Maps and inverse systems of metric spaces

by

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Abstract. Using methods of uniform spaces are proved some theorems giving a possibility to represent every map in the category of uniform spaces (or in the category of completely regular spaces) by a mapping between inverse systems of metrizable spaces satysfying some additional conditions.

1. Introduction. In this paper we shall prove that every continuous map $f\colon X \to Y$ of completely regular spaces is represented by a family of continuous maps $f_a\colon X_a \to Y_a$ of metric spaces. More precisely, we shall prove

THEOREM A. *For every completely regular space Z there exists an inverse system $S(Z)$ of metric spaces and there exists a dense embedding $g_Z\colon Z \to Z^*$, $Z^* = \varprojlim S(Z)$, such that if $f\colon X \to Y$ is a continuous map of completely regular spaces, then there exists a mapping $\bar{f}\colon S(X) \to S(Y)$ of inverse systems such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g_X}\downarrow & & \downarrow{\scriptstyle g_Y} \\ X^* & \xrightarrow[f^*]{} & Y^* \end{array} \qquad f^* = \varprojlim \bar{f}$$

is commutative.

THEOREM B. *If X is a completely regular space and $f\colon X \to X$ a continuous map, then there exists an inverse system $S_f = \{X_a, \pi^a_\beta, M\}$ of metric spaces X_a, $\dim X_a \leqslant \dim X$, $\operatorname{card} M \leqslant \operatorname{weight} X$, and there exist a dense embedding $g\colon X \to X^*$, $X^* = \varprojlim S_f$ and a family of continuous maps $f_a\colon X_a \to X_a$, $a \in M$, inducing a mapping $\bar{f}\colon S_f \to S_f$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ X^* & \xrightarrow[f^*]{} & X^* \end{array}, \qquad f^* = \varprojlim \bar{f}$$

is commutative.

Let $S = \{X_\alpha, \pi^\alpha_\beta, M\}$ and $S' = \{Y_{\alpha'}, \pi^{\alpha'}_{\beta'}, M'\}$ be given inverse systems.

We say that $\bar{f}\colon S \to S'$ is a *mapping of the systems* if $f = \{\varphi, f_{\alpha'}\}$ is a family consisting of a monotone function $\varphi\colon M' \to M$ and of continuous maps $f_{\alpha'}\colon X_{\varphi(\alpha')} \to Y_{\alpha'}$, $\alpha' \in M'$, such that $\pi^{\alpha'}_{\beta'} f_{\alpha'} = f_{\beta'} \pi^{\varphi(\alpha')}_{\varphi(\beta')}$.

Every mapping $\bar{f}\colon S \to S'$ of inverse systems induces a continuous map $f^* = \varprojlim \bar{f}\colon \varprojlim S \to \varprojlim S'$ of inverse limits; $f^*(\{x_\alpha\}) = \{f_{\alpha'}(x_{\varphi(\alpha')})\} = \{y_{\alpha'}\}$, $\{x_\alpha\} \in \varprojlim S$ and $\{y_{\alpha'}\} \in \varprojlim S'$ (Engelking [1], p. 89).

We say that a family of continuous maps $f_\alpha\colon X_\alpha \to X_\alpha$, $\alpha \in M$, *induces a mapping* $\bar{f}\colon S \to S$ if $\bar{f} = \{\alpha, f_\alpha\}$, i.e., if $\pi^\alpha_\beta f_\alpha = f_\beta \pi^\alpha_\beta$, $\alpha, \beta \in M$.

An analogous definition of a mapping of systems and a family of maps inducing a mapping of systems will be used in the category of uniform spaces with uniform maps.

Theorems A and B will be derived from analogous theorems 1 and 2, proved in the category of uniform spaces.

Theorem A has a simple topological proof, how it was observed by the referee. It suffices to embed βZ in I^m and to consider an inverse system of closed subspaces of finite products of I.

2. Preliminaries. We use some symbols and notations from [2] and [3].

Let us recall that in [2] a pseudouniformity $\mathfrak{U}$ on a set X is a filter of coverings of X directed with respect to the star refinement.

If a pseudouniformity $\mathfrak{U}$ satisfies the axiom of separation, then $\mathfrak{U}$ is said to be a *uniformity*.

A pair $(X, \mathfrak{U})$ is said to be a *pseudouniform space* and a *uniform space*, respectively.

A map $f\colon (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is said to be *uniform* iff $f^{-1}(Q) \in \mathfrak{U}$ for every $Q \in \mathfrak{V}$.

Symbols $P \succ Q$ $(P \underset{*}{\succ} Q)$ mean that P is a refinement (a star refinement) of Q.

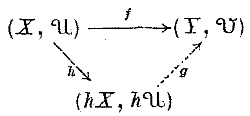
If for a pseudouniformity $\mathfrak{U}$ there exists a base of cardinality $\leqslant \gamma$ consisting of coverings of cardinality $\leqslant \tau$, then $\mathfrak{U}$ is said to be *of double weight* $\leqslant (\gamma, \tau)$, dweight $\mathfrak{U} \leqslant (\gamma, \tau)$.

If a pseudouniformity $\mathfrak{U}$ contains a base consisting of coverings of order $\leqslant n+1$, then it is said to be *of dimension* $\leqslant n$, $\dim \mathfrak{U} \leqslant n$.

The paper is based on a fundamental lemma proved in [2] and cited in another form in [3].

LEMMA 1. *Let $(X, \mathfrak{U})$ be a pseudouniform space. Then there exist a uniform space $(hX, h\mathfrak{U})$ and a uniform map $h\colon (X, \mathfrak{U}) \xrightarrow{\text{onto}} (hX, h\mathfrak{U})$ such that*

1. $\dim h\mathfrak{U} \leqslant \dim \mathfrak{U}$ *and* dweight $h\mathfrak{U} \leqslant$ dweight $\mathfrak{U}$,
2. *for every $P \in \mathfrak{U}$ there exists a $Q \in h\mathfrak{U}$ such that $h^{-1}(Q) \succ P$,*
3. *for every uniform map $f\colon (X, \mathfrak{U}) \to (Y, \mathfrak{V})$, where $\mathfrak{V}$ is a uniformity, there exists a unique map g such that a diagram*



is commutative.

The space $(hX, h\mathfrak{U})$ is said to be the *quotient of the space* $(X, \mathfrak{U})$ and the map $h\colon (X, \mathfrak{U}) \to (hX, h\mathfrak{U})$ is said to be the *quotient map*.

From Condition 3 of Lemma 1 it follows that the quotient is unique up to an isomorphism.

Let $f\colon (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ be a uniform map. Then a family $f^{-1}(\mathfrak{V}) = \{f^{-1}(Q)\colon Q \in \mathfrak{V}\}$ is a pseudouniformity contained in the uniformity $\mathfrak{U}$.

LEMMA 2. *Let $f\colon (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ be a uniform map onto a uniform space. Then the quotient of a pseudouniform space $(X, f^{-1}(\mathfrak{V}))$ is uniformly homeomorphic with the space $(Y, \mathfrak{V})$.*

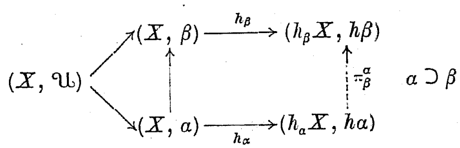
Proof. From Lemma 1 it follows that there exists a uniform map $g\colon (hX, hf^{-1}(\mathfrak{V})) \xrightarrow{\text{onto}} (Y, \mathfrak{V})$. We shall show that g is a uniform homeomorphism. Since the space $(hX, hf^{-1}(\mathfrak{V}))$ is uniform it suffices to show that for every $P \in hf^{-1}(\mathfrak{V})$ there exists a $Q \in \mathfrak{V}$ such that $g^{-1}(Q) \succ P$. Let $P \in hf^{-1}(\mathfrak{V})$. Then $h^{-1}(P) \in f^{-1}(\mathfrak{V})$ and there exists a $Q \in \mathfrak{V}$ such that $f^{-1}(Q) = h^{-1}(P)$. Since $f = g \circ h$ and h is onto, we have $P = g^{-1}(Q)$.

LEMMA 3. *Let $\mathfrak{U}$ be a uniformity on a set X and let M be a set of pseudouniformities contained in $\mathfrak{U}$ such that*

1. *the set M is directed with respect to inclusion,*
2. *$\bigcup M$ is a base for the uniformity $\mathfrak{U}$.*

Then the space $(X, \mathfrak{U})$ has a uniform dense embedding into an inverse limit of a system consisting of the quotients of the spaces belonging to the set M with the maps uniquely determined by Condition 3 of Lemma 1.

Proof. Let us consider the diagram



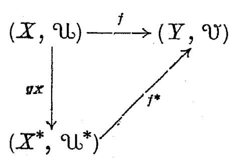
with identity uniform maps $(X, \mathfrak{U}) \to (X, \alpha)$, $(X, \mathfrak{U}) \to (X, \beta)$, $(X, \alpha) \to (X, \beta)$ and the quotient maps h_α and h_β. From Condition 3 of Lemma 1 it follows that there exists a unique uniform map π^α_β such that the diagram is commutative. Thus $S = \{(h_\alpha X, h\alpha), \pi^\alpha_\beta, \alpha, \beta \in M\}$ is an inverse system.

From Condition 2 of Lemma 1 and from Assumption 2 of Lemma 3 it follows that the maps $g_\alpha\colon (X, \mathcal{U}) \xrightarrow{\text{onto}} (X, \alpha) \xrightarrow{\text{onto}} (h_\alpha X, h\alpha)$ induce a uniform dense embedding $g\colon (X, \mathcal{U}) \to \underleftarrow{\lim}\, S$; $g(x) = \{g_\alpha(x)\}$.

3. Two theorems. A pseudouniformity is said to be *metrizable* iff it has a countable base.

Let $\mathcal{U}$ be a uniformity on a set X. Let us consider a set M of all metrizable pseudouniformities contained in $\mathcal{U}$. The set M is directed with respect to inclusion because for every pair $\alpha, \beta \in M$ a family $\alpha \cup \beta$ is a subbase for some metrizable pseudouniformity contained in $\mathcal{U}$. We have $\bigcup M = \mathcal{U}$. Thus from Lemma 3 it follows that the quotients of the spaces belonging to the set M with the maps uniquely determined by Lemma 1 form an inverse system $S(X, \mathcal{U})$ over the set M and the maps $g_\alpha\colon (X, \mathcal{U}) \to (X, \alpha) \to (h_\alpha X, h\alpha)$ induce a uniform dense embedding $g_X\colon (X, \mathcal{U}) \to (X^*, \mathcal{U}^*) = \underleftarrow{\lim}\, S(X, \mathcal{U})$; $g_X(x) = \{g_\alpha(x)\}$.

THEOREM 1. *Let $f\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniform map of uniform spaces, where the space $(Y, \mathcal{V})$ is the inverse limit of a system S of metrizable spaces. Then there exists a mapping $\bar f\colon S(X, \mathcal{U}) \to S$ of the inverse systems such that a diagram*



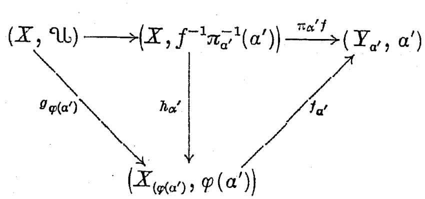
$f^* = \underleftarrow{\lim}\, \bar f$

is commutative.

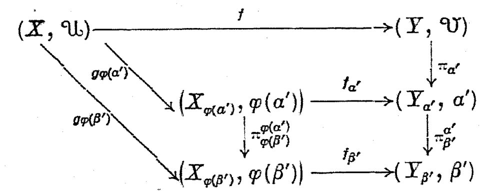
Proof. Let $S(X, \mathcal{U}) = \{(X_\alpha, \alpha), \pi^\alpha_\beta, M\}$ and $S = \{(Y_{\alpha'}, \alpha'), \pi^{\alpha'}_{\beta'}, M'\}$.

We define a monotone function $\varphi\colon M' \to M$; for every $\alpha' \in M'$, $\varphi(\alpha')$ is a metrizable uniformity in the space $(X_{\varphi(\alpha')}, \varphi(\alpha'))$ being the quotient of the space $(X, f^{-1}\pi^{-1}_{\alpha'}(\alpha'))$.

From Lemma 1 follows the existence of uniform maps $f_{\alpha'}\colon (X_{\varphi(\alpha')}, \varphi(\alpha')) \to (Y_{\alpha'}, \alpha')$ such that the diagram



is commutative. More precisely, the diagram



is commutative. The family $\bar f = \{\varphi, f_{\alpha'}\}$ forms a mapping $\bar f\colon S(X, \mathcal{U}) \to S$. Let $f^* = \underleftarrow{\lim}\, \bar f$. We have $f = f^* \circ g$. Indeed,

$$f^* g(x) = f^*\{g_\alpha(x)\} = \{f_{\alpha'} g_{\varphi(\alpha')}(x)\} = \{\pi_{\alpha'} f(x)\} = f(x)\,.$$

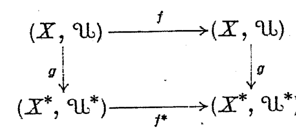
Notice that Lemma 2 implies that the maps $f_{\alpha'}$ are uniform embeddings.

THEOREM 2. *Let $f\colon (X, \mathcal{U}) \to (X, \mathcal{U})$ be a uniform map of a uniform space and let $\dim \mathcal{U} \leqslant n$ and $\operatorname{dweight} \mathcal{U} \leqslant (\gamma, \tau)$.*

Then there exist:

1. *an inverse system $S = \{(X_\alpha, \alpha), \pi^\alpha_\beta, M\}$ such that $\dim \alpha \leqslant n$, $\operatorname{dweight} \alpha \leqslant (\aleph_0, \tau)$ for $\alpha \in M$ and* $\operatorname{card} M = \operatorname{weight} \mathcal{U}$,
2. *a uniform dense embedding $g\colon (X, \mathcal{U}) \to (X^*, \mathcal{U}^*)$, $(X^*, \mathcal{U}^*) = \underleftarrow{\lim}\, S$,*
3. *uniform maps $f_\alpha\colon (X_\alpha, \alpha) \to (X_\alpha, \alpha)$ inducing a mapping $\bar f\colon S \to S$ of the system such that the diagram*

(*)



$f^* = \underleftarrow{\lim}\, \bar f$

is commutative.

Proof. For every $P \in \mathcal{U}$ let us write

$$f^{-0}(P) = P, \qquad f^{-(m+1)}(P) = f^{-1}(f^{-m}(P)), \qquad m = 0, 1, \ldots$$

Let $\mathcal{B}$ be a base for the uniformity $\mathcal{U}$ of cardinality $\leqslant \gamma$, consisting of coverings of cardinality $\leqslant \tau$ and of order $\leqslant n+1$ (see [3], Prop. 6). We shall show that for every $P \in \mathcal{B}$ there exists a metrizable pseudouniformity $\mathcal{V}$ having a base contained in $\mathcal{B}$ and such that $f^{-1}(\mathcal{V}) \subset \mathcal{V}$ and $P \in \mathcal{V}$. Put $W_1 = \{f^{-m}(P)\colon m = 0, 1, \ldots\}$. Let us assume that we have defined countable families $W_1, \ldots, W_{n-1}$. We choose a countable family W_n such that

(a) if $Q \in W_n$ then $f^{-1}(Q) \in W_n$,

(b) for every pair $P_1, P_2 \in \bigcup \{W_i\colon i = 1, \ldots, n-1\}$ there exists a $Q \in W_n \cap \mathcal{B}$ such that $Q \underset{*}{>} P_1$ and $Q \underset{*}{>} P_2$.

The choice of such a family is possible because the family $\bigcup\{W_i: i=1,\ldots,n-1\}$ is countable and $\mathcal{B}$ is a base for $\mathfrak{U}$. Let $\mathfrak{V}$ be a pseudouniformity induced by a base $\bigcup\{W_n: n=1,2,\ldots\}$.

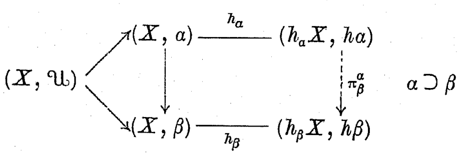
Thus, we can choose a family M' of pseudouniformities such that $\mathcal{B}\subset\bigcup M'\subset\mathfrak{U}$, card $M'=$ card $\mathcal{B}$, and for every $\alpha'\in M'$

(c) $\quad f^{-1}(\alpha')\subset\alpha'$, $\dim\alpha'\leqslant n$ and dweight $\alpha'\leqslant(\aleph_0,\tau)$.

Using the method as above, we may prove that for every pair $\alpha,\beta\in M'$ there exists a pseudouniformity $\gamma\supset\alpha\cup\beta$ having the property (c).

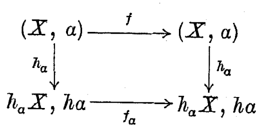
Hence, by induction, we can choose a directed set M of pseudouniformities such that $\mathcal{B}\subset\bigcup M\subset\mathfrak{U}$, card $\mathcal{B}=$ card M, if $\alpha\in M$ then $f^{-1}(\alpha)\subset\alpha$, $\dim\alpha\leqslant n$ and dweight $\alpha\leqslant(\aleph_0,\tau)$.

Let S be an inverse system of the quotients of spaces belonging to the set M with the maps π^α_β uniquely determined by Lemma 1. The diagram



with identity maps $(X,\mathfrak{U})\to(X,\alpha)$, $(X,\mathfrak{U})\to(X,\beta)$, $(X,\alpha)\to(X,\beta)$, the quotients maps h_α, h_β and maps π^α_β uniquely determined by Condition 3 of Lemma 1, is commutative.

From Lemma 1 and from the conditions $f^{-1}(\alpha)\subset\alpha$, $\alpha\in M$, follows the existence of maps f_α, $\alpha\in M$, such that the diagram



is commutative.

Considering the two diagrams, we conclude that maps f_α induce a mapping $f: S\to S$ such that the diagram (*) is commutative, where g is a map induced by maps g_α: $(X,\mathfrak{U})\to(X,\alpha)\overset{h_\alpha}{\to}(h_\alpha X, h\alpha)$.

4. Proof of Theorem A **and Theorem** B. In order to prove Theorem A and Theorem B it suffices to know some facts;

1. Every completely regular topological space has a finest compatible uniformity $\mathfrak{U}^*$.

2. If $f: X\to Y$ is a continuous map of completely regular spaces, then $f: (X,\mathfrak{U}^*)\to(Y,\mathfrak{V}^*)$ is a uniform map of the uniform spaces with the finest compatible uniformities.

3. $\dim X\leqslant n$ iff $\dim\mathfrak{U}^*\leqslant n$ (Pasynkov [5]).

4. If there exists a uniformity $\mathfrak{U}$ compatible with the topology on the space X such that $\dim\mathfrak{U}\leqslant n$ and dweight $\mathfrak{U}\leqslant(\aleph_0,\tau)$, τ arbitrary, then the topological space χ is metrizable and $\dim X\leqslant n$ (Nagata [4], p. 126).

Hence, applying Theorem 1 and Theorem 2 to the finest compatible uniformities, we receive topological corollaries: Theorems A and B, without the assertion of the cardinality of the system M. In order to receive this assertion we must apply Theorem 2 to a uniformity $\mathfrak{U}$ of $\dim\mathfrak{U}\leqslant\dim X$ and weight $\mathfrak{U}\leqslant$ weight X, compatible with the topology and such that $f: (X,\mathfrak{U})\to(X,\mathfrak{U})$ is uniform. The method of proving the existence of such uniformity is the same as in the proof of Theorem 2.

References

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