corresponding theorem on $A$-spaces. The proof is on similar lines hence it is omitted.

(5.6) Theorem. Every polyhedron (not necessarily finite) with Whitehead topology is a $\mu A$-space.

Let $\mathcal{A}$ be the full subcategory of $\mathcal{J}$ whose objects are metric ANR's and polyhedra (with Whitehead topology).

(5.7) Theorem. The category $\mathcal{A}$ is admissible.

Proof. Since every metric ANR is homotopically dominated by a polyhedron (an object of $\mathcal{J}$), it follows from Theorem (5.3) and Theorem (3.1) that $\mathcal{A}$ is admissible.

In the light of the result of $A$-spaces [4], a similar theorem on $\mu A$-spaces is as follows.

(5.8) Theorem. Every metric ANR is a $\mu A$-space.

On the insertion of Darboux, Baire-one functions

by

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Abstract. If $f$ and $g$ possess the Darboux property and are in the first class of Baire on an interval $I$ and if $f(x) > g(x)$ for all $x \in I$, there exists another Darboux function $h$ also in the first class of Baire, such that $f(x) > h(x) > g(x)$ for all $x$. Certain related statements are also valid.

1. Introduction. Let $f$ and $g$ be two real functions defined on a real interval $I$, each with the Darboux (i.e., intermediate value) property. If $g(x) < f(x)$ for all $x$ in $I$ one can ask whether there exists another Darboux function $h$ such that $g(x) < h(x) < f(x)$ for all $x$ in $I$. This question was answered negatively by Ceder and Weiss in [6]; they found, however, a useful sufficient condition in terms of the way in which $f$ and $g$ are separated by constant functions (see Section 4, below). They showed that this sufficient condition is satisfied when both $f$ and $g$ are in the first class of Baire. They also posed the problem of whether or not there exists a Darboux, Baire-one function between two comparable Darboux, Baire-one functions.

The purpose of this article is to show that the question has an affirmative answer (see Theorem 1). We also show that it is not possible in general to insert a Darboux function between comparable Darboux functions even if one is in the first class of Baire and the other in the second class of Baire. If, however, the first of these functions meets any of a number of additional "regularizing" conditions, such an insertion is always possible. We mention in passing that some extensions of results found in [6] are found in [5].

2. Notation and terminology. The set of real numbers will be denoted by $R$ and $I$ will be a fixed real interval. For a set $A \subseteq R$, $\overline{A}$ and $A'$ will denote the closure and interior of $A$. We will regard a real function as identical with its graph. If $f$ is a function $C(R, R(f))$ will denote the set of bilateral condensation points of $f$ (see [4], Lemma 1), and $C(f)$ will denote the set of $x \in R$ at which $f$ is continuous. Moreover, $F^+(f, a)$ and

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that one can construct partitions \([a_1, a_2, \ldots, a_N]\) of the interval \([a, b] = (a, b) \cup (b, a)\) and \([b_1, b_2, \ldots, b_N]\) of \([a, b]\) with the following properties:

1. For \(1 \leq i < N\), \(a_i \in \bar{B} \cap (f|P) \cap (g|P)\).
2. \(|b_{i+1} - b_i| < \min \left\{ \frac{b - r - r}{3}, \epsilon \right\} \) for all \(i\).
3. \(K_\delta(I) \cap (f|I) \cap (g|I)\).

Let \(A = \{a_1, a_2, \ldots, a_N\}\) be a set of points such that \(a_1 < a_2 < \cdots < a_N\).

To construct the desired intervals \((I_x)\) of \([a, b]\) we will construct such intervals in each subinterval \([a_i, a_{i+1}]\) and juxtapose them in the obvious way.

We fix \((a_1, a_2, \ldots, a_N)\) and let \(\epsilon > 0\) be given. We may pick disjoint closed intervals \((M_k, M_{k+1})\) in \([a_1, a_2]\) having the following properties where \(M_k = [c_k, d_k] = [c_k, c_{k+1}]\):

1. \(d_k < c_{k+1}\) for each \(k\);
2. \(c_k, d_k\) belong to \(\bar{B} \cap (f|P) \cap (g|P)\);
3. \(M_k\) contains only \(f\)-points of \(E\) or only \(g\)-points of \(E\);
4. \(M_k = [c_k, d_k] \supset E\).

Since \(W\) is even and \(M_i\) contains an \(f\)-point or a \(g\)-point, we have four cases to consider.

Case \(a\). \(W\) is even and \(M_i\) contains an \(f\)-point. Then we put
\[
I_1 = [a_1, d_1], I_2 = [d_1, c_1], I_3 = [c_1, d_2], I_4 = [d_2, c_2], I_5 = [c_2, d_3], \ldots
\]
and \(I_{4m-1} = [c_{4m-1}, d_{4m}], I_{4m} = [c_{4m}, d_{4m+1}]\).

Case \(b\). \(W\) is odd and \(M_i\) contains an \(f\)-point. Then we put
\[
I_1 = [a_1, d_1], I_2 = [d_1, c_1], I_3 = [c_1, d_2], I_4 = [d_2, c_2], I_5 = [c_2, d_3], \ldots
\]
and \(I_{4m-1} = [c_{4m-1}, d_{4m}], I_{4m} = [c_{4m}, d_{4m+1}]\).

Case \(c\). \(W\) is even and \(M_i\) contains a \(g\)-point. Then we put
\[
I_1 = [a_1, d_1], I_2 = [d_1, c_1], I_3 = [c_1, d_2], I_4 = [d_2, c_2], I_5 = [c_2, d_3], \ldots
\]
and \(I_{4m-1} = [c_{4m-1}, d_{4m}], I_{4m} = [c_{4m}, d_{4m+1}]\).

Case \(d\). \(W\) is odd and \(M_i\) contains a \(g\)-point. Then we put
\[
I_1 = [a_1, d_1], I_2 = [d_1, c_1], I_3 = [c_1, d_2], I_4 = [d_2, c_2], I_5 = [c_2, d_3], \ldots
\]
and \(I_{4m-1} = [c_{4m-1}, d_{4m}], I_{4m} = [c_{4m}, d_{4m+1}]\).
where $0 < \delta < \delta - \delta_0$ and $\delta$ is small enough so that both $I^1_0$ and $I^2_0$ hit $P$, and where $0 < \eta < \eta - \eta_0$ and $\eta$ is small enough so that $I^3_0$, $I^4_0$, and $P$ all hit $P$, and have then endpoints (except possibly $a_1$) in $P \cap C(f(P) \cup C(g(P))$.

Case 5. $W$ is odd and $M_1$ contains an $f$-point. The construction is similar to those of the other three cases and therefore is omitted.

Obviously parts (1) and (2) of the conclusion of the lemma are satisfied. To show part (3), let $(x, y) \in (f(P) - D(P))$. Then $(x, y) \in A \cap (g(P) = 0)$. Therefore, $(x, y)$ is within $\sqrt{2} \varepsilon$ of $(x_1, y_1)$. Since $x_1$ is an $f$-point we have by construction $x_1 \in \bigcup_{k=0}^{m-1} I_{4k+1}$. Therefore $(x, y)$ is within $\sqrt{2} \varepsilon$ of $(x_1, y_1)$. Likewise part (4) is satisfied, finishing the proof of the lemma.

**Lemma 2.** Let $0 < \epsilon < \frac{\delta - \delta_0}{3}$ and $c$ and $d$ belong to $\tilde{P} \cap C(f(P) \cup C(g(P)))$ with $c < d$. Then there exists a continuous, increasing (or decreasing) function $h$ on $P \cap [c, d]$ such that range $h = [g(c) - \epsilon, f(d) - \epsilon]$ (resp. $[f(c) - \epsilon, g(d) + \epsilon]$).

Proof. The vertical closed line segment joining $(c, g(c) - \epsilon)$ to $(d, c)$ can be covered by finitely many open disks, none of which intersects $(g(P) \cup f(P))$. The same can be said for the closed segment joining $(d, f(d) - \epsilon)$ to $(c, d)$. Letting $T$ be the rectangle with vertices $(d, c), (c, e), (d, e)$ and $(c, c)$, it is clear that within $T$ union these disks one can construct a Cantor-like function on $P \cap [c, d]$ satisfying the required properties.

**Lemma 3.** Let $\delta < 0$. Then there exists a function $h \in \mathcal{B}(P)$ such that $g < h < f$ on $P$ and each point of $\{f(P) \cup g(P)) \cup D(P)$ is within $\delta$ of some point of $h$.

Proof. Clearly one can find a partition of $[a, b] \cap P$ into finitely many non-empty relative, closed intervals of $P$, each a perfect set having endpoints in $P$, each two of which intersect at most once at a common endpoint, and moreover each having a convex hull less than $\min \left(\frac{1}{2}, \frac{\delta - \delta_0}{3}\right)$.

Let $\epsilon = \min\left(\frac{\delta - \delta_0}{3}, \frac{\delta - \delta_0}{3}\right)$ and let $Q$ be any such perfect subset of $P$.

Next, apply Lemma 1 to obtain the intervals $(I^k_0)_{k=0}^{M-1}$ for $Q$ and $\epsilon$.

We then define $h_0$ as follows:

$$h_0(x) = \begin{cases} f(x) - \epsilon & \text{for } x \in Q \cap \bigcup_{k=0}^{M-1} I_{4k+1}, \\ g(x) + \epsilon & \text{for } x \in Q \cap \bigcup_{k=0}^{M-1} I_{4k+1}. \end{cases}$$

Since the endpoints of each $I_{4k+1} = [a_{4k+1}, b_{4k+1}]$ belong to $Q$ we may choose a decreasing continuous function $h_0$ on the perfect set $Q \cup \{a_{4k+1}, b_{4k+1}\}$ as specified by Lemma 2. Similarly, we may choose an increasing continuous function $h_0$ on $Q \cup \{a_{4k+1}, b_{4k+1}\}$ for each $k$ by Lemma 2.

Let $h$ be the union of all such $h_0$ over all such $Q$ clearly, $h \in \mathcal{B}(P)$ and $g < h < f$ on $P$. Moreover, from the construction of $h$ and the properties of $(I^k_0)_{k=0}^{M-1}$, as given by Lemma 1, it is clear that each point of $\{(f(P) \cup g(P)) \cup D(P)$ is within $\delta$ of some point of $h$.

**Lemma 4.** Let $\epsilon > 0$ and let $J = (c, d)$ be an open interval with $c, d \in \tilde{P}$. Then there exists a function $h$ with domain $J \cap X$ such that $h \in \mathcal{B}(J \cap X)$ and $g < h < f$ on $J \cap P$. Moreover, each point of $\{(f(j \cap P) \cup g(J \cap P)) \cup D(P)$ is in $\epsilon$ of some point of $h$ and

$$K^+(h, c) = \inf \{K^+(g((J \cap P) \cup c), c), \sup \{K^+(f((J \cap P) \cup J), d)\} \},$$

$$K^-(h, d) = \inf \{K^+(g((J \cap P) \cup c), c), \sup \{K^+(f((J \cap P) \cup J), d)\} \}.$$

Proof. Let $(c_0)$ and $(d_0)$ be sequences in $J \cap \tilde{P} \cap C(f(P) \cup C(g(P))$ such that $c_{k+1} < c_k < d_k < d_{k+1}$ for each $k$ and $\lim_{k \to \infty} c_k = c$ and $\lim_{k \to \infty} d_k = d$.

Consider $[c_0, c_0] \cap P$. We may modify the proof of Lemma 3 relative to $P \cap [c_0, c_0]$ and $\epsilon$ to obtain a function $h_0 \in \mathcal{B}(P \cap [c_0, c_0])$ with the additional properties that

$$h_0(c_{k+1}) = \begin{cases} f(c_{k+1}) - \epsilon & \text{if } n \text{ is odd}, \\ g(c_{k+1}) + \epsilon & \text{if } n \text{ is even}. \end{cases}$$

and

$$h_0(d_{k+1}) = \begin{cases} g(d_{k+1}) + \epsilon & \text{if } n \text{ is odd}, \\ f(d_{k+1}) - \epsilon & \text{if } n \text{ is even}. \end{cases}$$

Likewise on each $[d_0, d_0] \cap P$ we may obtain a function $h_0$ such that

$$h_0(d_{k+1}) = \begin{cases} g(d_{k+1}) + \epsilon & \text{if } n \text{ is odd}, \\ f(d_{k+1}) - \epsilon & \text{if } n \text{ is even}. \end{cases}$$
The construction of $O_1$, $P_1$ and $h_1$. Applying the above construction, we can find a sequence of non-void open intervals \( \{G_n\}_{n=1}^{\infty} \) and real sequences \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) such that:

1. the endpoints of $G_n$ belong to $C(f) \cap C(g) \cap P_2$,
2. $G_n \cap G_m = \emptyset$ whenever $n \neq m$,
3. \( \bigcup_{n=1}^{\infty} G_n = P_0 \),
4. \( f \) and $g$ are bounded on each $G_n \cap P_0$,
5. $g < a_n < b_n < f$ on $G_n \cap P_0$.

Now put $O_1 = O_0 \cup \bigcup_{n=1}^{\infty} G_n$ and $P_1 = I - O_1$. Obviously $O_1$ is a dense open set and $P_1$ is a non-empty perfect set.

On each $G \cap P$ we construct $h^*$ according to Lemma 4 with respect to $P = P_2$, $J = G_0$ and $\varepsilon = \min \left( \frac{a_n - b_n}{3}, \text{length of } G_0 \right)$. Put $h_1 = h^* \cup \bigcup_{n=1}^{\infty} h_n$.

Since $h_n \in C_0(G_0)$ and $h^* \in C_0(G_0 \cap P)$ it follows that $h_1 \in C_0(G_1)$. It is also clear that $g < h_1 < f$ on $O_1$.

Next we show that $h_1 \in C_0(G_2)$. For this it suffices to show that for each $x \in O_1$, $h_1(x)$ is $K^*(h_1, x)$ and $K^*(h_1, x)$ (see [2]).

This statement is clearly valid whenever $x \in O_0$ since $h_1 \in C_0(G_0)$. So we may assume that $x \not\in O_0 \cap P_0$ for some $n$. We will show that $h_1(x)$ is $K^*(h_1, x)$. The proof that $h_1(x)$ is $K^*(h_1, x)$ is similar. We have two cases to consider:

Case 1. $x$ is the left endpoint of a component $J$ of $O_n$. By the definition of $h_n$ on $J$, $K^*(h_n, x) \supseteq \{f(x), g(x)\}$ (see Lemma 4). Hence $h_1(x) \in [f(x), g(x)] \subseteq K^*(h_0, x) \subseteq K^*(h_1, x)$.

Case 2. $x$ is a left-hand endpoint of a component of $O_n$. From the construction in Lemma 4 of $h_n$ on $O_n \cap P_0$ it follows that $x$ belongs to some $[a_n, b_n) \cap P_0$ where $h_n$ on $[a_n, b_n) \cap P_0$ satisfies one of the following conditions:

- $(a)$: $h_n = f - \varepsilon$ for some $\varepsilon > 0$;
- $(b)$: $h_n = g + \varepsilon$ for some $\varepsilon > 0$;
- $(c)$: $h_n$ is an increasing Cantor function;
- $(d)$: $h_n$ is a decreasing Cantor function.

In cases $(a)$ or $(b)$, since $x$ is a left-hand point of a component of $O_n$ it follows that $h_1(x)$ is a limit point of $h_n(x) \cap P_0$ so that $h_1(x) \in K^*(h_1, x)$.

Suppose that condition $(a)$ holds (the proof for condition $(b)$ is similar). Then there arise two subcases.

Subcase $(a_1)$. There exists a sequence $[x_0, b_n)_{n=1}^{\infty}$ in $(x, b) \cap O_n$ such that $x_n = x$ and $f(x_n) = f(x)$. Without loss of generality we may assume that $[x_0, b_n)_{n=1}^{\infty}$ is a sequence of distinct components of $O_n$ decreasing to $x$ such that $x_n \not\in I_n$ for each $n$. By Lemma 4 there exists a point $[w_0, b_n)$.
of \( h_\lambda I_0 \) whose distance from \( \{ a, f(\{ a \}) \} \) is less than the length of \( I_0 \). Since the length of \( I_0 \) approaches 0, it follows that \( f(x) \in K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \).

Clearly \( \{ a, h_\lambda \{ a \} \} \subset \) range \( h_\lambda I_0 \) so that \( \{ a, f(\{ a \}) \} \subset K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \).

However, \( \epsilon < - \frac{a - r_0}{3} \) and \( s_0 < f(x) \) so that \( \epsilon + r_0 < s_0 + \frac{a - r_0}{3} = s_0 < f(x) \).

Therefore, \( h_\lambda (x) \in K^+(h_\lambda, x) \).

Case (i). \( \beta \) is a non-limit ordinal. Then \( \beta = \alpha + 1 \) for some \( \alpha \). In case \( P_\infty = 0 \), then \( O_\alpha = I \) and \( h_\lambda \) is the desired \( D \omega \) function. So we may assume \( P_\infty \neq 0 \). We construct \( O_\alpha \) just as we constructed \( O_\alpha \), \( P_\alpha \) and \( h_\lambda \). The only essential difference is that in the proof that \( h_\lambda \) is \( D \omega \) one must pick the sequence \( \{ x_\alpha \} \) to avoid the countable set \( \bigcup \bigcup \{ D(P_\alpha) \} \). This is possible since the deletion of a countable set from the domain of a Darboux function doesn’t change the cluster sets.

Case (ii). \( \beta \) is a limit ordinal. Then \( I - \bigcup \{ O_\alpha \} \cap \bigcup \{ P_\alpha \} = \bigcup \{ O_\alpha \} \cap \bigcup \{ P_\alpha \} \bigcup \{ C_\alpha \} \). Put \( P_\infty = I - P_\infty \)

Therefore there exists a perfect set \( P_\infty \) (possibly empty) and a countable set \( C_\alpha \) such that \( C_\alpha \cap P_\infty = 0 \) and \( \bigcup \{ P_\alpha \} \cap \bigcup \{ C_\alpha \} \). Let \( O_\alpha = I - P_\alpha \)

It follows that \( h_\lambda \in D \omega \) and \( h_\lambda < f \in D \omega \).

Let us show that \( h_\lambda \in D \omega \). Let \( x \in O_\alpha \). If \( x \notin O_\alpha \), then \( h_\lambda (x) = h_\lambda (x) \in K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \).

Since \( h_\lambda \in D \omega \), we will show that \( g(x), f(x) \in K^+(h_\lambda, x) \).

By the proof for \( K^+(h_\lambda, x) \) is similar. Let \( f \) be the derivative of \( g(x), f(x) \) is similar. Since \( f \in D \omega \), there exists a sequence \( \{ x_\alpha \} \) in \( J \cap \{ x, \infty \} \) such that \( \{ x_\alpha, f(x_\alpha) \} \subset \bigcup \{ D(P_\alpha) \} 

We may assume that \( x \in O_\alpha \), a component of \( O_\alpha \) where \( a_\alpha < \beta \). If \( x \) is a left endpoint of some \( O_\alpha \) then the proof of Case (i) shows that \( g(x), f(x) \in K^+(h_\lambda, x) \).

Assuming, then, that \( x \) is not the left endpoint of any \( O_\alpha \), we may by Lemma 4 find a sequence \( \{ x_\alpha \} \) in \( \{ x, \infty \} \) such that \( x_\alpha < \beta \).

hence, \( h_\lambda (x) = h_\lambda (x) \in K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \subset K^+(h_\lambda, x) \).

This completes the inductive definition of \( \{ O_\alpha \} \in \omega \), \( \{ P_\alpha \} \in \omega \) and \( \{ h_\lambda \} \in \omega \). Clearly, conditions (1) through (5) of the inductive hypothesis are satisfied. Since \( P_\infty \) is a descending well-ordered chain of closed sets in the real line, there exists a \( \gamma \) such that \( P_\infty = 0 \). Let \( \gamma \) be the least such \( \gamma \). By condition (5) of the inductive hypothesis \( \lambda \) must be a limit ordinal. In this case \( I = \bigcup \{ O_\alpha \} = I = 0 \) and \( h_\lambda \) is the desired \( D \omega \) function inserted between \( f \) and \( g \).

4. Additional results. In this section we consider the possibility of inserting a Darboux function between two comparable functions which are not quite \( D \omega \) functions.

In [6] an example was given of two comparable \( D \omega \) functions admitting no Darboux function between them. We can improve this example to the following.

EXAMPLE. There exist two comparable Darboux functions one in Baire class one and the other in Baire class two, which admit no Darboux function between them.

Let \( g \) be any function in \( D \omega \), which is positive on a dense subset \( A \) of the real line \( R \). Define \( f(x) = 2g(x); \) on \( B \), \( f(x) = 0; \) on \( Z \), \( f \) takes on every positive real number in every interval. We can do this is such a way that \( f \in D \omega \) (see [6]).

Now, if \( h \) is between \( f \) and \( g \), then \( h > 0 \) on \( A \cup Z \) and \( h < 0 \) on \( B \), so \( f \) cannot have the Darboux property.

In the above example \( f \) and \( g \) mesh in such a way as to remove any possibility of inserting a Darboux function between them. Theorem 1 asserts that such behaviour is impossible if both functions belong to \( D \omega \). Theorem 2 states that we can drop all requirements (except, of course, the Darboux property) on one of the functions if we remove, in an appropriate way, some of the pathological behaviour of the other. The additional regularizing hypothesis in the statement of Theorem 2 occurs in a number of cases, some of which are listed in a corollary to the Theorem.
THEOREM 2. Let $f$ and $g$ be Darboux functions such that $g(x) < f(x)$ for all $x$ in $I$. If $g(H \cap C(y))$ is dense in $g(H)$ for every non-degenerate subinterval $H$ of $I$, then there exists a Darboux function $h$ between $f$ and $g$.

Proof. According to Theorem 1 of [6] it suffices to show that for any non-degenerate subinterval $(a, b)$ of $I$ and any number $\lambda$ for which $g(a) < \lambda < f(b)$, the set $\{x \in (a, b): g(x) < \lambda < f(x)\}$ has cardinality $c$. Let $\lambda, a, b$ be as above. By our hypothesis, there is a point $x_1 \in (a, b)$ such that $g(x_1) < \lambda$ and $g$ is continuous at $x_1$. Therefore there exists an open interval in $(a, b)$ on which $g < \lambda$. Let $J$ be a maximal such interval contained in $(a, b]$. Let $a$ be an endpoint of $J$. Then, since $g$ is Darboux, we must have $g(a) = \lambda < f(x)$. Since $f$ is Darboux on $J$ it follows that $\{x \in J: f(x) > \lambda\}$ has cardinality $c$. Therefore, $\{x \in J: g(x) < \lambda < f(x)\}$ has cardinality $c$, completing the proof.

COROLLARY. Let $f$ and $g$ be comparable Darboux functions on $I$. If $g$ meets any of the conditions below, there exists a Darboux function $h$ between $f$ and $g$ on $I$.

1. $g$ possesses Banach’s condition $T_2$ and is of Baire class one.
2. $g$ is continuous except on a denumerable set.
3. $g$ is quasi-continuous in the sense of Kempisty [7].

Proof. In each case, the hypothesis of Theorem 2 is met. (For condition (1), see [1, p. 19]; that conditions (2) and (3) suffice for the hypothesis of Theorem 2 follows directly from the definition of “Darboux” and “quasi-continuity”.)

We note that condition (2) automatically implies that $g \in \mathfrak{B}_1$, but condition (3) does not.

We close by posing the problem of determining necessary and sufficient conditions for inserting a Darboux function between two comparable Darboux functions.

References