References

[1] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Mathematics 223, Berlin 1971.
[2] T. Jech, *On models of set theory without* AC, Proceedings of the 1967 UCLA Summer Institute on Axiomatic Set Theory. A.M.S. Proceedings of Symposia in Pure Mathematics XIII, Part I, Providence (R.I.) 1971, pp. 135–141.
[3] W. Kinna and K. Wagner, *Über eine Abschwächung des Auswahlpostulates*, Fund. Math. 42 (1955), pp. 76–82.
[4] J. R. Shoenfield, *Unramified forcing*, Proceedings of the 1967 UCLA Summer Institute, pp. 357–381.
[5] P. Vopěnka and B. Balcar, *On complete models of the set theory*, Bull. Acad. Pol. Sci. sér. math. astron. phys. 15 (1967), pp. 839–841.

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Scott sentences and a problem of Vaught for mono-unary algebras

by

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Abstract. First we show that there is a countable ordinal α such that if $\langle A, \{f\}\rangle$ is a mono-unary algebra then one can find a Scott sentence (which describes $\langle A, \{f\}\rangle$ up to isomorphism) whose rank is less than α. Combining this result with Morley's we see that if a sentence of $\mathfrak{L}_{\omega_1\omega}$ for mono-unary algebras has more than denumerably many isomorphism types of countable models then it must have continuum many of these isomorphism types.

We wish to show that for a given countable mono-unary algebra $\mathfrak{A}$ we can construct a reasonably simple *Scott Sentence* $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega_1\omega}$, i.e. for any countable mono-unary algebra $\mathfrak{B}$, $\mathfrak{B} \models \varphi_{\mathfrak{A}}$ iff $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$. Then we apply the methods of Morley [1] to determine the possible number of isomorphism types which can be realized among the countable models of a $\mathfrak{L}_{\omega_1\omega}$ sentence for mono-unary algebras.

1. The Scott Sentence. In what follows we will always assume $\mathfrak{L}_{\omega_1\omega}$ involves one non-logical symbol, a unary operation symbol. Let $\mathfrak{L}$ be a subset of $\mathfrak{L}_{\omega_1\omega}$. Define $\boldsymbol{C}_0(\mathfrak{L})$ to be the closure of $\mathfrak{L}$ under $\wedge$, $\vee$, $\neg$, $\exists$ and $\forall$; define $\boldsymbol{C}_1(\mathfrak{L})$ to be $\mathfrak{L}$ union the set of formulas formed by taking the countable conjunction (or disjunction) of a set $\mathcal{F}$ of formulas in $\mathfrak{L}$, where the set of variables which occur free in members of $\mathcal{F}$ is finite.

Define a transfinite sequence $\mathfrak{L}_0 \subseteq \mathfrak{L}_1 \subseteq \ldots$ by the following inductive procedure:

$\mathfrak{L}_0$ is the usual first-order predicate calculus with one unary operation symbol,

$\mathfrak{L}_\xi = \bigcup_{\eta<\xi}\mathfrak{L}_\eta$ for limit ordinals ξ, $\xi<\omega_1$, and $\mathfrak{L}_{\xi+1} = \boldsymbol{C}_0\boldsymbol{C}_1(\mathfrak{L}_\xi)$ for $\xi<\omega_1$.

Then $\mathfrak{L}_{\omega_1\omega} = \bigcup_{\xi<\omega_1}\mathfrak{L}_\xi$.

THEOREM 1. *The isomorphism type of a countable mono-unary algebra $\mathfrak{A} = \langle A, \{f\}\rangle$ can be defined by a single sentence $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega+4}$.*

Proof. In the following we will introduce the notations which will be used to construct $\varphi_{\mathfrak{A}}$, and following each definition we will state its meaning as well as an $\mathfrak{L}_\xi$ to which it belongs. In much of what follows it will be helpful to visualize a mono-unary algebra $\mathfrak{A} = \langle A, \{f\}\rangle$ as a *directed graph* $\{\langle a, f(a)\rangle : a \in A\}$ (see Fig. 1). We will freely draw upon graph-theoretical terminology such as *predecessor*, *immediate predecessor*, *successor*, *component* and *loop*. Note that in the directed graph of a mono-unary algebra each component contains at most one loop.

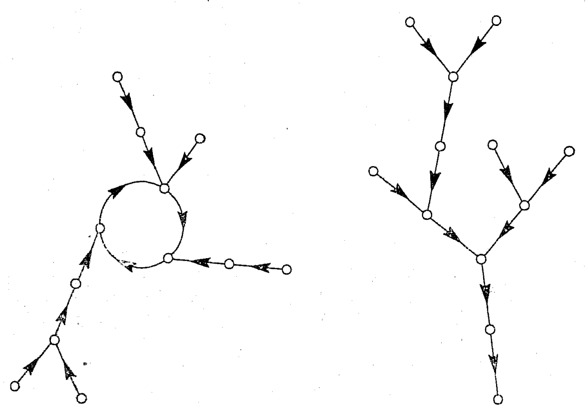

Fig. 1

(1) $$D(x_0, \ldots, x_n) = \bigwedge\nolimits_{0\leqslant i<j\leqslant n}(x_i \neq x_j)\,.$$

(This formula is in $\mathfrak{L}_0$ and expresses the predicate: $x_0, \ldots, x_n$ *are pairwise distinct.*)

(2) $$P(x_0, x_1) = D(x_0, x_1)\wedge(fx_0 = x_1)\,.$$

(This formula is in $\mathfrak{L}_0$ and says: x_0 *is an immediate predecessor of* x_1.)

(3) If $S(x_0)$ is any formula in $\mathfrak{L}_{\omega_1\omega}$ and $\alpha \leqslant \omega$, let

$$\exists^{\alpha}_{x_0} S(x_0) = \begin{cases} \neg\exists x_0 S(x_0)\,, & \text{if } \alpha = 0;\\ \exists x_0 \ldots \exists x_{\alpha-1}[D(x_0, \ldots, x_{\alpha-1})\wedge S(x_0)\wedge \ldots \wedge S(x_{\alpha-1})]\wedge & \\ \wedge\neg\exists x_0 \ldots \exists x_{\alpha}[D(x_0, \ldots, x_{\alpha})\wedge S(x_0)\wedge \ldots \wedge S(x_{\alpha})]\,, & \text{if } 1\leqslant\alpha<\omega;\\ \bigwedge_{1\leqslant\beta<\omega}\exists x_0 \ldots \exists x_{\beta}[D(x_0, \ldots, x_{\beta})\wedge S(x_0)\wedge \ldots \wedge S(x_{\beta})]\,, & \text{if } \alpha=\omega. \end{cases}$$

If $S(x_0)$ contains free variables other than x_0, then a suitable change of variables is employed to prevent them from becoming bound.

(This is in $\mathfrak{L}_{\beta+1}$ if $S(x_0)$ is in $\mathfrak{L}_{\beta}$ and says: *There are exactly* α x_0*'s such that* $S(x_0)$.)

(4) $$L(x_0) = \bigwedge\nolimits_{1\leqslant n<\omega}(f^n x_0 = x_0).$$

(This is in $\mathfrak{L}_1$ and says: x_0 *generates a loop.*)

(5) $$P^*(x_0, x_1) = P(x_0, x_1)\wedge\neg L(x_0)\,.$$

($P^*(x_0, x_1)$ is in $\mathfrak{L}_1$ and expresses: x_0 *immediately precedes* x_1 *and does not generate a loop.*)

(6) For $k<\omega$, $\psi \in \omega+1^k$ define

$$P^{\psi}(x_0) = \exists^{\psi(0)}_{x_1}[P^*(x_1, x_0)\wedge\exists^{\psi(1)}_{x_2}[P(x_2, x_1)\wedge\ldots\wedge\exists^{\psi(k-1)}_{x_k}P(x_k, x_{k-1})]\ldots]\,.$$

($P^{\psi}(x)$ is in $\mathfrak{L}_{k+1}$ and says: *There are exactly* $\psi(0)$ *immediate predecessors of* x_0 *which do not generate a loop, each of which has exactly* $\psi(1)$ *immediate predecessors, each of which ... each of which has exactly* $\psi(k-1)$ *immediate predecessors.*)

(7) Returning to our algebra $\mathfrak{A} = \langle A, \{f\}\rangle$, and focusing our attention on an element a in A, let

$$P_a(x_0) = \bigwedge\{P^{\psi}(x_0) : P^{\psi}(a) \text{ holds, where } \psi\in\omega+1^k,\ k<\omega\}\,.$$

($P_a(x_0)$ is in $\mathfrak{L}_{\omega+1}$ and *tells the structure of all predecessors of* a *which are not in a loop.*)

(8) Let $S_a(x_0)$ be whichever of the following formulas is true of a:

$$(f^m x_0 = f^{m+n}x_0)\wedge\bigwedge\{\neg(f^i x_0 = f^{i+j}x_0) : i\leqslant m,\ j\leqslant n,\ i+j<m+n\}\,,$$

where $n, j\geqslant 1$, or

$$\bigwedge\{\neg(f^m x_0 = f^n x_0) : m, n<\omega,\ m\neq n\}\,.$$

($S_a(x_0)\in\mathfrak{L}_1$ and *describes the structure of the successors of* a.)

We remark that if $\mathfrak{B} = \langle B, \{f\}\rangle$ is a countable mono-unary algebra and $b\in B$, then $P_a(b)$ implies b has the same predecessor structure as a, discarding those points which generate a loop. Likewise $S_a(b)$ implies a and b have the same successor structure.

(9) $$K_a(x_0) = S_a(x_0)\wedge\bigwedge\nolimits_{n<\omega}\{P_{f^n(a)}(f^n x_0)\}\,.$$

($K_a(x_0)\in\mathfrak{L}_{\omega+2}$ and *tells the structure of the component of* a *in* $\mathfrak{A}$.)

(10) $$DK(x_0, x_1) = \bigwedge\nolimits_{m,n<\omega}\neg(f^m x_0 = f^n x_1)\,.$$

($DK(x_0, x_1)\in\mathfrak{L}_1$ and says: x_0 and x_1 *belong to distinct components.*)

(11) Let I_a be whichever of the following is true of a:

$$\exists x_0 \ldots \exists x_{\alpha}[(\bigwedge\nolimits_{i\leqslant\alpha}K_a(x_i))\wedge(\bigwedge\nolimits_{0\leqslant i<j\leqslant\alpha}DK(x_i, x_j))]\wedge$$
$$\wedge\neg\exists x_0 \ldots \exists x_{\alpha+1}[(\bigwedge\nolimits_{i\leqslant\alpha+1}K_a(x_i))\wedge(\bigwedge\nolimits_{0\leqslant i<j\leqslant\alpha+1}DK(x_i, x_j))]\,,$$

where $\alpha<\omega$ or

$$\bigwedge\nolimits_{\alpha<\omega}\exists x_0 \ldots \exists x_{\alpha}[(\bigwedge\nolimits_{i\leqslant\alpha}K_a(x_i))\wedge(\bigwedge\nolimits_{0\leqslant i<j\leqslant\alpha}DK(x_i, x_j))]\,.$$

($I_a \in \mathfrak{L}_{\omega+3}$. *From I_a we can determine the number of components isomorphic to the component of a, as well as the structure of the component of a.*)

Finally, to describe the Scott Sentence, let $\{a_\lambda \colon \lambda \in \Lambda\}$ be a subset of A such that it contains exactly one element from each component of $\mathfrak{A}$. Then the sentence:

$$\varphi_{\mathfrak{A}} = (\bigwedge_{\lambda\in\Lambda} I_{a_\lambda}) \wedge \forall x_0 \exists x_1 [\neg DK(x_0, x_1) \wedge \bigvee_{\lambda\in\Lambda} K_{a_\lambda}(x_1)] \tag{12}$$

is readily seen to completely describe the isomorphism type of $\mathfrak{A}$, and is in $\mathfrak{L}_{\omega+4}$.

2. The number of isomorphism types. The remainder of the paper is an adaptation of Morley [1]. Let $\mathfrak{L}$ be a subset of $\mathfrak{L}_{\omega_1\omega}$. Suppose $\mathfrak{L}$ is closed under $\boldsymbol{C}_0$, substitution of one variable for another (with a suitable renaming of bound variables to prevent a clash), and contains all subformulas of its members. Then if $\mathfrak{L}$ is countable we will say that it is *regular*. If T is a theory of mono-unary algebras consisting of a sentence from $\mathfrak{L}_{\omega_1\omega}$, and $\boldsymbol{K}$ is the class of models of T which are countable, then we will say T is *scattered* if, for every regular $\mathfrak{L} \subseteq \mathfrak{L}_{\omega_1\omega}$ and $n < \omega$, $S_n(\mathfrak{L}, \boldsymbol{K})$ is countable, where $S_n(\mathfrak{L}, K)$ denotes the set of n-types in $\mathfrak{L}$ realized by models in $\boldsymbol{K}$.

Assume that T is a scattered theory of mono-unary algebras, and $\boldsymbol{K}$ its class of countable models. Let $\mathfrak{L}_0^*$ be a regular language containing $\mathfrak{L}_0$, $\mathfrak{L}(x_0)$, $DK(x_0, x_1)$, $P^*(x_0, x_1)$, $P^\psi(x_0)$ for all $\psi \in \omega+1^k$, $k < \omega$, and all possible $S_a(x_0)$ as described in (8).

Let $\mathfrak{A} = \langle A, \{f\}\rangle$ and $\mathfrak{B} = \langle B, \{f\}\rangle$ be two algebras in K, and let $a \in A$, $b \in B$. Returning to (7) one sees that either $P_a(x_0)$ is identical to $P_b(x_0)$, or $P_a(x_0) \wedge P_b(x_0)$ is always false. Since $P_a(x_0)$ is a conjunction of formulas in $\mathfrak{L}_0^*$, it follows that for some $\psi \in S_1(\mathfrak{L}_0^*, K)$, $\bigwedge \psi \to P_a(x_0)$, and if $P_a(x_0)$ is not identical to $P_b(x_0)$, then $\neg(\bigwedge\psi \to P_a(x_0))$. Since T is scattered it follows that there are only countably many formulas of the form $P_a(x_0)$, where $\mathfrak{A} = \langle A, \{f\}\rangle \in \boldsymbol{K}$ and $a \in A$. Let $\mathfrak{L}_1^*$ be a regular language containing $\mathfrak{L}_0^*$ and formulas of the form $P_a(x_0)$.

By an argument of the above style we can also conclude that there are only countably many formulas of the form $K_a(x_0)$. Let us denote them by $K_n(x_0)$, $n < a$, for a suitable $a \leqslant \omega$. Referring to (11) it is immediate that there are only countably many sentences of the form I_a. Let us introduce the notation $I_{i,j}$, $i < a$, $j \leqslant \omega$, where i refers to the isomorphism type described by $K_i(x_0)$, and j tells the number of components of this type. Let $\mathfrak{L}_2^*$ be a regular language containing $\mathfrak{L}_1^*$ and the $I_{i,j}$.

Let $\theta, \psi \in \omega+1^a$. If $\theta \neq \psi$ it is easy to verify that $\bigwedge_{i<a} I_{i,\theta(i)}$ and $\bigwedge_{i<\omega} I_{i,\psi(i)}$ are contradictory. Since $S_0(\mathfrak{L}_2^*, \boldsymbol{K})$ is countable, it will follow that there are only countably many $\theta \in \omega+1^a$ such that $\bigwedge_{i<a} I_{i,\theta(i)}$ is true of some model of T. Since the sentence $\bigwedge_{i<a} I_{i,\theta(i)}$ completely describes the isomorphism type of a model in $\boldsymbol{K}$ which satisfies it, $\boldsymbol{K}$ has only countably many different isomorphism types.

THEOREM 2. *The number of isomorphism types of countable mono-unary algebras which satisfy a sentence of $\mathfrak{L}_{\omega_1\omega}$ is either countable or 2^ω.* (*This answers a problem of Vaught — in the case of mono-unary algebras* (see [3]).)

Proof. In [1] Morley proved everything stated except he allowed the possibility of ω_1 isomorphism types in a scattered theory, and we have just finished excluding this.

In conclusion we remark that all of the possible numbers of isomorphism types can be realized by a suitable theory of mono-unary algebras. Also, by some obvious modifications Theorem 2 is still true if we add a finite number of constants to our language (which already involves one unary operation).

References

[1] M. Morley, *The number of countable models*, Journ. Symb. Logic 35 (1970), pp. 14–18.

[2] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, Theory of Models, Amsterdam (1965), pp. 329–341.

[3] R. L. Vaught, *Denumerable Models of Complete Theories*, Proceedings of the Symposium in Foundations of Mathematics, Infinitistic Methods, New York 1961, pp. 303–321.

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