Scott sentences and a problem of Vaught for mono-unary algebras

by

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Abstract. First we show that there is a countable ordinal $\alpha$ such that if $\langle A, f \rangle$ is a mono-unary algebra then one can find a Scott sentence (which describes $\langle A, f \rangle$ up to isomorphism) whose rank is less than $\alpha$. Combining this result with Morley's we see that if a sentence of $\mathcal{L}_{\mathcal{S}}$ for mono-unary algebras has more than denumerably many isomorphism types of countable models then it must have continuum many of these isomorphism types.

We wish to show that for a given countable mono-unary algebra $A$ we can construct a reasonably simple Scott sentence $\varphi_A$ in $\mathcal{L}_{\mathcal{S}}$, i.e. for any countable mono-unary algebra $B$, $B \models \varphi_A$ if and only if $B$ is isomorphic to $A$. Then we apply the methods of Morley [1] to determine the possible number of isomorphism types which can be realized among the countable models of a $\mathcal{L}_{\mathcal{S}}$ sentence for mono-unary algebras.

1. The Scott Sentence. In what follows we will always assume $\mathcal{L}_{\mathcal{S}}$ involves one non-logical symbol, a unary operation symbol. Let $\mathcal{L}$ be a subset of $\mathcal{L}_{\mathcal{S}}$. Define $C_0(\mathcal{L})$ to be the closure of $\mathcal{L}$ under $\land, \lor, \neg, \forall_1, \exists$, and $\forall_1$; define $C_0(\mathcal{L})$ to be $\mathcal{L}$ union the set of formulas formed by taking the countable conjunction (or disjunction) of a set $\mathcal{F}$ of formulas in $\mathcal{L}$, where the set of variables which occur free in members of $\mathcal{F}$ is finite.

Define a transfinite sequence $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots$ by the following inductive procedure:

$\mathcal{L}_0$ is the usual first-order predicate calculus with one unary operation symbol,

$\mathcal{L}_\xi = \bigcup_{\nu < \xi} \mathcal{L}_\nu$ for limit ordinals $\xi$, and $\mathcal{L}_{\xi+1} = C_0(\mathcal{L}_\xi)$ for $\xi < \omega_1$.

Then $\mathcal{L}_{\omega_1} = \bigcup_{\xi < \omega_1} \mathcal{L}_\xi$.

Theorem 1. The isomorphism type of a countable mono-unary algebra $A = \langle A, f \rangle$ can be defined by a single sentence $\varphi_A$ in $\mathcal{L}_{\omega_1}$. Proof. In the following we will introduce the notations which will be used to construct $\varphi_A$, and following each definition we will state its meaning as well as an $\mathcal{L}_\xi$ to which it belongs. In much of what follows
it will be helpful to visualize a mono-unary algebra $\mathfrak{A} = \langle A, (f) \rangle$ as a directed graph $\{(a, f(a)) : a \in A\}$ (see Fig. 1). We will freely draw upon graph-theoretical terminology such as predecessor, immediate predecessor, successor, component and loop. Note that in the directed graph of a mono-unary algebra each component contains at most one loop.

**Fig. 1**

$D(x_1, \ldots, x_n) = \bigwedge_{0 < i < n} (x_i \neq x_j) .

(1)

(This formula is in $\mathcal{L}_0$, and expresses the predicate: $x_1, \ldots, x_n$ are pairwise distinct.)

$P(x_0, x_1) = D(x_0, x_1) \land (f(x_0) = x_1) .

(2)

(This formula is in $\mathcal{L}_0$ and says: $x_0$ is an immediate predecessor of $x_1$.)

If $S(x_0)$ is any formula in $\mathcal{L}_{<\omega}$ and $\alpha < \omega$, let

$\forall x_0 \forall x_1 \ldots \exists x_{\alpha-1} \bigwedge (D(x_0, x_1, \ldots, x_{\alpha-1}) \land S(x_0) \land \ldots \land S(x_{\alpha-1})) \land

\bigwedge_{1 < \alpha < \omega} \exists x_0 \ldots \exists x_{\alpha} \forall x_{\alpha+1} \forall x_{\alpha+2} \ldots \exists x_{\alpha+\beta} \exists x_{\alpha+\beta+1} \ldots \exists x_{\alpha+n} (D(x_0, x_1, \ldots, x_{\alpha+n}) \land S(x_0) \land \ldots \land S(x_{\alpha+n}),

\text{ if } \alpha = 0 ;

\text{ if } 1 < \alpha < \omega ;

\text{ if } \alpha = \omega .

(3)

If $S(x_0)$ contains free variables other than $x_0$, then a suitable change of variables is employed to prevent them from becoming bound. (This is in $\mathcal{L}_{<\omega}$ and says: There are exactly $\alpha$ $x_0$'s such that $S(x_0)$.)

$L(x_0) = \bigwedge_{0 < \alpha < \omega} (f^\alpha x_0 = x_0).$

(4)

(This is in $\mathcal{L}_1$ and says: $x_0$ generates a loop.)

$P^\alpha(x_0, x_1) = P(x_0, x_1) \land \neg L(x_0).$

(5)

$P^\alpha(x_0, x_1)$ is in $\mathcal{L}_1$ and expresses: $x_0$ immediately precedes $x_1$ and does not generate a loop.

$P^\alpha(x_0, x_1)$ is in $\mathcal{L}_1 + \mathcal{L}_\omega$ and says: There are exactly $\alpha(0)$ immediate predecessors of $x_0$ which do not generate a loop, each of which has exactly $\alpha(1)$ immediate predecessors, each of which ... each of which has exactly $\alpha(\omega-1)$ immediate predecessors.

(6)

For $k < \omega, \alpha < \omega + 1^k$ define

$P^\alpha(x_0) = \forall \forall \exists \exists [P^\alpha(x_1, x_2) \land \ldots \land P^\alpha(x_{2^\alpha}, x_{2^\alpha})].$

$P^\alpha(x_0)$ is in $\mathcal{L}_{<\omega}$ and says: There are exactly $\alpha(0)$ immediate predecessors of $x_0$, each of which has exactly $\alpha(1)$ immediate predecessors, each of which ... each of which has exactly $\alpha(\omega-1)$ immediate predecessors.

(7)

Returning to our algebra $\mathfrak{A} = \langle A, (f) \rangle$, and focusing our attention on an element $a$ in $A$, let

$P_\alpha(a) = \bigwedge \{ P^\alpha(x_0) : P^\alpha(a) \}$ holds, where $\alpha < \omega + 1^k, k < \omega .

(8)

Let $S_\alpha(a)$ be whichever of the following formulas is true of $a$:

$(f^\alpha a = f^{\alpha+n} a) \land \bigwedge_{i < m, j < n, i + j < m+n} (f^\alpha a = f^{i+j} a) ;$

where $n, j > 1$, or

$\bigwedge \{ f^{\alpha+n} a = f^{m+n} a : m, n < \omega, m \neq n \} .

(S_\alpha(a) \in \mathcal{L}_1$ and describes the structure of the successors of $a$.)

We remark that if $\mathfrak{B} = \langle B, (f) \rangle$ is a countable mono-unary algebra and $b \in B$, then $P_\alpha(b)$ implies $b$ has the same predecessor structure as $a$, discarding those points which generate a loop. Likewise $S_\alpha(b)$ implies $a$ and $b$ have the same successor structure.

(9)

$L_\alpha(a) = S_\alpha(a) \land \bigwedge_{\alpha < \omega} (P^{\alpha}(f^\alpha a)).$

$L_\alpha(a) \in \mathcal{L}_{<\omega}$ and tells the structure of the component of $a$ in $\mathfrak{B}$. (10)

$D_\alpha(a, x_0, x_1) \in \mathcal{L}_1 + \mathcal{L}_\omega$, and says: $x_0$ and $x_1$ belong to distinct components.)

Let $L_\alpha$ be whichever of the following is true of $a$:

$\bigwedge_{0 < \alpha < \omega} (f^\alpha a = f^{\alpha+n} a) \land

\bigwedge_{0 < \alpha < \omega} (f^\alpha a = f^{m+n} a) ;$

where $n, j > 1$, or

$\bigwedge \{ f^{\alpha+n} a = f^{m+n} a : m, n < \omega, m \neq n \} .

(11)$

$L_\alpha(a) \in \mathcal{L}_1 + \mathcal{L}_\omega$, and says: $x_0$ and $x_1$ belong to distinct components.)
From $I_0$ we can determine the number of components (isomorphic to the component of $a$, as well as the structure of the component of $a$.)

Finally, to describe the Scott Sentence, let $\{a_i : \lambda \in \Lambda\}$ be a subset of $A$ such that it contains exactly one element from each component of $A$. Then the sentence:

$$\theta_{\Lambda} = (\forall \lambda \in \Lambda) \forall x_i \exists a_i [\sim DK_\lambda(x_i, a_i) \land \forall \varepsilon < \omega \exists a_i (x_i)]$$

is readily seen to completely describe the isomorphism type of $A$, and is in $\Sigma_0^1$.

2. The number of isomorphism types. The remainder of the paper is an adaptation of Morley [1]. Let $C$ be a subset of $\Sigma_{\omega^1}$. Suppose $C$ is closed under $C_0$, substitution of one variable for another (with a suitable renaming of bound variables to prevent a clash), and contains all subformulas of its members. Then if $C$ is countable we will say that it is regular. If $T$ is a theory of mono-unary algebras consisting of a sentence from $\Sigma_{\omega^1}$ and $K$ is the class of models of $T$ which are countable, then we will say $T$ is scattered if, for every regular $\Lambda \subseteq \Sigma_{\omega^1}$ and $a < \omega$, $S_\Lambda(\Lambda, K)$ is countable, where $S_\Lambda(\Lambda, K)$ denotes the set of $\Lambda$-types in $\Lambda$ realized by models in $K$.

Assume that $T$ is a scattered theory of mono-unary algebras, and $K$ is its class of countable models. Let $\xi_\Lambda$ be a regular language containing $\xi_\Lambda(\Lambda, a_i \in A, DK(\Lambda, a_i), P^*(\Lambda, a_i), P^*(\Lambda)_{\forall \varepsilon < \omega}, k < \omega$, and all possible $S_\Lambda(\Lambda, K)$ as described in (8).

Let $\mathcal{H} = \langle A, \xi \rangle$ and $\mathcal{B} = \langle B, \xi \rangle$ be two algebras in $K$, and let $a \in A$, $b \in B$. Returning to (7) one sees that either $P^*(\Lambda)(a)$ is identical to $P^*(\Lambda)(b)$, $P^*(\Lambda)(a)$ is always false. Since $P^*(\Lambda)$ is a conjunction of formulas in $\xi_\Lambda$, it follows that for some $y \in S_\Lambda(\xi_\Lambda, K)$, $b \rightarrow P^*(\Lambda)$, and if $P^*(\Lambda)(a)$ is not identical to $P^*(\Lambda)(b)$, then $\sim (\forall y \rightarrow P^*(\Lambda))$. Since $T$ is scattered it follows that there are only countably many formulas of the form $P^*(\Lambda)(a)$, where $\mathcal{H} = \langle A, \xi \rangle \in K$ and $a \in A$. Let $\xi_\Lambda^*$ be a regular language containing $\xi_\Lambda$ and formulas of the form $P^*(\Lambda)$.

By an argument of the above style we can also conclude that there are only countably many formulas of the form $K^*(\Lambda)$, let us denote them by $K^*(\Lambda)(a)$, $a < \omega$, for a suitable $a < \omega$. Referring to (11) it is immediate that there are only countably many sentences of the form $I_\lambda$. Let us introduce the notation $I_{\lambda,i}$, $i < \omega$, $j \in \omega$, where $i$ refers to the isomorphism type described by $K^*(\Lambda)(a)$, and $j$ tells the number of components of this type. Let $\xi_\Lambda^*$ be a regular language containing $\xi_\Lambda$ and the $I_{\lambda,i}$.

Let $\theta \neq \mu$ it is easy to verify that $\land_{\theta \neq \mu} I_{\lambda,i}$ and $\land_{\theta \neq \mu} I_{\lambda,i}$ are contradictory. Since $S_\Lambda(\xi_\Lambda^*), K$ is countable, it will follow that there are only countably many $\theta \in \omega^1$, such that $\land_{\theta \neq \mu} I_{\lambda,i}$ is true of some model of $T$. Since the sentence $\land_{\theta \neq \mu} I_{\lambda,i}$ completely describes the isomorphism type of a model in $K$ which satisfies it, $K$ has only countably many different isomorphism types.

Theorem 2. The number of isomorphism types of countable mono-unary algebras which satisfy a sentence of $\Sigma_{\omega^1}$ is either countable or $2^\omega$. This answers a problem of Vaught — in the case of mono-unary algebras (see [3]).

Proof. In [1] Morley proved everything stated except he allowed the possibility of $\omega_1$ isomorphism types in a scattered theory, and we have just finished excluding this.

In conclusion we remark that all of the possible numbers of isomorphism types can be realized by a suitable theory of mono-unary algebras. Also, by some obvious modifications Theorem 2 is still true if we add a finite number of constants to our language (which already involves one unary operation).

References


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