mathfrak{S} \precsim^{s} \mathfrak{S}' & \rightarrow & \mathfrak{S} \precsim \mathfrak{S}' \\ \downarrow & & \downarrow \\ \mathfrak{S} \precsim^{s}_{w} \mathfrak{S}' & \leftrightarrow & \mathfrak{S} \precsim_{w} \mathfrak{S}' \end{array}$$

PROBLEM 1. How to complete the diagram? Which implications hold and which fail?

We have proved the upper and lower Löwenheim-Skolem-Tarski theorems for $\precsim$. Thus from Theorem 5 it follows that both theorems hold for $\precsim_w$ and $\precsim^s_w$. From Guzicki's example it follows that the lower Löwenheim-Skolem-Tarski theorem fails for $\precsim_s$ and therefore for $\precsim^s_s$. However, if we introduce into the lower Löwenheim-Skolem-Tarski theorem some stronger assumptions (see [1] Theorem 4.3.1), it will hold for $\precsim_s$.

PROBLEM 2. Does the lower Löwenheim-Skolem-Tarski theorem hold for $\precsim^s$? Does the upper Löwenheim-Skolem-Tarski theorem hold for $\precsim^s$, $\precsim_s$ and $\precsim^s_s$?

A partial answer to Problem 2 is given in [1] (see page 63). Namely, for any Boolean-valued relational system $\mathfrak{S}$ and any cardinal number μ there is a Boolean-valued relational system $\mathfrak{S}'$ of the power at least μ such that $\mathfrak{S} \precsim_s \mathfrak{S}'$.

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A non-symmetric generalization of the Borsuk-Ulam theorem (*)

by

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Abstract. The following generalization of the well-known Borsuk-Ulam theorem is proved. Theorem: Let X be a compact subset of the Euclidean space R^{n+1} which disconnects R^{n+1} in such a way that the origin is in a bounded component of $R^{n+1} - X$ and let $f\colon X \to R^n$ be a map. Then there exist two points x, y in X, lying on opposite rays from the origin (i.e. $y = -\lambda x$ for some $\lambda > 0$), such that $f(x) = f(y)$. This provides an affirmative answer to a question of Borsuk. The proof is based on P. A. Smith's theory of the index of a periodic transformation acting on a topological space, Yang's result about maps from such spaces to the Euclidean spaces and the technique of approximating the set X by a special class of polyhedra, the so-called "regular polyhedra" defined in the paper. The special cases $n = 1$ or 2 of the theorem were proved earlier by Sieklucki by a different argument.

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1. Introduction. The classic Borsuk-Ulam theorem [2] states that if f is a map from the n-dimensional sphere S^n into the n-dimensional Euclidean space R^n then there exists a pair of antipodal points $\{x, -x\}$ on S^n such that $f(x) = f(-x)$. Several generalizations of this theorem, proceeding in various directions, have been obtained among others by Agoston [1], Jaworowski [8], Yang [14], Granas [6] who extended the result to infinite-dimensional Banach spaces and Munkholm [9] who considered Z_p-actions on a homology sphere for a prime p. In many of

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