Decomposable cardinals

by

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Abstract. An infinite cardinal \( m \) is decomposable if there exist cardinals \( p, q < m \) such that \( p + q = m \). Well-orderable cardinals are not decomposable, so in the presence of the axiom of choice there are no decomposable cardinals. Let \( N \) be the Halpern-Lévy model for the independence of the axiom of choice from the Boolean prime ideal theorem. We show that in \( N \) every non-well-orderable cardinal is decomposable; in particular, \( N \models 2^\omega \) is decomposable.

An infinite cardinal \( m \) is decomposable if there exist cardinals \( p, q < m \) such that \( p + q = m \). Alephs are not decomposable, so in the presence of the axiom of choice there are no decomposable cardinals. In this paper we prove that in the Halpern-Lévy model for set theory without the axiom of choice every non-well-orderable cardinal is decomposable; in particular \( 2^\omega \) is decomposable in the model.

The Halpern-Lévy model is described in [1]; we give here another version of the model. If \( \kappa \) is an aleph we define

\[ S_\kappa(A) = \{X \subseteq A : |X| < \kappa\} \]

and

\[ H_\kappa(A, B) = \{f : \text{dom}(f) \in S_\kappa(A) \text{ and \ ran}(f) \subseteq B\} \cdot \]

Let \( M \) be a countable transitive model of \( \text{ZF} + \forall \in L_1 \), and let \( G \) be \( H_\omega(\omega \times \omega, 2) \)-generic over \( M \). For \( i < \omega \) define

\[ G_i = \{\langle n, k \rangle : \langle i, n, k \rangle \in G\} \cdot \]

Then each \( G_i \) is an element of \( \kappa \). Set \( G^* = \{G_i : i < \omega\} \) and \( N = (L(G^*))^{M[G]} \). \( N \) is our version of the Halpern-Lévy model. We note that every element of \( N \) is constructible from \( G^* \), some \( G_{i_1}, \ldots, G_{i_n} \) and an element \( \pi \in M \).

(*) The results here are taken from the author's Ph. D. thesis (University of Bristol 1971), which was supervised by Dr. P. Rowbottom. The author was supported by a Monash University Traveling Scholarship.

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THEOREM 1. Let \( \psi \) be a ZF-formula, \( x \in M \). Then \( N \models \) if \( r, r_1, \ldots, r_n \in G^* \) and \( \psi(G^*, r_1, \ldots, r_n, x, r) \) and \( r \notin \{r_1, \ldots, r_n\} \) then there is a \( h \in H_\omega(\omega, 2) \) such that \( r \geq h \) and for all \( r' \geq h \)

\[ r' \in G^* \Rightarrow \psi(G^*, r_1, \ldots, r_n, x, r') \]

Proof. See [1], p. 133. The proof there is easily adapted to our version of the model.

THEOREM 2. If \( y \in N \) there is a unique smallest subset \( a \) of \( G^* \) such that \( y \) is constructible from \( G^* \), \( a \) and some \( x \in M \).

Proof. See [1], p. 137. Of course \( a \) is finite.

We call a the support of \( y \), written \( \text{supp}(y) \).

It is an immediate consequence of Theorem 1 that \( G^* \), and thence \( \omega_2 \), cannot be well-ordered in \( N \).

For the rest of the paper \( X \) is a fixed element of \( N \) such that \( X \) cannot be well-ordered in \( N \). We will work in \( N \) throughout.

We observe first that

\[ \{\text{supp}(x) : x \in X\} \text{ is infinite.} \]

For if \( \{\text{supp}(x) : x \in X\} \) is finite it can be well-ordered, and any such well-ordered immediately induces a well-ordering on \( X \). (See [1], p. 138.)

Write \( \alpha_x \) for \( \text{supp}(x) \). From \( \alpha_x \) we define an injection

\[ \theta : X \rightarrow S_\omega(G^* - \alpha_x) \times ON \]

as follows. For \( x \in X \)

\[ \theta(x) = (\text{supp}(x), \alpha_x, x) \]

where \( \alpha \) is the least ordinal such that \( x \) is the \( \alpha \)th set constructed from \( \text{supp}(x) \cup \alpha_x \) and \( G^* \). Set \( Y = \theta'X \); we note that \( Y \) is constructible from \( \alpha_x \) and \( G^* \). Set

\[ A = \bigcup \{a \in S_\omega(G^*) : (H_\alpha)((a, \alpha, \alpha) \in Y)\} \]

It follows from (1) that \( A \) is infinite. Clearly \( A \) is constructible from \( \alpha_x \) and \( G^* \), and \( A \cap \alpha_x = \emptyset \). So by an application of Theorem 1 (to the sentence \( \alpha \in A \)) we have

\[ (\forall r \in A) (\exists p \in H_\omega(\omega, 2)) (r \geq p) \text{ and } (\exists s \in G^*) (s \supseteq p \in A) \]

We now define by induction two sequences \( \{A_n\}_{n \in \omega} \) and \( \{B_n\}_{n \in \omega} \) of nonempty subsets of \( A \) and a sequence \( \{p_{n}\}_{n \in \omega} \) of elements of \( H_\omega(\omega, 2) \). (Here \( n \) is either finite or equal to \( \omega \).) Let \( \rho \) be a particular well-order in type \( \omega \) of \( H_\omega(\omega, 2) \).

\( p_0 \) is the \( \rho \)-first element \( p \) of \( H_\omega(\omega, 2) \) such that for all \( r \in G^* \), \( r \geq p \rightarrow r \in A \) (such \( p \) exists by (2)). We set

\[ A_0 = \{ r \in G^* : r \supseteq p_0 \} \quad B_0 = A - A_0 \]

\( p_{n+1} \) is the \( \rho \)-first element \( p \) of \( H_\omega(\omega, 2) \) such that

\[ (\forall r \in G^*) (r \supseteq p \rightarrow r \in B_n) \]

Such exist if \( B_n \) is non-empty, because if \( x \in B_n \) by (2) there is some \( p \) such that \( x \supseteq p \) and \( (\forall r \in G^*) (r \supseteq p \rightarrow r \in A) \). Now extend \( p \) to \( p' \) incompatible with each of \( p_0, \ldots, p_n \). Then

\[ r \supseteq p' \rightarrow r \in A_0 \cup \ldots \cup A_n \]

so \( r \in G^* \) and \( r \supseteq p' \rightarrow r \in B_0 \). We set

\[ A_{n+1} = \{ r \in G^* : r \supseteq p_{n+1} \} \quad B_{n+1} = B_n - A_{n+1} \]

If \( B_n \) is empty we stop. Clearly \( i, j \leq n \implies A_i \cap A_j = \emptyset \) and \( \bigcup_{\alpha \leq n} A_\alpha = A \).

We now split \( A \) into two halves \( A_0, A_1 \) as follows. Set \( m_n = (\text{dom}(p_n) \cup \omega) \). Then

\[ A_0 \cap A_1 = \{ r \in A_n : (x \geq m_n)(|r(m) - 1|) \text{ is even} \} \]

\[ A_1 \cap A_0 = \{ r \in A_n : (x \geq m_n)(|r(m) - 1|) \text{ is odd} \} \]

We use \( A_0 \) and \( A_1 \) to split \( X \) into two halves \( X_0, X_1 \). Thus, if \( a \in S_\omega(G^*) \)

(3) \( \text{we write } a_0 \text{ for the lexicographically first element of } a \). Take \( \langle a, a \rangle \in Y \),

\[ a = \emptyset \rightarrow \langle a, a \rangle \in X_0 \]

a \( \neq \emptyset \) and \( a_0 \in A_0 \rightarrow \langle a, a \rangle \in X_0 \),

a \( \neq \emptyset \) and \( a_0 \in A_1 \rightarrow \langle a, a \rangle \in X_1 \).

LEMMA 3. \( \{Y_0\} \) and \( \{Y_1\} \) are incomparable.

Proof. Suppose that \( f : Y_0 \rightarrow Y_1 \) is an injection and \( f \) has support \( c \in S_\omega(G^*) \). Set \( d = c - A \), \( e = c \cap A \). Then \( e \cap A_n \) for some \( n < \omega \), so \( \rho \supseteq p_n \). It is clear that

\[ (3) \text{ in } A_n \text{ there are elements of both } A_0 \text{ and } A_1 \text{ which lexicographically precede } c_n. \]

We choose \( c, c_n \in Y \) such that \( a_0 \leq_{\text{lex}} e_0 \) and \( a_0 \leq c_n \). (Here \( \leq_{\text{lex}} \) is the lexicographic order on \( \omega \).) This implies that \( \langle c, a \rangle \in X_0 \), so set

\[ f(\langle c, a \rangle) = \langle b, b \rangle \]

(4) \( a \geq b \) we may hold \( b, c \) and \( a \) \( - \) \( \langle a \rangle \) fixed and apply Theorem 1 to (4) to move \( a \). We conclude that \( f \) is not 1-1, contrary to our hypothesis.

If \( a_0 \notin c_n \), certainly \( a_0 \neq b_0 \) (as \( a_0 \in H_\omega, b_0 \in H_\omega \)). So \( b_0 \leq_{\text{lex}} a_0 \) whence \( b_0 \leq c \) and \( b_0 \notin a \). Thus we may hold \( a, c \) and \( b_0 \notin a \) fixed and apply Theorem 1 to (4) to move \( b_0 \). We conclude that \( f \) is not a function, again a contradiction.
This shows that \(|X_4| \not< |Y_1|\). A similar proof shows that \(|Y_1| \not< |Y_0|\).

**Theorem 4.** \(N = \{\text{every non-well-orderable cardinal can be decomposed into incomparable cardinals.}\)**

**Proof.** Suppose \(N \models X\) is not well-orderable. Then in the notation used above, \(|X| = |Y| = |X_0| + |Y_1|\). Since \(X_2, X_3 \subseteq Y, Y_0 \subseteq X_0, Y_1 \subseteq Y\).

However \(|X_4| \not= |Y|\) (for otherwise \(|Y_1| \not< |Y_0|\), contradicting Lemma 5), and similarly \(|Y_3| \not= |Y|\).

It is not a theorem of ZF that every decomposable cardinal can be decomposed into incomparable cardinals. For it is consistent with ZF that there exist infinite sets \(X\) with only finite and cofinite subsets. For such \(X\), \(|X|\) is decomposable but the two cardinals of a decomposition are always comparable.

It is not a theorem of ZF (at any rate if the existence of an inaccessible cardinal is consistent) that every non-well-orderable cardinal is decomposable. For in [2] a model of ZF is constructed (assuming an inaccessible) in which \(2^n\) is neither an aleph nor decomposable.

**References**


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**Models of ZF with the same sets of ordinals**

by

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**Abstract.** In 1967 Vopěnka and Balcar showed that if \(M_1\) and \(M_2\) are transitive models of Zermelo-Fraenkel set theory (ZF) and one of \(M_1, M_2\) satisfies the axiom of choice, then \(M_1 = M_2\). Jech then constructed two distinct models of ZF with the same sets of ordinals. In this paper we exhibit an \(\omega\)-sequence of transitive models of ZF such that the \(k\)th and \((k+1)\)st models have the same sets of sets... (\(k\) times) of ordinals. The construction is by the method of forcing, each model being a generic extension of its predecessor in the sequence.

**Introduction.** In this paper we exhibit an \(\omega\)-sequence of transitive models of Zermelo-Fraenkel set theory (ZF) such that the \(k\)th and \((k+1)\)th models have the same sets of sets... (\(k\) times) of ordinals. The construction of the sequence of models proceeds by forcing, each model being a generic extension of its predecessor.

Vopěnka and Balcar showed in [5] that if \(M_1\) and \(M_2\) are transitive models of ZF with the same sets of ordinals and one of \(M_1, M_2\) satisfies the axiom of choice, then \(M_1 = M_2\). Jech then constructed two distinct transitive models of ZF with the same sets of ordinals. The models are symmetric submodels of \(V\)-models. Jech left open the problem of constructing two distinct symmetric models with the same sets of sets of ordinals. Since the models in this paper are not symmetric models Jech's problem is still unsolved.

**A condition for two models with the same sets of sets of... of ordinals to be equal.** We begin by introducing some weak axioms of choice. If \(X\) is a proper class, denote by \(S(X)\) the class of all subsets of \(X\). We set

\[
S^0(ON) = ON, \quad S^{k+1}(ON) = S(S^k(ON)) \quad \text{for} \quad k < \omega.
\]

We define \(\text{KWP}^k\) as the axiom

\[
(\forall x)(\exists f)(f: x \rightarrow S^k(ON) \text{ and } f \text{ is } 1:1).
\]

(*) The results here are taken from the author's Ph. D. thesis (University of Bristol 1971), which was supervised by Dr P. Rowbottom. The author was supported by a Monash University Travelling Scholarship.