Fixed points of symmetric product mappings of polyhedra and metric absolute neighborhood retracts

by

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Abstract. Let X be a topological space and X^n the nth cartesian product. Let G be a group of permutation of the numbers $[1, 2, \ldots, n]$. The group G acts on X^n as a group of homeomorphism by defining, for $g \in G$ and $(x_1, x_2, \ldots, x_n) \in X^n$, $g(x_1, x_2, \ldots, x_n) = (x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})$. The orbit space under the action G is denoted by X^n/G (C. N. Maxwell, *Fixed points of symmetric product mappings*, Proc. Amer. Math. Soc. 8 (1957), pp. 808–815). A point $x \in X$ is called a *fixed point of the symmetric product map*, f: $X \to X^n/G$, if x is a coordinate of $f(x)$. In this paper Lefschetz number $L(f)$ of the map f in the case when X is an absolute neighborhood retract is defined and it is proved that if $L(f) \neq 0$ then f has a fixed point. An outline is drawn to extend the result in the case when X is a polyhedron (not necessarily finite) or a metric absolute neighborhood retract and f is a compact map. A complete proof of the latter can be found in the author's dissertation at Indiana University, Bloomington, Indiana.

1. Introduction. Let G be a group of permutations of the numbers $[1, 2, \ldots, n]$. Let X be a topological space and X^n the nth cartesian product of X with product topology. The group G acts on X^n as a group of homeomorphisms by defining, for $(x_1, x_2, \ldots, x_n) \in X^n$ and $g \in G$, $g(x_1, x_2, \ldots, x_n) = (x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})$. The orbit space with identification topology is called the *G-product of X* and it is denoted by X^n/G. Let η: $X^n \to X^n/G$ be the identification map.

A map f: $X \to X^n/G$ is called a *symmetric product map of X*. A point $x \in X$ is said to be a *fixed point of the map f*, if for $(x_1, x_2, \ldots, x_n) \in X^n$ and $\eta(x_1, x_2, \ldots, x_n) = f(x)$ implies that $x = x_i$ for some i, $i = 1, 2, \ldots, n$.

C. N. Maxwell defined the Lefschetz number $L(f)$ of a map f: $X \to X^n/G$, when X is a compact polyhedron, and showed that $L(f) \neq 0$ implies that f has a fixed point [5].

In this paper the results of Maxwell [5] are generalized in the case when

(1) X is a compact ANR (Absolute Neighborhood Retract).

(2) X is a polyhedron (not necessarily finite) with Whitehead topology and f: $X \to X^n/G$ is a compact map.

(3) X is a metric ANR and f is a compact map. The proofs are, however, given only in the case when X is a compact ANR.

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2. Let X be a metric space with metric d and X^n has the usual Euclidean metric, denoted by d'. A metric $\tilde{d}$ on X^n/G can be defined as follows:

$$\tilde{d}\big(\eta(z), \eta(z')\big) = \operatorname{Inf}\{d'(z, gz')|\ g \in G\}$$

where $z, z' \in X^n$. There exists a real valued function ω on $X \times X^n/G$ defined by

$$\omega\big(x, \eta(z)\big) = \operatorname{Inf}\{d\big(x, \pi_i(z)\big)|\ i = 1, 2, \ldots, n\},$$

where $x \in X$, $z \in X^n$ and π_i: $X^n \to X$ is the projection onto the ith factor. The function ω satisfies the following inequality:

$$\omega(x, y) \leqslant \omega(x, y') + \tilde{d}(y, y')$$

where $x \in X$ and $y, y' \in X^n/G$ [5].

Let $\mathfrak{J}$ be the category of all topological spaces and homotopy classes of maps between topological spaces. For a map f: $X \to Y$, let $[f]$ denote the homotopy class of f. Let f^n: $X^n \to Y^n$ and $\tilde{f}$: $X^n/G \to Y^n/G$ be the maps induced by f, i.e.,

$$f^n(x_1, x_2, \ldots, x_n) = \big(f(x_1), f(x_2), \ldots, f(x_n)\big) \quad \text{and} \quad \eta^Y f^n = \tilde{f}\eta^X$$

for $(x_1, x_2, \ldots, x_n) \in X^n$, where η^X, η^Y are identification maps.

Let Γ, Γ_G: $\mathfrak{J} \to \mathfrak{J}$ be defined as follows: for an object X of and a map f: $X \to Y$,

$$\Gamma(X) = X^n,\ \Gamma([f]) = [f^n] \quad \text{and} \quad \Gamma_G(X) = X^n/G,\ \Gamma_G([f]) = [\tilde{f}]\,.$$

Then Γ, Γ_G are covariant functors on the category $\mathfrak{C}$.

Let H_* be the singular homology theory with coefficients in the rational field. Let η_*: $H_*\Gamma \Rightarrow H_*\Gamma_G$ and Π_*: $H_*\Gamma \Rightarrow H_*$ be the natural transformations defined by the identification map and the projections as follows: for an object X of $\mathfrak{J}$,

$$\eta_* = H_*(\eta^X) \quad \text{and} \quad \Pi_* = \sum_{i=1}^{n} H_*(\pi_i) = \sum_{i=1}^{n} \pi_{i*}$$

where η^X: $X^n \to X^n/G$ is the identification map and π_i^X: $X^n \to X$, $i = 1, 2, \ldots, n$, are the projections.

Let $\mathfrak{CP}$ be the full subcategory of $\mathfrak{J}$ whose objects are compact polyhedra. Maxwell showed the existence of a natural transformation μ: $H_*\Gamma_G \Rightarrow H_*$ on the category $\mathfrak{CP}$ such that $\mu\eta_* = \Pi_*$.

Let X be a compact polyhedron and f: $X \to X^n/G$ a map, then the Lefschetz number $L(f)$ of the map f is defined by

$$L(f) = \sum_{p=0}^{\infty} (-1)^p \operatorname{Trace}(\mu_p^X f_{*p})$$

where, μ_p^X: $H_p(X^n/G) \to H_p(X)$, is defined by the natural transformation μ and f_{*p}: $H_p(X) \to H_p(X^n/G)$ is the homology homomorphism induced by the map f.

(2.1) Theorem [5]. *If X is a compact polyhedron and f: $X \to X^n/G$ a map such that $L(f) \neq 0$, then f has a fixed point.*

The above results were proved by Maxwell for simplicial homology theory, however, it is easy to show that they hold for singular homology theory with rational coefficients.

3. Fixed point theorem of symmetric product mappings of compact ANR's. Let $\mathfrak{CA}$ be the full subcategory of $\mathfrak{J}$ whose objects are compact ANR's. The category $\mathfrak{CP}$ is a full subcategory of $\mathfrak{CA}$.

(3.1) Theorem. *Let $\mathfrak{C}$ be a subcategory of $\mathfrak{J}$ and $\mathfrak{B}$ be a fullsubcategory of $\mathfrak{J}$ such that $\mathfrak{B} \subset \mathfrak{C}$. If every object of $\mathfrak{C}$ is homotopically dominated by some object of $\mathfrak{B}$, then every natural transformation,*

$$\mu\colon H_*\Gamma_G \Rightarrow H_*\,, \quad \text{such that } \mu\eta_* = \Pi_*$$

defined on $\mathfrak{B}$, has an extension to the category $\mathfrak{C}$, i.e., there exists a natural transformation μ: $H_*\Gamma_G \Rightarrow H_*$ on $\mathfrak{C}$ which coincides with the given μ on $\mathfrak{B}$.

Proof. Let X be an object of $\mathfrak{C}$. Let Y be an object of $\mathfrak{B}$ and p: $X \to Y$, q: $Y \to X$ be the maps such that $qp \simeq 1_X$. Define

$$\mu^X\colon H_*(X^n/G) \to H_*(X)$$

as the composition of

$$H_*(X^n/G) \xrightarrow{\tilde{p}_*} H_*(Y^n/G) \xrightarrow{\mu^Y} H_*(Y) \xrightarrow{q_*} H_*(X)\,.$$

We claim that μ^X defined above is independent of the choice of Y and the maps q and p and that it defines a natural transformation on $\mathfrak{C}$.

Let X' be another object of $\mathfrak{C}$ and f: $X \to X'$ be a map such that $[f]$ is a morphism in $\mathfrak{C}$. Let Y' be an object of $\mathfrak{B}$ and p': $X' \to Y'$, q': $Y' \to X'$ be the maps such that $q'p' \simeq 1_{X'}$. Consider the following diagram

$$\begin{array}{ccccccc}
H_*(Y^n/G) & \xrightarrow{\tilde{q}_*} & H_*(X^n/G) & \xrightarrow{\tilde{f}_*} & H_*(X'^n/G) & \xrightarrow{\tilde{p}_*} & H_*(Y'^n/G) \\
\downarrow{\scriptstyle \mu^Y} & & \downarrow{\scriptstyle \mu^X} & & \downarrow{\scriptstyle \mu^{X'}} & & \downarrow{\scriptstyle \mu^{Y'}} \\
H_*(Y) & \xrightarrow[q_*]{} & H_*(X) & \xrightarrow[f_*]{} & H_*(X') & \xrightarrow[p_*]{} & H_*(Y')
\end{array}$$

where, $\mu^X = q_*\mu^Y\tilde{p}_*$ and $\mu^{X'} = q'_*\mu^{Y'}\tilde{p}'_*$.

Since $p'fq$: $Y \to Y'$ is a map of objects of $\mathfrak{B}$ and $\mathfrak{B}$ is a full subcategory of $\mathfrak{J}$, it follows from the naturality of μ on $\mathfrak{B}$ that $\mu^{Y'}\tilde{p}_*\tilde{f}_*\tilde{q}_* = p'_*f_*q_*\mu^Y$.

Since $qp \simeq 1_X$ and $q'p' \simeq 1_{X'}$ it follows that $\tilde{q}\tilde{p} \simeq (1_{X^n/G}$ and $\tilde{q}\tilde{p} \simeq 1_{X'^n/G}$. Hence $q'_*p'_* = (1_{X'})_*$ and $\tilde{q}_*\tilde{p}_* = (1_{X^n/G})_*$. It follows that

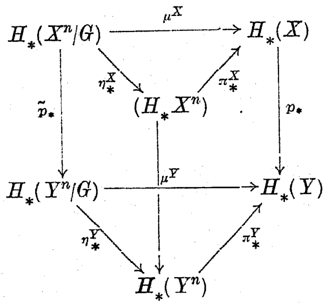
$$q'_*\mu^Y\tilde{p}'_*\tilde{f}_*\tilde{q}_*\tilde{p}_* = q'_*p'_*f_*q_*\mu^Y p_*$$

i.e.,

$$\mu^X f_* = f_*\mu^{X'}.$$

It follows from the above equation that μ^X is independent of the choice of Y and defines a natural transformation on $\mathcal{C}$.

To show that $\mu\eta_* = \Pi_*$ holds on $\mathcal{C}$, we consider the following diagram.



It is clear from the naturality of μ and the definitions of p^n and $\tilde{p}$ that all sides of the above diagram commute except possibly the top triangle. The commutativity of the top triangle follows from the commutativity of the other faces and the fact that p_* is a monomorphism. Hence $\mu^X\eta^X_* = \Pi^X_*$. Q.E.D.

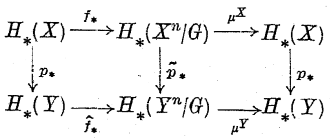
(3.2) Corollary. *The natural transformation μ of section two defined on the category $\mathcal{CF}$ has an extension $\mu\colon H_*\Gamma_G \Rightarrow H_*$ to the category $\mathcal{CA}$ such that $\mu\eta_* = \Pi_*$.*

Proof. Follows from Lemma (3.3) and the fact that every ANR is homotopically dominated by a compact polyhedron [1].

Since the homology of a compact ANR is of finite type, it is possible to define the Lefschetz number of a map $f\colon X \to X^n/G$, when X is a compact ANR, as in section two.

(3.3) Lemma. *Let X be a compact ANR and $f\colon X \to X^n/G$ be a given map. Let Y be a compact polyhedron and $q\colon Y \to X$, $p\colon X \to Y$ be maps such that $qp \simeq 1_X$. If $\hat{f} = \tilde{p}fq\colon Y \to Y^n/G$, then $L(f) = L(\hat{f})$.*

Proof. Consider the following diagram



Clearly the above diagram commutes. It follows by property of Trace function that

$$\begin{aligned}\mathrm{Trace}(\mu^Y_m\hat{f}_{*m}) &= \mathrm{Trace}(\mu^Y_m\tilde{p}_{*m}f_{*m}q_{*m})\\ &= \mathrm{Trace}(p_{*m}\mu^X_m f_{*m}q_{*m})\\ &= \mathrm{Trace}(\mu^X_m f_{*m}q_{*m}p_{*m})\\ &= \mathrm{Trace}(\mu^X_m f_{*m}) \qquad (q_*p_* = 1).\end{aligned}$$

Hence $L(f) = L(\hat{f})$.

(3.4) Definition [1]. Let X and Y be metric space. A homotopy $h\colon X\times I \to Y$ is said to be an *ε-homotopy*, $\varepsilon > 0$, if $d(h(x,t),t) < \varepsilon$ for all $x \in X$ and $t \in I$.

(3.5) Theorem. *If X is a compact ANR and $f\colon X \to X^n/G$ a map such that $L(f) \neq 0$, then f has a fixed point.*

Proof. Suppose f has no fixed point. Then $\omega(x, f(x)) \neq 0$ for all $x \in X$. Since X is compact, there exists a number $\delta > 0$ such that $\omega(x, f(x)) > \delta$ for all $x \in X$.

There exists a compact polyhedron Y and maps $q\colon Y \to X$, $p\colon X \to Y$ such that $qp \simeq 1_X$ through a $\delta/\sqrt{n}$-homotopy [3]. It follows that $\tilde{q}\tilde{p} \simeq 1_{X^n/G}$ through a δ-homotopy. Let $\hat{f} = \tilde{p}fq$. By Lemma (3.3) it follows that $L(f) = L(\hat{f})$. By Theorem (2.1) $L(f) \neq 0$ implies that f has a fixed point. Let $y \in Y$ be a fixed point of f. Let $z = (z_1, z_2, \ldots, z_n) \in X^n$ such that $\eta^Y(z) = \hat{f}(y)$, then it follows that $y = z_i$ for some i, $1 \leqslant i \leqslant n$. Since $\eta^X q^n = \tilde{q}\eta^Y$ and $q(y) = q(z_i)$, it follows that

$$\omega(q(y), \eta^X q^n(z)) = 0\,.$$

Since $\eta^X q^n(z) = \tilde{q}\eta^Y(z) = \tilde{q}\hat{f}(y) = \tilde{q}\tilde{p}fq(y)$ and $\tilde{q}\tilde{p} \simeq 1_{X^n/G}$ through a δ-homotopy, it follows that

$$\begin{aligned}\omega(q(y), fq(y)) &\leqslant \omega(q(y), \tilde{q}\tilde{p}fq(y)) + \tilde{d}(\tilde{q}\tilde{p}fq(y), fq(y))\\ &< 0 + \delta.\end{aligned}$$

Which is a contradiction. Hence f has a fixed point. Q.E.D.

(3.6) Corollary. *If X is an acyclic compact ANR, then every map $f\colon X \to X^n/G$ has a fixed point.*

Proof. It is easy to see that $\mu^X_0 f_{*0} = n$ and $\mu^X_p f_{*p} = 0$, for all $p \neq 0$. Hence $L(f) = n \neq 0$. It follows that f has a fixed point.

(3.7) Theorem. *If X is a compact connected absolute retract, then every map $f\colon X \to X^n/G$ has a fixed point.*

Proof. Follows from the fact that a compact connected AR is a contractible ANR [1].

4. Let X be a compact connected ANR and x_0 be an arbitrary point of X. For an integer k, $1 \leqslant k \leqslant n$, let $d_k\colon X \to X^n$ be define as follows

$$\pi_i^X d_k = 1_X\,, \quad 1 \leqslant i \leqslant k\,,$$
$$\pi_i^X d_k = x_0\,, \quad \text{for all } x \in X \text{ and } k < i \leqslant n\,.$$

Let $\tilde{d}_k = \eta^X d_k\colon X \to X^n/G$.

By the fact that a compact connected ANR is path connected, it follows that the homotopy class of the map $\tilde{d}_k$ is independent of the choice of the point x_0.

(4.1) Theorem. *Let X be a compact connected ANR. Let $\chi(X)$ be the Euler characteristics of X. If $\chi(X) \neq (k-n)/n$, then every map $f\colon X \to X^n/G$ homotopic to $\tilde{d}_k$ has a fixed point.*

The proof of this theorem is analogous to the corresponding theorem for compact polyhedron [5].

5. Fixed point of symmetric product mappings of polyhedron and metric ANR's.

(5.1) Definition. A subcategory $\mathfrak{C}$ of $\mathfrak{I}$ is said to be *admissible* if

(i) $\mathfrak{CS}$ is a full subcategory of $\mathfrak{C}$.

(ii) The natural transformation $\mu\colon H_*\Gamma_G \Rightarrow H_*$ defined on $\mathfrak{CS}$ has an extension to $\mathfrak{C}$ such that $\mu\eta_* = \Pi_*$.

Let $X = |K|$ be a polyhedron (not necessarily finite) with Whitehead topology. Let $\{K_\alpha\}_{\alpha\in I}$ be the collection of all finite subpolyhedra of K. Let $i_\alpha\colon K_\alpha \to K$ and $i_\alpha^\beta\colon K_\alpha \to K_\beta$, where $K_\alpha \subset K_\beta$, be the inclusion maps. It is easy to see that $\{H_*(|K_\alpha|), i_{\alpha*}^\beta\}$ and $\{H_*(K_\alpha^n/G), \tilde{i}_{\alpha*}^\beta\}$ are direct systems and $H_*(X) = \varinjlim H_*(|K_\alpha|)$.

It is easy to show that the map $\eta\colon X^n \to X^n/G$ is proper and every compact subset of X^n/G is contained in a set of the form $|K_\alpha|^n/G$. It follows that $H_*(X^n/G) = \varinjlim H_*(K_\alpha^n/G)$.

Let $\eta^\alpha\colon K_\alpha^n \to K_\alpha^n/G$ be the identification map and $\pi_i^\alpha\colon K_\alpha^n \to K_\alpha$, the ith projection. Then it is easy to show $\{\eta_*^\alpha\}$, $\{\pi_{i*}^\alpha\}$ define maps of direct systems and

$$\varinjlim \eta_*^\alpha = \eta_*^X\,, \quad \varinjlim \pi_{i*}^\alpha = \pi_{i*}^X\,.$$

For each $\alpha \in I$, let $\mu^\alpha\colon H_*(|K_\alpha|^n/G) \to H_*(|K_\alpha|)$ is the homomorphism defined by the natural transformation μ which is defined on the category $\mathfrak{CS}$. By naturality of μ, it follows that $\{\mu^\alpha\}$ define a map of direct systems. Let $\mu^X = \varinjlim \mu^\alpha\colon H_*(X^n/G) \to H_*(X)$. Then it is easy to see that $\mu^X \eta_*^x = \Pi_*^X$.

Let $Y = |L|$ be another polyhedron. Let $\{L'_\nu\}'_{\nu\in J}$ be the corresponding direct system of finite subpolyhedron of L. Let $f\colon X \to Y$ be a map. For each $\alpha \in I$, $f(K_\alpha)$ is contained in a finite subpolyhedron of L. Let $L_{\alpha'}$ be the smallest subpolyhedron of L containing $f(K_\alpha)$. Let $f_\alpha\colon K_\alpha \to L_{\alpha'}$ be the map define by f. Then it is easy to see that $\{f_{\alpha*}\}$ and $\{\tilde{f}_{\alpha*}\}$ are maps of direct systems and $f_* = \varinjlim f_{\alpha*}$, $\tilde{f}_* = \varinjlim \tilde{f}_{\alpha*}$.

(5.2) Lemma. *If X and Y are polyhedra as in above discussion and $f\colon X \to Y$ is a map then $f_*\mu^X = \mu^Y f_*^n$, where μ^X and μ^Y are defined by using direct limits as above.*

Proof. Since μ is a natural transformation on the category $\mathfrak{CS}$, it follows that the following diagram commutes

$$\begin{array}{ccc} H_*(|K_\alpha^n|/G) & \xrightarrow{\mu^\alpha} & H_*(|K_\alpha|) \\ \tilde{f}_{\alpha*}\big\downarrow & & \big\downarrow f_{\alpha*} \\ H_*(|L_\alpha^n|/G) & \xrightarrow[\mu^{\alpha'}]{} & H_*(|L_{\alpha'}|) \end{array}$$

Hence it follows that

$$\varinjlim f_{\alpha*} \varinjlim \mu^\alpha = \varinjlim \mu^{\alpha'} \varinjlim \tilde{f}_{\alpha*}$$

i.e.,

$$f_*\mu^X = \mu^Y \tilde{f}_*\,.$$

(5.3) Theorem. *The category $\mathfrak{S}$ of all polyhedra is admissible.*

Proof. For each polyhedron X, let, μ^X be defined as above. Then it follows from Lemma (5.2) that μ is a natural transformation and μ^X is independent of the choice of the triangulation K of X. Q.E.D.

Let $\mathfrak{C}$ be an admissible category and X be an object of $\mathfrak{C}$. Let $f\colon X \to X^n/G$ be a map such that the homomorphism $\mu^X f_*\colon H_*(X) \to H_*(X)$ of graded vector space $H_*(X)$ is of finite type i.e., $\mu_p^X f_{*p} = 0$, for all, except for a finite number of p and $\mu_p^X f_{*p}$ is of finite type for all p [4]. In the case when $\mu^X f_*$ is of finite type, the Lefschetz number $L(f)$ of the map f is defined by

$$L(f) = \sum_{p=0}^{\infty} (-1)^p \operatorname{Trace}(\mu_p^X f_{*p})\,.$$

(5.4) Definition. Let X be an object of an admissible category $\mathfrak{C}$. A map f: $X \to X^n/G$ is said to be a *$\mu\Lambda$-map* if $L(f)$ is defined and $L(f) \neq 0$ implies that f has a fixed point (in particular $\mu^X f_*$ is of finite type).

(5.5) Definition. An object X of an admissible category is said to be a *$\mu\Lambda$-space* if every compact map $f\colon X \to X^n/G$ is a $\mu\Lambda$-map.

The above definitions are analogous to the corresponding definitions of Λ-map and Λ-space [4]. The following theorem is analoguous to the corresponding theorem on A-spaces. The proof is on similar lines hence it is omitted.

(5.6) THEOREM. *Every polyhedron (not necessarily finite) with Whitehead topology is a μA-space.*

Let $\mathfrak{TA}$ be the full subcategory of $\mathfrak{T}$ whose objects are metric ANR's and polyhedra (with Whitehead topology).

(5.7) THEOREM. *The category $\mathfrak{TA}$ is admissible.*

Proof. Since every metric ANR is homotopically dominated by a polyhedron (an object of $\mathfrak{T}$), it follows from Theorem (5.3) and Theorem (3.1) that $\mathfrak{TA}$ is admissible.

In the light of the result of A-spaces [4], a similar theorem on μA-spaces is as follows.

(5.8) THEOREM. *Every metric ANR is a μA-space.*

References

[1] K. Borsuk, *Theory of Retracts*, Warszawa 1967.
[2] S. Eilenberg and N. Steenrod, *Foundation of Algebraic Topology*, 1952, p. 224.
[3] S. T. Hu, *Theory of Retracts*, Detroit 1965.
[4] J. W. Jaworowski and M. J. Powers, *A-spaces and fixed point theorems*, Fund. Math. 64 (1969), pp. 157–162.
[5] C. N. Maxwell, *Fixed points of symmetric product mappings*, Proc. Amer. Math. Soc. 8 (1957), pp. 808–815.
[6] E. H. Spanier, *Algebraic Topology*, New York 1960.

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On the insertion of Darboux, Baire-one functions

by

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Abstract. If f and g possess the Darboux property and are in the first class of Baire on an interval I and if $f(x) > g(x)$ for all $x \in I$, there exists another Darboux function h, also in the first class of Baire, such that $f(x) > h(x) > g(x)$ for all x. Certain related statements are also valid.

1. Introduction. Let f and g be two real functions defined on a real interval I, each with the Darboux (i.e., intermediate value) property. If $g(x) < f(x)$ for all x in I one can ask whether there exists another Darboux function h such that $g(x) < h(x) < f(x)$ for all x in I. This question was answered negatively by Ceder and Weiss in [6]; they found, however, a useful sufficient condition in terms of the way in which f and g are separated by constant functions (see Section 4, below). They showed that this sufficient condition is satisfied when both f and g are in the first class of Baire. They also posed the problem of whether or not there exists a Darboux, Baire-one function between two comparable Darboux, Baire-one functions.

The purpose of this article is to show that the question has an affirmative answer (see Theorem 1). We also show that it is not possible in general to insert a Darboux function between comparable Darboux functions even if one is in the first class of Baire and the other in the second class of Baire. If, however, the first of these functions meets any of a number of additional "regularizing" conditions, such an insertion is always possible. We mention in passing that some extensions of results found in [6] are found in [5].

2. Notation and terminology. The set of real numbers will be denoted by R and I will be a fixed real interval. For a set $A \subseteq R$, $\bar{A}$ and A^0 will denote the closure and interior of A. We will regard a real function as identical with its graph. If f is a function $\subseteq R^2$, $B(f)$ will denote the set of bilateral condensation points of f (see [4], Lemma 1), and $C(f)$ will denote the set of $x \in R$ at which f is continuous. Moreover, $K^+(f, a)$ and

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