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Decomposable cardinals

by

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Abstract. An infinite cardinal m is *decomposable* if there exist cardinals $p, q < m$ such that $p+q = m$. Well-orderable cardinals are not decomposable, so in the presence of the axiom of choice there are no decomposable cardinals. Let N be the Halpern-Lévy model for the independence of the axiom of choice from the Boolean prime ideal theorem. We show that in N every non-well-orderable cardinal is decomposable; in particular, $N \models 2^\omega$ is decomposable.

An infinite cardinal m is *decomposable* if there exist cardinals $p, q < m$ such that $p+q = m$. Alephs are not decomposable, so in the presence of the axiom of choice there are no decomposable cardinals. In this paper we prove that in the Halpern-Lévy model for set theory without the axiom of choice every non-well-orderable cardinal is decomposable; in particular 2^ω is decomposable in the model.

The Halpern-Lévy model is described in [1]; we give here another version of the model. If $\varkappa$ is an aleph we define

$$S_\varkappa(A) = \{X \subseteq A\colon\ |X| < \varkappa\}$$

and

$$H_\varkappa(A, B) = \{f \text{ a function: } \operatorname{dom}(f) \in S_\varkappa(A) \text{ and } \operatorname{ran}(f) \subseteq B\}.$$

Let M be a countable transitive model of $ZF + V = L$, and let G be $H_\omega(\omega\times\omega, 2)$-generic over M. For $i < \omega$ define

$$G_i = \{\langle n, k\rangle\colon\ \{\langle i, n, k\rangle\} \in G\}.$$

Then each G_i is an element of ${}^\omega 2$. Set $G^* = \{G_i\colon i < \omega\}$ and $N = \big(L(G^*)\big)^{M[G]}$. N is our version of the Halpern-Lévy model. We note that every element of N is constructible from G^*, some $G_{i_1}, \ldots, G_{i_n}$ and an element $x \in M$.

(*) The results here are taken from the author's Ph. D. thesis (University of Bristol 1971), which was supervised by Dr F. Rowbottom. The author was supported by a Monash University Travelling Scholarship.

THEOREM 1. *Let ψ be a ZF-formula, $x \in M$. Then $N \models$ if $r, r_1, \ldots, r_n \in G^*$ and $\psi(G^*, r_1, \ldots, r_n, x, r)$ and $r \notin \{r_1, \ldots, r_n\}$ then there is $h \in H_\omega(\omega, 2)$ such that $r \supseteq h$ and for all $r' \supseteq h$*

$$r' \in G^* \rightarrow \psi(G^*, r_1, \ldots, r_n, x, r') .$$

Proof. See [1], p. 133. The proof there is easily adapted to our version of the model.

THEOREM 2. *If $y \in N$ there is a unique smallest subset a of G^* such that y is constructible from G^*, a and some $x \in M$.*

Proof. See [1], p. 137. Of course a is finite.

We call a the *support* of y, written $\mathrm{supp}(y)$.

It is an immediate consequence of Theorem 1 that G^*, and thence ${}^\omega 2$, cannot be well-ordered in N.

For the rest of the paper X is a fixed element of N such that X cannot be well-ordered in N. We will work in N throughout.

We observe first that

(1) $$\{\mathrm{supp}(x):\ x \in X\} \text{ is infinite.}$$

For if $\{\mathrm{supp}(x):\ x \in X\}$ is finite it can be well-ordered, and any such well-ordering immediately induces a well-ordering on X. (See [1], p. 138).

Write a_X for $\mathrm{supp}(X)$. From a_X we define an injection

$$\theta:\ X \rightarrow S_\omega(G^* - a_X) \times \mathrm{ON}$$

as follows. For $x \in X$

$$\theta(x) = \langle \mathrm{supp}(x) - a_X,\ \alpha \rangle$$

where α is the least ordinal such that x is the αth set constructed from $\mathrm{supp}(x) \cup a_X$ and G^*. Set $Y = \theta'' X$; we note that Y is constructible from a_X and G^*. Set

$$A = \bigcup \{a \in S_\omega(G^*):\ (\exists \alpha)(\langle a, \alpha \rangle \in Y)\} .$$

It follows from (1) that A is infinite. Clearly A is constructible from a_X and G^*, and $A \cap a_X = \emptyset$. So by an application of Theorem 1 (to the sentence "$r \in A$") we have

(2) $$(\forall r \in A)\big(\exists p \in H_\omega(\omega, 2)\big)\big(r \supseteq p \text{ and } (\forall s \in G^*)(s \supseteq p \rightarrow s \in A)\big) .$$

We now define by induction two sequences $(A_n)_{n<n_0}$ and $(B_n)_{n<n_0}$ of subsets of A and a sequence $(p_n)_{n<n_0}$ of elements of $H_\omega(\omega, 2)$. (Here n_0 is either finite or equal to ω.) Let ϱ be a particular well-order in type ω of $H_\omega(\omega, 2)$.

p_0 is the ϱ-first element p of $H_\omega(\omega, 2)$ such that for all $r \in G^*$, $r \supseteq p \rightarrow r \in A$ (such p exist, by (2)). We set

$$A_0 = \{r \in G^*:\ r \supseteq p_0\} , \qquad B_0 = A - A_0 .$$

p_{n+1} is the ϱ-first element p of $H_\omega(\omega, 2)$ such that

$$(\forall r \in G^*)(r \supseteq p \rightarrow r \in B_n) .$$

Such exist if B_n is non-empty, because if $s \in B_n$ by (2) there is some p such that $s \supseteq p$ and $(\forall r \in G^*)(r \supseteq p \rightarrow r \in A)$. Now extend p to p' incompatible with each of $p_0, \ldots, p_n$. Then

$$r \supseteq p' \rightarrow r \notin A_0 \cup \ldots \cup A_n ,$$

so $r \in G^*$ and $r \supseteq p' \rightarrow r \in B_n$. We set

$$A_{n+1} = \{r \in G^*:\ r \supseteq p_{n+1}\} , \qquad B_{n+1} = B_n - A_{n+1} .$$

If B_n is empty we stop. Clearly $i, j < n_0 \rightarrow A_i \cap A_j = \emptyset$ and $\bigcup_{n<n_0} A_n = A$.

We now split A into two halves H_0, H_1 as follows. Set $m_n = (\mu m)(\mathrm{dom}(p_n) \subseteq m)$. Then

$$H_0 \cap A_n = \big\{r \in A_n:\ (\mu m \geqslant m_n)\big(r(m) = 1\big) \text{ is even}\big\} ,$$

$$H_1 \cap A_n = \big\{r \in A_n:\ (\mu m \geqslant m_n)\big(r(m) = 1\big) \text{ is odd}\big\} .$$

We use H_0 and H_1 to split Y into two halves Y_0, Y_1 thus. If $a \in S_\omega(G^*) - \{\emptyset\}$ we write a_0 for the lexicographically first element of a. Take $\langle a, \alpha \rangle \in Y$,

$$a = \emptyset \rightarrow \langle a, \alpha \rangle \in Y_0 ,$$

$$a \neq \emptyset \quad \text{and} \quad a_0 \in H_0 \rightarrow \langle a, \alpha \rangle \in Y_0 ,$$

$$a \neq \emptyset \quad \text{and} \quad a_0 \in H_1 \rightarrow \langle a, \alpha \rangle \in Y_1 .$$

LEMMA 3. *$|Y_0|$ and $|Y_1|$ are incomparable.*

Proof. Suppose that $f:\ Y_0 \rightarrow Y_1$ is an injection and f has support $c \in S_\omega(G^*)$. Set $d = c - A$, $e = c \cap A$. Then $e_0 \in A_n$ for some $n < n_0$, so $e_0 \supseteq p_n$. It is clear that

(3) in A_n there are elements of both H_0 and H_1 which lexicographically precede e_0.

We choose $\langle a, \alpha \rangle \in Y$ such that $a_0 <_{\mathrm{lex}} e_0$ and $a_0 \in H_0$. (Here $<_{\mathrm{lex}}$ is the lexicographic order on ${}^\omega 2$.) This implies that $\langle a, \alpha \rangle \in Y_0$, so set

(4) $$f(\langle a, \alpha \rangle) = \langle b, \beta \rangle .$$

If $a_0 \notin b$ we may hold b, c and $a - \{a_0\}$ fixed and apply Theorem 1 to (4) to move a_0. We conclude that f is not 1:1, contrary to our hypothesis.

If $a_0 \in b$, certainly $a_0 \neq b_0$ (as $a_0 \in H_0$, $b_0 \in H_1$). So $b_0 <_{\mathrm{lex}} a_0$, whence $b_0 \notin c$ and $b_0 \notin a$. Thus we may hold a, c and $b - \{b_0\}$ fixed and apply Theorem 1 to (4) to move b_0. We conclude that f is not a function, again a contradiction.

This shows that $|Y_0| \nleqslant |Y_1|$. A similar proof shows that $|Y_1| \nleqslant |Y_0|$.

THEOREM 4. $N \models$ *every non-well-orderable cardinal can be decomposed into incomparable cardinals.*

Proof. Suppose $N \models X$ is not well-orderable. Then in the notation used above, $|X| = |Y| = |Y_0| + |Y_1|$. Since $Y_0, Y_1 \subseteq Y$, $|Y_0|, |Y_1| \leqslant |Y|$. However $|Y_0| \neq |Y|$ (for otherwise $|Y_1| \leqslant |Y_0|$, contradicting Lemma 3), and similarly $|Y_1| \neq |Y|$.

It is not a theorem of ZF that every decomposable cardinal can be decomposed into incomparable cardinals. For it is consistent with ZF that there exist infinite sets X with only finite and cofinite subsets. For such X, $|X|$ is decomposable but the two cardinals of a decomposition are always comparable.

It is not a theorem of ZF (at any rate if the existence of an inaccessible cardinal is consistent) that every non-well-orderable cardinal is decomposable. For in [2] a model of ZF is constructed (assuming an inaccessible) in which 2^{ω} is neither an aleph nor decomposable.

References

[1] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Mathematics 223, Berlin 1971.

[2] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. Math. 92 (1970), pp. 1–56.

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Models of ZF with the same sets of sets of ordinals

by

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Abstract. In 1967 Vopěnka and Balcar showed that if M_1 and M_2 are transitive models of Zermelo-Fraenkel set theory (ZF) and one of M_1, M_2 satisfies the axiom of choice, then $M_1 = M_2$. Jech then constructed two distinct models of ZF with the same sets of ordinals. In this paper we exhibit an ω-sequence of transitive models of ZF such that the kth and $(k+1)$st models have the same sets of sets of sets... (k times) of ordinals. The construction is by the method of forcing, each model being a generic extension of its predecessor in the sequence.

Introduction. In this paper we exhibit an ω-sequence of transitive models of Zermelo-Fraenkel set theory (ZF) such that the kth and $(k+1)$-st models have the same sets of sets... (k times) of ordinals. The construction of the sequence of models proceeds by forcing, each model being a generic extension of its predecessor.

Vopěnka and Balcar showed in [5] that if M_1 and M_2 are transitive models of ZF with the same sets of ordinals and one of M_1, M_2 satisfies the axiom of choice, then $M_1 = M_2$. Jech [2] then constructed two distinct transitive models of ZF with the same sets of ordinals. The models are symmetric submodels of V-models. Jech left open the problem of constructing two distinct symmetric models with the same sets of sets of ordinals. Since the models in this paper are not symmetric models Jech's problem is still unsolved.

A condition for two models with the same sets of sets of ...of ordinals to be equal. We begin by introducing some weak axioms of choice. If X is a proper class, denote by $S(X)$ the class of all subsets of X. We set

$$S^0(\mathrm{ON}) = \mathrm{ON}\,, \quad S^{k+1}(\mathrm{ON}) = S\big(S^k(\mathrm{ON})\big) \quad \text{for} \quad k < \omega\,.$$

We define KWP^k as the axiom

$$(\forall x)(\exists f)(f\colon x \to S^k(\mathrm{ON}) \text{ and } f \text{ is } 1{:}1)\,.$$

(*) The results here are taken from the author's Ph. D. thesis (University of Bristol 1971), which was supervised by Dr F. Rowbottom. The author was supported by a Monash University Travelling Scholarship.