corresponding theorem on Λ-spaces. The proof is on similar lines hence it is omitted.

(5.6) THEOREM. *Every polyhedron (not necessarily finite) with Whitehead topology is a $\mu\Lambda$-space.*

Let $\mathfrak{T}\mathcal{A}$ be the full subcategory of $\mathfrak{I}$ whose objects are metric ANR's and polyhedra (with Whitehead topology).

(5.7) THEOREM. *The category $\mathfrak{T}\mathcal{A}$ is admissible.*

Proof. Since every metric ANR is homotopically dominated by a polyhedron (an object of $\mathfrak{T}$), it follows from Theorem (5.3) and Theorem (3.1) that $\mathfrak{T}\mathcal{A}$ is admissible.

In the light of the result of Λ-spaces [4], a similar theorem on $\mu\Lambda$-spaces is as follows.

(5.8) THEOREM. *Every metric ANR is a $\mu\Lambda$-space.*

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On the insertion of Darboux, Baire-one functions

by

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Abstract. If f and g possess the Darboux property and are in the first class of Baire on an interval I and if $f(x) > g(x)$ for all $x \in I$, there exists another Darboux function h, also in the first class of Baire, such that $f(x) > h(x) > g(x)$ for all x. Certain related statements are also valid.

1. Introduction. Let f and g be two real functions defined on a real interval I, each with the Darboux (i.e., intermediate value) property. If $g(x) < f(x)$ for all x in I one can ask whether there exists another Darboux function h such that $g(x) < h(x) < f(x)$ for all x in I. This question was answered negatively by Ceder and Weiss in [6]; they found, however, a useful sufficient condition in terms of the way in which f and g are separated by constant functions (see Section 4, below). They showed that this sufficient condition is satisfied when both f and g are in the first class of Baire. They also posed the problem of whether or not there exists a Darboux, Baire-one function between two comparable Darboux, Baire-one functions.

The purpose of this article is to show that the question has an affirmative answer (see Theorem 1). We also show that it is not possible in general to insert a Darboux function between comparable Darboux functions even if one is in the first class of Baire and the other in the second class of Baire. If, however, the first of these functions meets any of a number of additional "regularizing" conditions, such an insertion is always possible. We mention in passing that some extensions of results found in [6] are found in [5].

2. Notation and terminology. The set of real numbers will be denoted by R and I will be a fixed real interval. For a set $A \subseteq R$, $\bar{A}$ and A^0 will denote the closure and interior of A. We will regard a real function as identical with its graph. If f is a function $\subseteq R^2$, $B(f)$ will denote the set of bilateral condensation points of f (see [4], Lemma 1), and $C(f)$ will denote the set of $x \in R$ at which f is continuous. Moreover, $K^+(f, a)$ and

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$K^-(f, a)$ will designate the right-hand and left-hand cluster sets of f at a (see [2]).

If $G \subseteq I$ and domain $f = G$, then we write (1) $f \in \mathcal{B}_1(G)$ provided $f^{-1}(O)$ is an F_σ set in I for each open set O, and (2) $f \in \mathcal{D}(G)$ if for all $x \in G$, $f(x) \in K^+(f, x) \cap K^-(f, x)$. The set of functions of Baire class one on I will be denoted by $\mathcal{B}_1$ and the set of Darboux functions on I will be denoted by $\mathcal{D}$ and $\mathcal{B}_1 \cap \mathcal{D}$ will be written as $\mathcal{D}\mathcal{B}_1$. Likewise $\mathcal{D}\mathcal{B}_2$ is the class of all Darboux functions of Baire class 2. If $G = I$, then clearly $\mathcal{B}_1(I) = \mathcal{B}_1$. Moreover, if $G = I$, then $\mathcal{D}(I) \cap \mathcal{B}_1(I) = \mathcal{D}\mathcal{B}_1$ (see [2]). A function has (Banach's) property T_2 if almost every level set is countable. (See [8], page 277.)

Throughout the sequel f and g will be functions having domain I such that $g < f$. Associated with f and g and a subset $P \subseteq I$ is the countable set

$$D(P) = [(f|P) - B(f|P)] \cup [(g|P) - B(g|P)].$$

3. The main result. This section is devoted to proving the following result.

THEOREM 1. *Suppose $g(x) < f(x)$ for all x in I and f and g are Darboux, Baire one functions. Then, there exists a Darboux Baire-one function h such that $g(x) < h(x) < f(x)$ for all x in I.*

Before proceeding with the long and involved proof, we mention that the natural candidate for such a function h, namely the average of f and g will not, in general, work. In fact, *any* Baire-one function is the average of two appropriately chosen functions in $\mathcal{D}\mathcal{B}_1$ (see [3]).

Basic to the proof of Theorem 1 is Lemma 4 below which is itself preceded by three lemmas. In the statement of each lemma it is assumed that

I is a closed interval,
f and g belong to $\mathcal{D}\mathcal{B}_1$,
P is a non-empty perfect subset of I having convex hull $[a, b]$,
$\tilde{P}$ consists of the bilateral limit points of P together with a and b,
u, v, r, s are real numbers such that $u < g < r < s < f < v$ on P.

LEMMA 1. *Suppose $b - a < \varepsilon$. Then there exists a partition $[e_1, e_2, \ldots, e_{4m+1}]$ of $[a, b]$ such that if $I_i = [e_i, e_{i+1}]$ the following hold:*

(1) *$I_i^0 \cap P \neq \emptyset$ for all i;*

(2) *$e_i \in \tilde{P} \cap C(f|P) \cup C(g|P)$ for $1 < i < 4m+1$;*

(3) *each point of $(f|P) - D(P)$ is within $\varepsilon\sqrt{2}$ of some point of $f|\big(\bigcup_{j=0}^{m-1} I_{4j+1}\big)$;*

(4) *each point of $(g|P) - D(P)$ is within $\varepsilon\sqrt{2}$ of some point of $g|\big(\bigcup_{j=0}^{m-1} I_{4j+3}\big)$.*

Proof. Since $F - B(F)$ is countable for any function $F \subseteq R^2$ (see [4]), $P - \tilde{P}$ is countable, and $C(F)$ is residual in P whenever $F \in \mathcal{B}_1$, it is clear that one can construct partitions $[a_1, a_2, \ldots, a_N]$ of the interval $\big[a - \frac{1}{2}(\varepsilon - (b-a)), b + \frac{1}{2}(\varepsilon - (b-a))\big]$ and $[b_1, b_2, \ldots, b_N]$ of $[u, v]$ with the following properties:

(1) For $1 < i < N$, $a_i \in \tilde{P} \cap C(f|P) \cap C(g|P)$.

(2) $b_{i+1} - b_i < \min\left(\dfrac{s-r}{3}, \varepsilon\right)$ for all i.

(3) If $R_{ij} = [a_i, a_{i+1}) \times [b_j, b_{j+1})$ intersects $(f|P) - D(P)$ (or $(g|P) - D(P)$), then R_{ij}^0 intersects $(f|P) - D(P)$ (resp. $(g|P) - D(P)$).

Let $\mathcal{A}$ consist of all R_{ij} such that R_{ij}^0 intersects $\big((f|P) \cup (g|P)\big) - D(P)$. For each $A \in \mathcal{A}$ we may pick an $(x_A, y_A) \in A^0 \cap [(f|P) \cup (g|P)] - D(P)$ in such a way that $A_1 \neq A_2$ implies $x_{A_1} \neq x_{A_2}$. We will say that x_A is an *f-point* if $(x_A, y_A) \in f$ and a *g-point* if $(x_A, y_A) \in g$.

To construct the desired intervals $\{I_k\}$ of $[a, b]$ we will construct such intervals in each subinterval $[a_i, a_{i+1}]$ and juxtapose them in the obvious way.

Let us fix $[a_i, a_{i+1}]$ and let $\mathcal{E}$ consist of all those $A \in \mathcal{A}$ of the form $A = [a_i, a_{i+1}) \times [b_j, b_{j+1})$ for some j. Clearly $\mathcal{E} \neq \emptyset$. Now consider the set $E = \{x_A \colon A \in \mathcal{E}\}$. We may pick disjoint closed intervals $\{M_k\}_{k=1}^W$ in (a_i, a_{i+1}) having the following properties where $M_k = [c_k, d_k]$:

(i) $d_k < c_{k+1}$ for each k;

(ii) c_k, d_k belong to $\tilde{P} \cap C(f|P) \cap C(g|P)$;

(iii) each M_k contains only f-points of E or only g-points of E;

(iv) if M_k contains an f-point of E, then M_{k-1} and M_{k+1} contain g-points of E.

Since W is even or odd and M_1 contains an f-point or a g-point, we have four cases to consider.

Case α. W is even and M_1 contains an f-point. Then we put

$$I_1 = [a_i, d_1],\ I_2 = [d_1, c_2],\ I_3 = [c_2, d_2],\ I_4 = [d_2, c_3],\ I_5 = [c_3, d_3], \ldots$$
$$\ldots, I_{4(W/2)-1} = [c_W, d_W],\ I_{4(W/2)} = [c_W, a_{i+1}].$$

Case β. W is odd and M_1 contains an f-point. Then we put

$$I_1 = [a_i, d_1],\ I_2 = [d_1, c_2],\ I_3 = [c_2, d_2],\ I_4 = [d_2, c_3],\ I_5 = [c_3, d_3], \ldots$$
$$\ldots, I_{2W-1} = [c_W, d_W],\ I_{2W} = [d_W, d_W + \delta],\ I_{2W+1} = [d_W + \delta, d_W + 2\delta],$$
$$I_{2W+2} = [d_W + 2\delta, a_{i+1}],$$

where $0 < \delta < a_{i+1} - d_W$ and δ is so small that $I_{2W}^0, I_{2W+1}^0, I_{2W+2}^0$ all hit P and have their endpoints (except a_{i+1}) in $\tilde{P} \cap C(f|P) \cap C(g|P)$.

Case γ. W is even and M_1 contains a g-point. Then we put

$$I_1 = [a_i, d_1 - \delta],\ I_2 = [d_1 - \delta, c_1],\ I_3 = [c_1, d_1],\ I_4 = [d_1, c_2],\ I_5 = [c_2, d_2], \ldots$$
$$\ldots, I_W = [c_W, d_W],\ I_{W+1} = [d_W, d_W + \eta],\ I_{W+2} = [d_W + \eta, d_W + 2\eta],$$
$$I_{W+3} = [d_W + 2\eta, a_{i+1}],$$

where $0 < \delta < d_1 - a_i$ and δ is small enough so that both I_1^0 and I_2^0 hit P, and where $0 < \eta < a_{i+1} - d_W$ and η is small enough so that I_{W+1}^0, I_{W+2}^0 and I^0 all hit P, and have then endpoints (except possibly a_i) in $\widetilde{P} \cap C(f|P) \cap C(g|P)$.

Case ξ. W is odd and M_1 contains an f-point. The construction is similar to those of the other three cases and therefore is omitted.

Obviously parts (1) and (2) of the conclusion of the lemma are satisfied. To show part (3), let $(x, y) \in (f|P) - D(P)$. Then $(x, y) \in$ some $A \in \mathcal{A}$ where $A \cap (g|P) = \emptyset$. Therefore (x, y) is within $\sqrt{2}\varepsilon$ of (x_A, y_A). Since x_A is an f-point we have by construction $x_A \in \bigcup_{k=0}^{m-1} I_{4k+1}$. Therefore (x, y) is within $\sqrt{2}\varepsilon$ of $f|(\bigcup_{k=0}^{m-1} I_{4k+1})$. Likewise part (4) is satisfied, finishing the proof of the lemma.

Lemma 2. *Let $0 < \varepsilon < \frac{s-r}{3}$ and c and d belong to $\widetilde{P} \cap C(f|P) \cap C(g|P)$ with $c < d$. Then there exists a continuous, increasing (or decreasing) function h on $P \cap [c, d]$ such that range $h = [g(c)+\varepsilon, f(d)-\varepsilon]$ (resp. $[f(c)-\varepsilon, g(d)+\varepsilon]$).*

Proof. The vertical closed line segment joining $(c, g(c)+\varepsilon)$ to (c, r) can be covered by finitely many open disks, none of which intersects $(g|P) \cup (f|P)$. The same can be said for the closed segment joining $(d, f(d)-\varepsilon)$ to (d, s). Letting T be the rectangle with vertices (d, s), (c, s), (d, r) and (c, r), it is clear that within T union these disks one can construct a Cantor-like function on $P \cap [c, d]$ satisfying the required properties.

Lemma 3. *Let $\delta < 0$. Then there exists a function $h \in \mathcal{B}_1(P)$ such that $g < h < f$ on P and each point of $((f|P) \cup (g|P)) - D(P)$ is within δ of some point of h.*

Proof. Clearly one can find a partition of $[a, b] \cap P$ into finitely many non-empty relative, closed intervals of P, each a perfect set having endpoints in P, each two of which intersect at most once at a common endpoint, and moreover each having a convex hull of length less than $\min\left(\frac{\delta}{2}, \frac{s-r}{3}\right)$.

Let $\varepsilon = \min\left(\frac{\delta}{2}, \frac{s-r}{3}\right)$ and let Q be any such perfect subset of P. Next, apply Lemma 1 to obtain the intervals $\{I_k\}_{k=1}^{4M}$ for Q and ε.

We then define h_Q as follows:

$$h_Q(x) = \begin{cases} f(x) - \varepsilon & \text{for} \quad x \in Q \cap \bigcup_{k=0}^{M-1} I_{4k+1}, \\ g(x) + \varepsilon & \text{for} \quad x \in Q \cap \bigcup_{k=0}^{M-1} I_{4k+3}. \end{cases}$$

Since the endpoints of each $I_{4k+2} = [a_{4k+2}, b_{4k+2}]$ belong to Q we may choose a decreasing continuous function h_Q on the perfect set $Q \cap [a_{4k+2}, b_{4k+2}]$ as specified by Lemma 2. Similarly, we may choose an increasing continuous function h_Q on $Q \cap [a_{4k}, b_{4k}]$ for each k by Lemma 2.

Let h be the union of all such h_Q over all such Q. Clearly, $h \in \mathcal{B}_1(P)$ and $g < h < f$ on P. Moreover, from the construction of h and the properties of $\{I_k\}_{k=1}^{4M}$ as given by Lemma 1, it is clear that each point of $((f|P) \cup (g|P)) - D(P)$ is within δ of some point of h.

Lemma 4. *Let $\varepsilon > 0$ and let $J = (c, d)$ be an open interval with $c, d \in \widetilde{P}$. Then there exists a function h with domain $J \cap P$ such that $h \in \mathcal{B}_1(J \cap P)$ and $g < h < f$ on $J \cap P$. Moreover, each point of $((f|(J \cap P)) \cup (g|(J \cap P))) - D(P)$ is within ε of some point of h and*

$$K^+(h, c) = [\inf K^+(g|(J \cap P), c), \sup K^+(f|(J \cap P), c)],$$

$$K^-(h, d) = [\inf K^-(g|(J \cap P), c), \sup K^-(f|(J \cap P), d)].$$

Proof. Let $\{c_n\}$ and $\{d_n\}$ be sequences in $J \cap \widetilde{P} \cap C(f|P) \cap C(g|P)$ such that $c_{k+1} < c_k < c_1 = d_1 < d_k < d_{k+1}$ for each k and $\lim_{k\to\infty} c_k = c$ and $\lim_{k\to\infty} d_k = d$.

Consider $[c_{n+1}, c_n] \cap P$. We may modify the proof of Lemma 3 relative to $P \cap [c_{n+1}, c_n]$ and ε to obtain a function $h_n \in \mathcal{B}_1(P \cap [c_{n+1}, c_n])$ with the additional properties that

$$h_n(c_{n+1}) = \begin{cases} f(c_{n+1}) - \dfrac{\varepsilon}{n+1} & \text{if } n \text{ is odd}, \\ g(c_{n+1}) + \dfrac{\varepsilon}{n+1} & \text{if } n \text{ is even} \end{cases}$$

and

$$h_n(c_n) = \begin{cases} g(c_n) + \dfrac{\varepsilon}{n} & \text{if } n \text{ is odd}, \\ f(c) - \dfrac{\varepsilon}{n} & \text{if } n \text{ is even}. \end{cases}$$

Likewise on each $[d_n, d_{n+1}] \cap P$ we may obtain a function k_n such that

$$k_n(d_n) = \begin{cases} g(d_n) + \dfrac{\varepsilon}{n} & \text{if } n \text{ is odd}, \\ f(d_n) - \dfrac{\varepsilon}{n} & \text{if } n \text{ is even} \end{cases}$$

and

$$k_n(d_{n+1}) = \begin{cases} f(d_{n+1}) = \dfrac{\varepsilon}{n+1} & \text{if } n \text{ is odd}, \\ g(d_{n+1}) + \dfrac{\varepsilon}{n+1} & \text{if } n \text{ is even}. \end{cases}$$

Now put $h = (\bigcup_{n=1}^{\infty} h_n) \cup (\bigcup_{n=1}^{\infty} k_n)$ to obtain the desired function h.

Having established the four preliminary lemmas we proceed with the proof of the theorem. First of all, we may assume without loss of generality that the domain interval I of f and g is a finite closed interval.

The outline of the proof is as follows: for each $\alpha < \Omega$, the first uncountable ordinal, we construct sets O_α, P_α and h_α with the following properties:

(1) $I - O_\alpha = P_\alpha$,

(2) O_α is a dense open (relative to I) set and P_α is a perfect set,

(3) $\{O_\alpha\}_{\alpha<\Omega}$ is an ascending chain and $\{P_\alpha\}_{\alpha<\Omega}$ is a descending chain,

(4) $P_\xi \neq \emptyset$ implies $P_{\xi+1} \neq \emptyset$ and $P_{\xi+1} \neq P_\xi$,

(5) h_α is a function with domain O_α such that $h_\alpha \in \mathfrak{D}(O_\alpha) \cap \mathcal{B}_1(O_\alpha)$ and $g < h_\alpha < f$ on O_α.

Having done this, we show that for some limit ordinal λ, $O_\lambda = I$ so that h_λ is the desired $\mathfrak{D}\mathcal{B}_1$ function inserted between f and g.

The construction of O_0, P_0 and h_0. First we observe that if Q is any perfect set and $x \in C(f|Q) \cap C(g|Q)$, then there exists an open interval G containing x and real numbers r and s such that f and g are bounded on $G \cap Q$ and $g < r < s < f$ on $G \cap Q$.

Using this fact, we may construct a sequence of non-void open intervals $\{G_n\}_{n=1}^{\infty}$ and sequences of reals $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ such that

(1) the endpoints of G_n belong to $C(f) \cap C(g)$,

(2) $\overline{G}_n \cap \overline{G}_m = \emptyset$ whenever $n \neq m$,

(3) $\bigcup_{n=1}^{\infty} G_n = I$,

(4) f and g are bounded on each G_n,

(5) $g < r_n < s_n < f$ on G_n.

Then putting $O_0 = \bigcup_{n=1}^{\infty} G_n$ and $P_0 = I - O_0$, O_0 will be a dense open subset of I and P_0 will be a non-void perfect subset of I.

For each n we may choose by Lemma 4 a function h^n satisfying the conclusion of Lemma 4 relative to the values $J = G_n$, $P = I$, and $\varepsilon = \min\left(\dfrac{s_n - r_n}{3}, \text{length of } G_n\right)$. Putting $h_0 = \bigcup_{n=1}^{\infty} h^n$ it is easy to verify that $h_0 \in \mathfrak{D}(O_0) \cap \mathcal{B}_1(O_0)$ and $g < h_0 < f$ on O_0.

The construction of O_1, P_1 and h_1. Applying the above construction, we can find a sequence of non-void open intervals $\{G_n\}_{n=1}^{\infty}$ and real sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ such that

(1) the end points of G_n belong to $C(f|P_0) \cap C(g|P_0)$,

(2) $\overline{G}_n \cap \overline{G}_m = \emptyset$ whenever $n \neq m$,

(3) $\bigcup_{n=1}^{\infty} (G_n \cap P_0) = P_0$,

(4) f and g are bounded on each $G_n \cap P_0$,

(5) $g < r_n < s_n < f$ on $G_n \cap P_0$.

Now put $O_1 = O_0 \cup (\bigcup_{n=1}^{\infty} G_n)$ and $P_1 = I - O_1$. Obviously O_1 is a dense open set and P_1 is a non-empty perfect set.

On each $G \cap P$ we construct h^n according to Lemma 4 with respect to $P = P_0$, $J = G_n$ and $\varepsilon = \min\left(\dfrac{s_n - r_n}{3}, \text{length of } G_n\right)$. Put $h_1 = h_0 \cup (\bigcup_{n=1}^{\infty} h^n)$.

Since $h_0 \in \mathcal{B}_1(O_0)$ and $h^n \in \mathcal{B}_1(G_n \cap P)$ it follows that $h_1 \in \mathcal{B}_1(O_1)$. It is also clear that $g < h_1 < f$ on O_1.

Next we show that $h_1 \in \mathfrak{D}(O_1)$. For this it suffices to show that for each $x \in O_1$, $h_1(x) \in K^+(h_1, x) \cap K^-(h_1, x)$ (see [2]).

This statement is clearly valid whenever $x \in O_0$ since $h_1 \in \mathfrak{D}(O_0)$. So we may assume that $x \in G_n \cap P_0$ for some n. We will show that $h_1(x) \in K^+(h_1, x)$. The proof that $h_1(x) \in K^-(h_1, x)$ is similar. We have two cases to consider:

Case I. x is the left endpoint of a component J of O_0. By the definition of h_0 on J, $K^+(h_0, x) \supseteq [f(x), g(x)]$ (see Lemma 4). Hence $h_1(x) \in [f(x), g(x)] \subset K^+(h_0, x) \subseteq K^+(h_1, x)$.

Case II. x is not the left-hand endpoint of a component of O_0. From the construction in Lemma 4 of h_1 on $G_n \cap P_0$ it follows that x belongs to some $[a, b) \cap P_0$ where h_1 on $[a, b) \cap P_0$ satisfies one of the following conditions: (α) $h_1 = f - \varepsilon$, for some $\varepsilon > 0$; (β) $h_1 = g + \varepsilon$, for some $\varepsilon > 0$; (γ) h_1 is an increasing Cantor function; (ξ) h_1 is a decreasing Cantor function.

In cases (γ) or (ξ), since x is not a left-hand point of a component of O_0 it follows that $(x, h_1(x))$ is a limit point of $h_1|(x, b) \cap P_0$ so that $h_1(x) \in K^+(h_1, x)$.

Suppose that condition (α) holds (the proof for condition (β) is similar). Then there arise two subcases.

Subcase (α_1). There exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $(x, b) \cap O_0$ such that $x_k \to x$ and $f(x_k) \to f(x)$. Without loss of generality we may assume that $\{I_k\}_{k=1}^{\infty}$ is a sequence of distinct components of O_0 decreasing to x such that $x_k \in I_k$ for each n. By Lemma 4 there exists a point $(w_k, h_0(w_k))$

of $h_0|I_k$ whose distance from $(x_k, f(x_k))$ is less than the length of I_k. Since the length of I_k approaches 0, it follows that $f(x) \in K^+(h_0, x) \subseteq K^+(h_1, x)$.

Clearly $[r_n, h_0(w_k)] \subseteq \operatorname{range} h_0|I_k$ so that $[r_n, f(x)] \subseteq K^+(h_0, x) \subseteq K^+(h_1, x)$. However, $\varepsilon < \frac{s_n - r_n}{3}$ and $s_n < f(x)$ so that $\varepsilon + r_n < r_n + \frac{s_n - r_n}{3} < s_n < f(x)$ and $r_n < f(x) - \varepsilon = h_1(x)$. Therefore, $h_1(x) \in K^+(h_1, x)$.

Subcase (α_2). There exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $(x, b) \cap P_0$ such that $x_k \to x$ and $f(x_k) \to f(x)$. Since $h_1(x) = f(x) - \varepsilon$ on $[x, b) \cap P_0$, we have $h_1(x_n) = f(x_n) - \varepsilon \to f(x) - \varepsilon = h_1(x)$. Therefore, $h_1(x) \in K^+(h_1, x)$.

Construction of O_α, P_α and h_α with $\alpha > 2$. For this we proceed by induction: assume that we have defined O_β, P_β and h_β for each $\beta < \alpha$ to satisfy the inductive hypotheses (1) through (5). We have two cases to consider:

Case (i). β is a non-limit ordinal. Then $\beta = \alpha + 1$ for some α. In case $P_\alpha = \emptyset$, then $O_\alpha = I$ and h_α is the desired $\mathfrak{D}\mathcal{B}_1$ function. So we may assume that $P_\alpha \neq \emptyset$. We construct $O_{\alpha+1}$, $P_{\alpha+1}$ and $h_{\alpha+1}$ just as we constructed O_1, P_1 and h_1. The only essential difference is that in the proof that $h_{\alpha+1} \in \mathfrak{D}(O_{\alpha+1})$ one must pick the sequence $\{x_k\}_{k=1}^{\infty}$ to avoid the countable set $\bigcup_{\beta<\alpha} D(P_\beta)$. This is possible since the deletion of a countable set from the domain of a Darboux function does not change the cluster sets.

Case (ii). β is a limit ordinal. Then $I - \bigcup_{\alpha<\beta} O_\alpha = \bigcap_{\alpha<\beta} P_\alpha$ is closed. Therefore there exists a perfect set P_β (possibly empty) and a countable set C_β such that $C_\beta \cap P_\beta = \emptyset$ and $\bigcap_{\alpha<\beta} P_\alpha = P_\beta \cup C_\beta$. Now put $O_\beta = I - P_\beta = (\bigcup_{\alpha<\beta} O_\alpha) \cup C_\beta$. Define h_β to be h_α on O_α and on the set C_β let h_β be $\frac{f+g}{2}$. Obviously $h_\beta \in \mathcal{B}_1(O_\beta)$ and $g < h_\beta < f$ on O_β.

Let us show that $h_\beta \in \mathfrak{D}(O_\beta)$. Let $x \in O_\beta$. If $x \in O_\alpha$ for some $\alpha < \beta$, then

$$h_\beta(x) = h_\alpha(x) \in K^+(h_\alpha, x) \cap K^-(h_\alpha, x) \subset K^+(h_\beta, x) \cap K^-(h_\beta, x)$$

since $h_\alpha \in \mathfrak{D}(O_\alpha)$.

So let us assume that $x \in C_\beta$. We will show that $[g(x), f(x)] \subseteq K^+(h_\beta, x)$; the proof for $K^-(h_\beta, x)$ is similar. Let J be the component of O_β in which x lies and let $\lambda \in K^+(f, x)$. Since $f \in \mathfrak{D}\mathcal{B}_1$ there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $J \cap (x, \infty) - C_\beta$ such that $(x_k, f(x_k)) \notin \bigcup_{\alpha<\beta} D(P_\beta)$ and $(x_k, f(x_k)) \to (x, \lambda)$. We may assume that $x_k \in G_{\alpha_k}$, a component of O_{α_k} where $\alpha_k < \beta$. If x is a left endpoint of some O_{α_k}, then the proof of Case (i) shows that $[g(x), f(x)] \subseteq K^+(h_\beta, x)$. Assuming, then, that the x is not the left endpoint of any O_{α_k}, we may by Lemma 4 find a sequence $\{w_n\}_{n=1}^{\infty}$ where $w_n \in O_{\alpha_n}$ such that the distance from $(w_n, h_\beta(w_n))$ to $f|G_{\alpha_n}$ is less than the length of G_{α_n}. Therefore $h_\beta(x_n) \to \lambda$ since $\lim_{n\to\infty} \operatorname{length} G_{\alpha_n} = 0$. Thus $K^+(f, x) \subseteq K^+(h_\beta, x)$. Similarly, $K^+(g, x) \subseteq K^+(h_\beta, x)$.

Moreover, we know that $\operatorname{range} h_\beta|G_{\alpha_n} \supseteq [\inf g|G_{\alpha_n}, \sup f|G_{\alpha_n}]$. It follows that all points "between" $K^+(f, x)$ and $K^+(g, x)$ also belong to $K^+(h_\beta, x)$. Hence,

$$h_\beta(x) = \frac{f(x)+g(x)}{2} \in [f(x), g(x)] \subseteq K^+(h_\beta, x).$$

This completes the inductive definition of $\{O_\alpha\}_{\alpha<\Omega}$, $\{P_\alpha\}_{\alpha<\Omega}$ and $\{h_\alpha\}_{\alpha<\Omega}$. Clearly, conditions (1) through (5) of the inductive hypothesis are satisfied. Since $\{P_\alpha\}_{\alpha<\Omega}$ is a descending well-ordered chain of closed sets in the real line, there exists a γ such that $P_\gamma = \emptyset$. Let λ be the least such γ. By condition (5) of the inductive hypothesis λ must be a limit ordinal. In this case $I - O_\lambda = P_\lambda = \emptyset$ and $O_\lambda = I$ and h_λ is the desired $\mathfrak{D}\mathcal{B}_1$ function inserted between f and g.

4. Additional results. In this section we consider the possibility of inserting a Darboux function between two comparable functions which are not quite $\mathfrak{D}\mathcal{B}_1$ functions.

In [6] an example was given of two comparable $\mathfrak{D}\mathcal{B}_2$ functions admitting no Darboux function between them. We can improve this example to the following.

EXAMPLE. *There exist two comparable Darboux functions, one in Baire class one and the other in Baire class two, which admit no Darboux function between them.*

Let g be any function in $\mathfrak{D}\mathcal{B}_1$ which is positive on a dense subset A of the real line R and negative on another dense set B. (For example, g can be taken to be the derivative of a nowhere monotone differentiable function.) Then $g = 0$ on a dense set Z of type G_δ. Define f as follows: On A, $f(x) = 2g(x)$; on B, $f(x) = 0$; on Z, f takes on every positive real number in every interval. We can do this is such a way that $f \in \mathcal{B}_2$ (see [6]). Now, if h is between f and g, then $h > 0$ on $A \cup Z$ and $h < 0$ on B, so f cannot have the Darboux property.

In the above example f and g mesh in such a way as to remove any possibility of inserting a Darboux function between them. Theorem 1 asserts that such behaviour is impossible if both functions belong to $\mathfrak{D}\mathcal{B}_1$. Theorem 2 states that we can drop all requirements (except, of course, the Darboux property) on one of the functions if we remove, in an appropriate way, some of the pathological behaviour of the other. The additional regularizing hypothesis in the statement of Theorem 2 occurs in a number of cases, some of which are listed in a corollary to the Theorem.

THEOREM 2. *Let f and g be Darboux functions such that $g(x) < f(x)$ for all x in I. If $g(H \cap C(g))$ is dense in $g(H)$ for every non-degenerate subinterval H of I, then there exists a Darboux function h between f and g.*

Proof. According to Theorem 1 of [6] it suffices to show that for any non-degenerate subinterval (x_1, x_2) of I and any number λ for which $g(x_1) < \lambda < f(x_2)$, the set $\{x \in (x_1, x_2): g(x) < \lambda < f(x)\}$ has cardinality c. Let, then, λ, x_1 and x_2 be as above. By our hypothesis, there is a point $x_3 \in (x_1, x_2)$ such that $g(x_3) < \lambda$ and g is continuous at x_3. Therefore there exists an open interval in (x_1, x_2) on which $g < \lambda$. Let J be a maximal such interval contained in $[x_1, x_2]$. Let a be an endpoint of J. Then, since g is Darboux, we must have $g(a) = \lambda < f(x)$. Since f is Darboux on J it follows that $\{x \in J: f(x) > \lambda\}$ has cardinality c. Therefore, $\{x \in J: g(x) < \lambda < f(x)\}$ has cardinality c, completing the proof.

COROLLARY. *Let f and g be comparable Darboux functions on I. If g meets any of the conditions below, there exists a Darboux function h between f and g on I.*

(1) *g possesses Banach's condition T_2 and is of Baire class one.*

(2) *g is continuous except on a denumerable set.*

(3) *g is quasi-continuous in the sense of Kempisty* [7].

Proof. In each case, the hypothesis of Theorem 2 is met. (For condition (1), see [1, p. 19]; that conditions (2) and (3) suffice for the hypothesis of Theorem 2 follows directly from the definition of "Darboux" and "quasi-continuity".)

We note that condition (2) automatically implies that $g \in \mathcal{B}_1$, but condition (3) does not.

We close by posing the problem of determining necessary and sufficient conditions for inserting a Darboux function between two comparable Darboux functions.

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