This shows that $|Y_0| \nleqslant |Y_1|$. A similar proof shows that $|Y_1| \nleqslant |Y_0|$.

THEOREM 4. $N \models$ *every non-well-orderable cardinal can be decomposed into incomparable cardinals.*

Proof. Suppose $N \models X$ is not well-orderable. Then in the notation used above, $|X| = |Y| = |Y_0| + |Y_1|$. Since $Y_0, Y_1 \subseteq Y$, $|Y_0|, |Y_1| \leqslant |Y|$. However $|Y_0| \neq |Y|$ (for otherwise $|Y_1| \leqslant |Y_0|$, contradicting Lemma 3), and similarly $|Y_1| \neq |Y|$.

It is not a theorem of ZF that every decomposable cardinal can be decomposed into incomparable cardinals. For it is consistent with ZF that there exist infinite sets X with only finite and cofinite subsets. For such X, $|X|$ is decomposable but the two cardinals of a decomposition are always comparable.

It is not a theorem of ZF (at any rate if the existence of an inaccessible cardinal is consistent) that every non-well-orderable cardinal is decomposable. For in [2] a model of ZF is constructed (assuming an inaccessible) in which 2^{ω} is neither an aleph nor decomposable.

References

[1] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Mathematics 223, Berlin 1971.

[2] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. Math. 92 (1970), pp. 1–56.

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Models of ZF with the same sets of sets of ordinals

by

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Abstract. In 1967 Vopěnka and Balcar showed that if M_1 and M_2 are transitive models of Zermelo-Fraenkel set theory (ZF) and one of M_1, M_2 satisfies the axiom of choice, then $M_1 = M_2$. Jech then constructed two distinct models of ZF with the same sets of ordinals. In this paper we exhibit an ω-sequence of transitive models of ZF such that the kth and $(k+1)$st models have the same sets of sets of sets ... (k times) of ordinals. The construction is by the method of forcing, each model being a generic extension of its predecessor in the sequence.

Introduction. In this paper we exhibit an ω-sequence of transitive models of Zermelo-Fraenkel set theory (ZF) such that the kth and $(k+1)$-st models have the same sets of sets ... (k times) of ordinals. The construction of the sequence of models proceeds by forcing, each model being a generic extension of its predecessor.

Vopěnka and Balcar showed in [5] that if M_1 and M_2 are transitive models of ZF with the same sets of ordinals and one of M_1, M_2 satisfies the axiom of choice, then $M_1 = M_2$. Jech [2] then constructed two distinct transitive models of ZF with the same sets of ordinals. The models are symmetric submodels of V-models. Jech left open the problem of constructing two distinct symmetric models with the same sets of sets of ordinals. Since the models in this paper are not symmetric models Jech's problem is still unsolved.

A condition for two models with the same sets of sets of ...of ordinals to be equal. We begin by introducing some weak axioms of choice. If X is a proper class, denote by $S(X)$ the class of all subsets of X. We set

$$S^0(\mathrm{ON}) = \mathrm{ON}\,, \quad S^{k+1}(\mathrm{ON}) = S\big(S^k(\mathrm{ON})\big) \quad \text{for} \quad k < \omega\,.$$

We define KWP^k as the axiom

$$(\forall x)(\exists f)(f\colon x \to S^k(\mathrm{ON}) \text{ and } f \text{ is } 1{:}1)\,.$$

(*) The results here are taken from the author's Ph. D. thesis (University of Bristol 1971), which was supervised by Dr F. Rowbottom. The author was supported by a Monash University Travelling Scholarship.

Then $KWP^0 = AC$ and KWP^1 is a form of the Kinna-Wagner Principle (see [3], or [1], p. 124).

LEMMA 1. (i) $j \leqslant k \rightarrow S^j(ON) \subseteq S^k(ON)$. (ii) *Each* $S^k(ON)$ *is transitive.*

By $Seq_\omega(X)$ we mean the set (or class) of finite sequences of members of X.

LEMMA 2. *For each* $k < \omega$ *there is a canonical coding* $\langle ... \rangle^k: Seq_\omega(S^k(ON)) \rightarrow S^k(ON)$.

Proof. We proceed by induction on k. The lemma is certainly true for $k = 0$. Suppose that we have a coding for $S^k(ON)$; we will construct one for $S^{k+1}(ON)$.

Let $A_1, \ldots, A_n \in S^{k+1}(ON)$. Then $A_1, \ldots, A_n \subseteq S^k(ON)$. For $1 \leqslant m \leqslant n$ set

$$A'_m = \begin{cases} \{\langle m, a\rangle^k:\ a \in A_m\} & \text{if } A_m \neq \emptyset, \\ \{\langle 0, m\rangle^k\} & \text{if } A_m = \emptyset. \end{cases}$$

Set $\langle A_1, \ldots, A_n\rangle^{k+1} = A'_1 \cup \ldots \cup A'_n$.

THEOREM 3. *Let* M_1, M_2 *be two transitive models of* ZF *such that* $M_1 \models KWP^k$ *and* $M_1 \cap S^{k+1}(ON) = M_2 \cap S^{k+1}(ON)$. *Then* $M_1 = M_2$.

Proof. The proof in [5] of the result of Vopěnka and Balcar mentioned above generalizes easily to this case.

For a set X we define the *transitive closure* of X $(TC(X))$ as follows

$$TC(X) = \{X\} \cup X \cup (\bigcup X) \cup (\bigcup\bigcup X) \ldots,$$

$TC(X)$ is the smallest transitive set containing X.

We recall some of the properties of relative constructibility. If M is a transitive proper class which is a model of ZF and X is a transitive set, $M(X)$ is the smallest transitive class N such that $M \subseteq N$, $X \in N$ and $N \models ZF$. There is a canonical function

$$F:\ M \times Seq_\omega(X) \rightarrow M(X)$$

which is onto. F is defined only from M and X and is absolute for any transitive model containing X and including M; in particular

$$M(X) \models F:\ M \times Seq_\omega(X) \rightarrow V \text{ and } F \text{ is onto}.$$

If $x \in M(X)$ and $x = F(m, z)$ we say x is *constructed by* m, X and z. If X is not transitive we take $M(X)$ to be $M(TC(X))$.

THEOREM 4. *If* $X \in S^k(ON)$, *then* $L(X) \models KWP^k$.

Proof. Suppose $y \in L(X)$. y is constructed by X, an ordinal β and a finite sequence $\langle x_1, \ldots, x_n\rangle$ of elements of $TC(X) - \{X\}$. $x_1, \ldots, x_n \in S^{k-1}(ON)$; set $y' = \langle \beta, x_1, \ldots, x_n\rangle^{k-1}$. We call $\langle \beta, x_1, \ldots, x_n\rangle$ "y' uncoded". Set

$$A_y = \{z \in S^{k-1}(ON):\ y \text{ is constructed by } X \text{ and } z \text{ uncoded}\}.$$

Let B_y be the set of elements of A_y of least rank. Then $B_y \in S^k(ON)$ and the map $y \rightarrow B_y$ is 1:1. So in $L(X)$ there is an injection of the universe into $S^k(ON)$. KWP^k then follows immediately.

Construction of the models. Before describing the construction of our promised sequence of models we make some remarks on forcing.

We follow the approach of [4], with one exception concerning names in the forcing language. In general we say that if $x \in M[G]$ $\boldsymbol{x}$ shall be a name for x, where $\boldsymbol{x}$ is a symbol which is an element of M. However if in fact $x \in M$ we take x as a name for x. This does not introduce any ambiguity, as if $x \in M$ $\boldsymbol{x}$ is interpreted in all $M[G]$ as x. If however the interpretation of $\boldsymbol{x}$ lies in $M[G] - M$ for some G, then the interpretation of $\boldsymbol{x}$ will change with G.

Shoenfield restricts himself to the case where the ground model M satisfies ZFC. However the proof in [4] of the fundamental theorem applies equally well to the case where $M \models ZF$ only. We use $\Vdash$ to represent the weak forcing relation. The following notation (borrowed from [4]) is useful for defining notions of forcing.

$$D_\varkappa(A, B) = \{f:\ f \text{ is a function, } |dom(f)| < \varkappa \text{ and } f \subseteq A \times B\}.$$

(Here $\varkappa$ is an aleph.)

We now give the construction of the models. Let M be a countable transitive model of $ZF + V = L$. Set $M^0 = M$, $J^0 = \omega$. M^{k+1} is obtained from M^k as follows. Let H^{k+1} be $D_\omega(J^k \times J^k, 2)$-generic over M^k. For $r \in J^k$ set

$$H_r^{k+1} = \{\langle s, a\rangle:\ \langle r, s, a\rangle \in \bigcup(H^{k+1})\}.$$

(Then $H_r^{k+1}:\ J^k \rightarrow 2$.) Set

$$J^{k+1} = \{H_r^{k+1}:\ r \in J^k\}, \qquad M^{k+1} = \big(M^k(J^{k+1})\big)^{M^k[H^{k+1}]}.$$

In particular each H_i^1 $(i < \omega)$ is a Cohen-generic real, and M^1 is a version of the model used by Halpern and Lévy to show that the Boolean prime ideal theorem does not imply the axiom of choice.

THEOREM 5. $M^k \models KWP^{k+1}$.

Proof. $M^k = M^{k-1}(J^k) = M^0(J^k)$. The result now follows from Theorem 4 and the observation that J^k is nearly an element of $S^{k+1}(ON)$ and can easily be converted into an actual element of $S^{k+1}(ON)$ by the canonical coding of Lemma 2.

LEMMA 6. *Suppose* ψ *is a* ZF*-formula,* $x \in M^{k-1}$. *Then* $M^k \models$ *if* $r_1, \ldots, r_n, s_1, \ldots, s_m \in J^k$ *and* $\{r_1, \ldots, r_n\} \cap \{s_1, \ldots, s_m\} = \emptyset$ *and* $\psi(J^k, r_1, \ldots, r_n, x, s_1, \ldots, s_m)$ *then there are* $f_i \in D_\omega(J^{k-1}, 2)$ $(1 \leqslant i \leqslant m)$ *such that if* $A_i = \{r \in J^k: r \supseteq f_i\}$ *then* $s_i \in A_i$, $i \neq j \rightarrow A_i \cap A_j = \emptyset$ *and*

$$(\forall t_1 \in A_1) \ldots (\forall t_m \in A_m)\, \psi(J^k, r_1, \ldots, r_n, x, t_1, \ldots, t_m).$$

Proof. A proof of the corresponding lemma for the model of Halpern and Lévy is given in [1], p. 133. The proof there may be transferred to M^k with little change. Of course in the above $s_1, \ldots, s_m$ must all be different.

LEMMA 7. *Suppose* $P \in D_\omega(J^k \times J^k, 2)$, ψ *is a* ZF-*formula and*

(1) $$P \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, \boldsymbol{H}^{k+1}_{r_n}, s_1, \ldots, s_m, x)$$

where $r_1, \ldots, r_n, s_1, \ldots, s_m \in J^k$ *and* $x \in M^{k-1}$. (*Here the ground model is* M^k.) *Then*

$$P | \{r_1, \ldots, r_n, s_1, \ldots, s_m\}^2 \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x) .$$

Proof. Suppose that

$$\operatorname{dom}(P) \subseteq \{r_1, \ldots, r_n, s_1, \ldots, s_m, t_1, \ldots, t_d\}$$

where $\{t_1, \ldots, t_d\} \cap \{r_1, \ldots, r_n, s_1, \ldots, s_m\} = \emptyset$. (1) is a statement of M^k, so we may apply Lemma 6 to it. We obtain $f_1, \ldots, f_d \in D_\omega(J^{k-1}, 2)$ such that if $A_i = \{r \in J^k\colon r \supseteq f_i\}$ and $t'_i \in A_i$ for $1 \leqslant i \leqslant d$ then

(2) $$P(t'_1, \ldots, t'_d) \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x)$$

where $P(t'_1, \ldots, t'_d)$ is obtained from P by substituting t'_i for t_i $(1 \leqslant i \leqslant d)$. Since each A_i is infinite (this is easily proved) it follows from (2) that

$$\{Q \in D_\omega(J^k \times J^k, 2)\colon Q \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x)\}$$

is dense below $P | \{r_1, \ldots, r_n, s_1, \ldots, s_m\}^2$, and thus that

$$P | \{r_1, \ldots, r_n, s_1, \ldots, s_m\}^2 \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x) .$$

THEOREM 8. $M^k \cap S^{k-1}(\mathrm{ON}) = M^{k-1} \cap S^{k-1}(\mathrm{ON})$.

Proof. We proceed by induction on k. The theorem is clearly true for $k = 1$. Suppose it true for k; we will prove it for $k+1$. In fact we prove by induction on j that for $j \leqslant k$

(1) $$X \in S^j(\mathrm{ON}) \quad \text{and} \quad X \in M^{k+1} \to X \in M^k.$$

(1) is certainly true for $j = 0$. Assume (1) true for $j-1$; we will prove it for j.

Take $X \in S^j(\mathrm{ON}) \cap M^{k+1}$. X is constructed by $J^{k+1}, H^{k+1}_{r_1}, \ldots, H^{k+1}_{r_n} \in \boldsymbol{J}^{k+1}$, $s_1, \ldots, s_m \in J^k$ and $x \in M^{k-1}$, so we may write

$$X = \{y \in S^{j-1}(\mathrm{ON})\colon \psi(J^{k+1}, H^{k+1}_{r_1}, \ldots, H^{k+1}_{r_n}, s_1, \ldots, s_m, x, y)\} .$$

By (1) for $j-1$, $y \in S^{j-1}(\mathrm{ON}) \cap M^{k+1} \to y \in M^k$ and, by the theorem for k, $y \in S^{j-1}(\mathrm{ON}) \cap M^k \to y \in M^{k-1}$ (as $j-1 \leqslant k-1$). So

$$X = \{y \in S^{j-1}(\mathrm{ON}) \cap M^{k-1}\colon \psi(J^{k+1}, H^{k+1}_{r_1}, \ldots, x, y)\} .$$

Now suppose $y \in X$. Then for some $P \in H^{k+1}$

$$P \Vdash \psi(\boldsymbol{J}^{k+1}, \ldots, \boldsymbol{H}^{k+1}_{r_1}, \boldsymbol{H}^{k+1}_{r_n} s_1, \ldots, s_m, x, y)$$

(here M^k is the ground model). By Lemma 7 in fact

$$P | \{r_1, \ldots, r_n, s_1, \ldots, s_m\}^2 \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x, y) .$$

So if we set $P_0 = (\bigcup H^{k+1}) | \{r_1, \ldots, r_n, s_1, \ldots, s_m\}^2$

$$X = \{y \in S^{j-1}(\mathrm{ON}) \cap M^{k-1}\colon P_0 \Vdash \psi(\boldsymbol{J}^{k+1}, \boldsymbol{H}^{k+1}_{r_1}, \ldots, x, y)\} .$$

This shows that X is an element of M^k. So we have proved (1) for j.

Thus (1) is true for all $j \leqslant k$. The case $j = k$ is just the theorem for $k+1$, so the induction step is completed.

Conclusion. We summarize what we have done. For $k < \omega$ we have given a condition on M_1, M_2 for $M_1 \cap S^k(\mathrm{ON}) = M_2 \cap S^k(\mathrm{ON})$ to imply $M_1 = M_2$, the condition being that one of M_1, M_2 satisfies KWP^{k-1}. For $k < \omega$ we have constructed two models $M_1 = M^k$ and $M_2 = M^{k+1}$ such that $M_1 \neq M_2$ but $M_1 \cap S^k(\mathrm{ON}) = M_2 \cap S^k(\mathrm{ON})$. Also $M_1 \models \mathrm{KWP}^{k+1}$. As a corollary

THEOREM 9. *No* KWP^k *is a theorem of* ZF. *Indeed* $\mathrm{KWP}^{k+1} \nleftrightarrow \mathrm{KWP}^{k-1}$.

We close with some remarks on the strength of the axioms KWP^k. KWP^1 is treated extensively in [1]; among results given there are

$\mathrm{KWP}^1 \to$ every set can be ordered,

$\mathrm{KWP}^1 \leftrightarrow$ the order extension principle,

$\mathrm{KWP}^1 \leftrightarrow$ the countable axiom of choice (C^ω).

For KWP^2, consider the following construction. Set

$$X = \{r \in {}^\omega 2\colon r \text{ differs only finitely from some } H^1_i\} ,$$
$$N = \big(M^0(X)\big)^{M^0(H^1)} .$$

Then from Theorem 4 $N \models \mathrm{KWP}^2$. However D. H. Stewart (unpublished) has shown that

$N \models$ there is an infinite set which has only finite and cofinite subsets .

It follows that in N C^ω and the order principle fail, and it can easily be shown that the principle "there is a choice function for any family of countable sets" also fails in N. So for $k \geqslant 2$ KWP^k is a very weak axiom.

References

[1] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Mathematics 223, Berlin 1971.

[2] T. Jech, *On models of set theory without* AC, Proceedings of the 1967 UCLA Summer Institute on Axiomatic Set Theory. A.M.S. Proceedings of Symposia in Pure Mathematics XIII, Part I, Providence (R.I.) 1971, pp. 135–141.

[3] W. Kinna and K. Wagner, *Über eine Abschwächung des Auswahlpostulates*, Fund. Math. 42 (1955), pp. 76–82.

[4] J. R. Shoenfield, *Unramified forcing*, Proceedings of the 1967 UCLA Summer Institute, pp. 357–381.

[5] P. Vopěnka and B. Balcar, *On complete models of the set theory*, Bull. Acad. Pol. Sci. sér. math. astron. phys. 15 (1967), pp. 839–841.

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Scott sentences and a problem of Vaught for mono-unary algebras

by

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Abstract. First we show that there is a countable ordinal α such that if $\langle A, \{f\}\rangle$ is a mono-unary algebra then one can find a Scott sentence (which describes $\langle A, \{f\}\rangle$ up to isomorphism) whose rank is less than α. Combining this result with Morley's we see that if a sentence of $\mathfrak{L}_{\omega_1\omega}$ for mono-unary algebras has more than denumerably many isomorphism types of countable models then it must have continuum many of these isomorphism types.

We wish to show that for a given countable mono-unary algebra $\mathfrak{A}$ we can construct a reasonably simple *Scott Sentence* $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega_1\omega}$, i.e. for any countable mono-unary algebra $\mathfrak{B}$, $\mathfrak{B} \models \varphi_{\mathfrak{A}}$ iff $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$. Then we apply the methods of Morley [1] to determine the possible number of isomorphism types which can be realized among the countable models of a $\mathfrak{L}_{\omega_1\omega}$ sentence for mono-unary algebras.

1. The Scott Sentence. In what follows we will always assume $\mathfrak{L}_{\omega_1\omega}$ involves one non-logical symbol, a unary operation symbol. Let $\mathfrak{L}$ be a subset of $\mathfrak{L}_{\omega_1\omega}$. Define $\boldsymbol{C}_0(\mathfrak{L})$ to be the closure of $\mathfrak{L}$ under $\wedge$, $\vee$, $\neg$, $\exists$ and $\forall$; define $\boldsymbol{C}_1(\mathfrak{L})$ to be $\mathfrak{L}$ union the set of formulas formed by taking the countable conjunction (or disjunction) of a set $\mathcal{F}$ of formulas in $\mathfrak{L}$, where the set of variables which occur free in members of $\mathcal{F}$ is finite.

Define a transfinite sequence $\mathfrak{L}_0 \subseteq \mathfrak{L}_1 \subseteq \ldots$ by the following inductive procedure:

$\mathfrak{L}_0$ is the usual first-order predicate calculus with one unary operation symbol,

$\mathfrak{L}_\xi = \bigcup_{\eta<\xi} \mathfrak{L}_\eta$ for limit ordinals ξ, $\xi < \omega_1$, and $\mathfrak{L}_{\xi+1} = \boldsymbol{C}_0\boldsymbol{C}_1(\mathfrak{L}_\xi)$ for $\xi < \omega_1$.

Then $\mathfrak{L}_{\omega_1\omega} = \bigcup_{\xi<\omega_1} \mathfrak{L}_\xi$.

THEOREM 1. *The isomorphism type of a countable mono-unary algebra $\mathfrak{A} = \langle A, \{f\}\rangle$ can be defined by a single sentence $\varphi_{\mathfrak{A}}$ in $\mathfrak{L}_{\omega+4}$.*

Proof. In the following we will introduce the notations which will be used to construct $\varphi_{\mathfrak{A}}$, and following each definition we will state its meaning as well as an $\mathfrak{L}_\xi$ to which it belongs. In much of what follows