Spaces whose connected expansions preserve connected subsets

by

J. A. Guthrie and **H. E. Stone** (Pittsburgh, Penn.)

Abstract. A space is *essentially connected* if every finer connected topology determines the same connected subsets. This class includes the maximally connected spaces and the connected weakly linearly orderable topological spaces (WLOTS). A characterization of the class among WLOTS-wise connected spaces is obtained. Essential and maximal connectedness are inherited by connected subspaces, preserved by hereditarily quotient monotone maps, and destroyed by products. A concluding result is relevant to the maximally connected Hausdorff space problem: No space with a dispersion point has a finer maximally connected topology.

§ 1. Introduction. A topology finer than a topology τ is called an *expansion* of τ and a space X furnished with such a finer topology is called an expansion of (X, τ). In this paper we study spaces X whose connected subsets remain connected as subspaces of every expansion in which X remains connected. We call such spaces *essentially* connected.

Hildebrand observed in [7] that the unit interval is essentially connected in the process of constructing an intriguing connected expansion of it. This follows already from the result of Eilenberg [5] that connected weakly ordered spaces are essentially connected. Tanaka [9] generalized Hildebrand's observation as follows: *For a connected, locally connected, compact, second countable Hausdorff space X to be essentially connected, it is necessary and sufficient that* (i) *X contain no simple closed curve, and* (ii) *each simple arc in X contain at most a finite number of branch points.* Our Theorem 9 gives a characterization of essential connectedness in a large class of spaces, and includes the results of Tanaka and of Eilenberg as special cases. We also study products, mappings, and subspaces of essentially connected spaces.

A connected space having no connected expansion is *maximally connected*; such spaces have been studied in [10] and [6]. Maximal connectedness is studied here as an interesting special case of essential connectedness, and Theorem 11 identifies a class of spaces in which these concepts coincide.

Following the general terminology of Cameron [4], we call a space and its topology *strongly connected* if it has a maximally connected expansion. It is an open question whether there exist maximally connected Hausdorff spaces other than singletons. On the other hand, we show that a connected Hausdorff space with dispersion point is not strongly connected.

If (X, τ) is a topological space and $S \subseteq X$, we denote by $\tau|S$ the subspace topology induced on S by τ. The topology for X whose subbase is $\tau \cup \{S\}$ is the *simple expansion of* τ *by* S, and is denoted $\tau(S)$. The interior, closure, and boundary of S will be denoted by $\mathrm{Int}(S)$, $\mathrm{Cl}(S)$, and ∂S respectively, if necessary with a subscript indicating with respect to what topology these operations are being performed. Other conventions and terminology will be consistent with the usage of Willard [11].

§ 2. Essential connectedness. Let C be a connected subset of an essentially connected space X. If C is either open or closed in X, the proofs of Lemmas 1 and 2 of [6] show that every connected expansion of C is a subspace of some connected expansion of X. If now K is a connected subset of C and some connected expansion of C disconnects K, then two successive simple expansions give a connected topology for C which disconnects K. Then the corresponding sequence of simple expansions of X gives a connected expansion of X which disconnects K, contradicting the essential connectedness of X.

For arbitrary connected C, $\mathrm{Cl}(C)$ is connected, and hence essentially connected by what we have just said. Since C is dense in $\mathrm{Cl}(C)$, by Anderson [1] the simple expansion of $\mathrm{Cl}(C)$ by C is connected. The subspace topology of C is not altered by this process, and since C is now an open subset of an essentially connected space, the argument of the preceeding paragraph shows that C is essentially connected. Hence we have established the following result.

THEOREM 1. *Every connected subspace of an essentially connected space is essentially connected.*

A *submaximal space* is one in which every dense set is open; it is shown in [6] and follows from [1] that every connected space has a connected submaximal expansion in which a given filter of dense sets is made open. This fact allows us to prove another suggestive result.

THEOREM 2. *Let (X, τ) be an essentially connected space, and let C be a connected subset having at least two points. Then $\mathrm{Int}_\tau(C) \neq \emptyset$, and if (X, τ) is a T_1 space, $\mathrm{Int}_\tau(C)$ is dense in C.*

Proof. Suppose $I = \mathrm{Int}_\tau(C) = \emptyset$. Then $X - C$ is τ-dense, and there is a submaximal expansion μ of τ in which all sets containing $X - C$ are open. Then C is closed and discrete as a μ-subspace, and hence μ-disconnected. But τ is essentially connected, so that C must be μ-connected. Hence $I \neq \emptyset$.

Now suppose (X, τ) is T_1. If I is not dense in C, then $C - I$ has $\tau|C$-interior. But $I \cup (X - C)$ is τ-dense, and there is a submaximal expansion μ in which all sets containing $I \cup (X - C)$ are open. Then $C - I$ is discrete as a μ-subspace and has $\mu|C$-interior. Hence C contains a μ-isolated point, again contradicting the μ-connectedness of C.

The following theorem gives conditions under which the intersection of two connected sets is connected in an essentially connected space. We do not know whether this condition is necessary; we have no example of an essentially connected space in which two connected sets have disconnected intersection.

THEOREM 3. *Let (X, τ) be essentially connected, and let A and B be connected subsets of X. If $A \cap B$ is not connected, then the components of $A \cap B$ do not form a closure preserving collection.*

Proof. Suppose $A \cap B$ is disconnected. Then $A \cap B \neq \emptyset$, and by Theorem 1 we may take $X = A \cup B$. Let P and Q be distinct components of $A \cap B$ and let x be a cluster point of each. Then $P \cup \{x\} \cup Q = A'$ is connected, and by the distinctness of P and Q, $x \notin A \cap B$. For definiteness let $x \in A$. Then $A' \cap B$ has exactly two components, P and Q. Let $V = X - (\mathrm{Cl}(P) - \{x\})$, and let $\tau' = \tau(V)$. Then $\tau'|B = \tau|B$ and $\tau'|A' - P = \tau|A' - P$, so that $A' \cup B$ is τ'-connected. But A' is now τ'-disconnected, contradicting the essential connectedness. Hence the components of $A \cap B$ have disjoint closures. If the components of $A \cap B$ form a closure preserving family then in fact they form a discrete family, and adjoining $\mathrm{Cl}(A \cap B)$ to each of A and B gives connected sets whose intersection is closed and has the same component structure. Hence we assume $A \cap B$ is closed.

Let R be the component of $A - P$ containing Q. Then $A - R$ is connected in A. We suppose first that R is closed in A, so that $A - R$ is open in A. Let $V = (B - A) \cup R$ and let $\tau' = \tau(V)$. Then $B \cap V = (B - A) \cup (B \cap R) = B - (A - R)$, and since the components of $A \cap B$ are a discrete collection of closed sets, $B \cap V \in \tau|B$. Further, $R \cap V = R$ and $(A - R) \cap V = \emptyset$, so that B, R, and $A - R$ are τ'-connected. Since $B \cap R \neq \emptyset \neq B \cap (A - R)$, $X = B \cup R \cup (A - R)$ is τ'-connected. But R is a $\tau'|A$ open and closed subset of A, so that A is τ'-disconnected, a contradiction. Hence R clusters in P.

If S is the component of $B - P$ containing Q, the same argument shows that S clusters in P. Again by Theorem 1 take $A = P \cup R$ and $B = P \cup S$. Let $V = (B - A) \cup P$ and set $\tau' = \tau(V)$. Then $V \cap B \in \tau|B$, $V \cap R = V \cap (A - P) \in \tau|R$, and τ' is a connected expansion. But $V \cap A = P \in \tau'|A$, so that A is τ'-disconnected. This contradiction shows that

$A \cap B$ cannot have two distinct components, contrary to assumption. Hence the components of $A \cap B$ cannot form a closure-preserving collection.

COROLLARY 3A. *If $A \cap B$ is disconnected, it must have nonempty interior.*

Proof. If $A \cap B$ has empty interior, it is closed and discrete in some submaximal expansion of X. Then its components are singletons and form a discrete collection in contradiction to Theorem 3.

We now employ Theorem 3 to show that no nontrivial product space can be essentially connected. This situation is inherent in the notion of product, and not dependent on the use of the standard product topology, since the proof below applies equally to any topology in which every factor X is homeomorphic to each of its products with any point from the product of the remaining factors.

THEOREM 4. *A product space is essentially connected if and only if each factor is essentially connected and all but one is a singleton.*

Proof. Suppose factors X and Y are not singletons, and let $x, x' \in X$ and $y, y' \in Y$ be distinct. If neither $\{x, x'\}$ nor $\{y, y'\}$ is discrete, it is easy to see that $\{x, x'\} \times \{y, y'\}$ is a connected but not essentially connected subspace of the product. If at least one is discrete, set $A = (\{x\} \times Y) \cup (X \times \{y\})$, $B = (\{x'\} \times Y) \cup (X \times \{y'\})$. Then A and B are connected, while $A \cap B = \{(x, y'), (x'y)\}$ is discrete. This contradicts Theorems 1 and 3.

We now turn our attention to mapping theorems. The following characterization of hereditarily quotient maps has independent interest.

THEOREM 5. *Let $f: (X, \tau) \to (Y, \sigma)$ be continuous. Then f is hereditarily quotient with respect to τ and σ if and only if for every $A \subseteq Y$, f is quotient with respect to $\tau(f^{-1}(A))$ and $\sigma(A)$.*

Proof. Let S be a saturated subset of X, let $A = f(S)$. If f is a quotient map with respect to $\tau(f^{-1}(A))$ and $\sigma(A)$, then $f|S$ is a quotient map since $S \in \tau(f^{-1}(A))$. Since the subspace topologies on S and $f(S)$ are not altered by the expansions, $f|S$ is quotient with respect to τ and σ. Hence f is hereditarily quotient with respect to τ and σ.

Conversely, let f be hereditarily quotient with respect to τ and σ, and let $A \subseteq Y$. Clearly f is continuous with respect to $\tau(f^{-1}(A))$ and $\sigma(A)$. Let $S \subseteq Y$ with $f^{-1}(S) = V \cup (W \cap f^{-1}(A)) \in \tau(f^{-1}(A))$, where $V, W \in \tau$. Then $S = f(V) \cup (f(W) \cap A)$. Let $V' = f(V) - \partial f(V) \in \sigma$. If $x \in \partial f(V) \cap S$, then $f^{-1}(x) \nsubseteq V$, since f is pseudo-open. Thus $f^{-1}(x)$ meets $W \cap f^{-1}(A)$, and we conclude that $S = V' \cup (f(W) \cap A)$. Now $S \cap A = [f(V) \cup f(W)] \cap A$, and $f^{-1}(S \cap A) = f^{-1}(S) \cap f^{-1}(A) = (V \cup W) \cap f^{-1}(A) \in \tau|f^{-1}(A)$,

Since $f|f^{-1}(A)$ is a quotient map, $S \cap A \in \sigma|A$, so $S \cap A = W' \cap A$ for some $W' \in \sigma$. Then

$$f(W) \cap A \subseteq S \cap A = W' \cap A \subseteq S, \quad \text{so} \quad S = V' \cup (W' \cap A) \in \sigma(A).$$

Therefore f is a quotient map with respect to $\tau(f^{-1}(A))$ and $\sigma(A)$.

COROLLARY 5A. *The function f is hereditarily quotient with respect to $\tau(f^{-1}(A))$ and $\sigma(A)$.*

THEOREM 6. *Let f be a hereditarily quotient map of X onto Y with connected fibers. If X is essentially or maximally connected, then so is Y.*

Proof. If X is connected, so is Y, and if C is any connected subset of Y, $f|f^{-1}(C)$ is a quotient map and thus $f^{-1}(C)$ is connected.

Let σ'' be a connected expansion of the topology σ of Y, and suppose σ'' disconnects some σ-connected $C \subseteq Y$. If $A, B \in \sigma''$ induce a separation of C, set $\sigma' = (\sigma(A))(B)$ and $\tau' = [\tau(f^{-1}(A))](f^{-1}(B))$. Then σ' is a connected expansion of σ and f is hereditarily quotient with respect to τ' and σ'. Since the expansion of τ was made by including saturated sets, the fibers of f are τ'-connected. Thus τ' is a connected expansion of τ.

Now if (X, τ) is maximally connected, $\tau' = \tau$ so that $\sigma' = \sigma$. This shows that σ admits no proper connected expansion, so (Y, σ) is maximally connected.

If (X, τ) is essentially connected, then $f^{-1}(C)$ is τ'-connected, whence C is σ'-connected, contradicting the construction of σ'. Therefore C was σ''-connected in the first place, whence (Y, σ) is essentially connected.

§ 3. A characterization theorem. Let (X, τ) be a topological space. If there is a linear order $\leqslant$ on X whose open interval topology coincides with τ, then (X, τ) is a *linearly orderable topological space* (LOTS). If τ is finer than some order topology, (X, τ) is a *weakly linearly orderable topological space* (WLOTS). If every two points of X are contained in some connected subspace which is a WLOTS, then (X, τ) is WLOTS-wise connected.

We call (X, τ) *irreducibly connected* if every two points $x, y \in X$ belong to a subspace $[x, y]$ minimum with respect to being connected and containing x and y. This is evidently equivalent to the property that arbitrary intersections of connected sets are connected. It is known [3] that if X is irreducibly connected, then each *segment* $[x, y]$ is closed in X and is a WLOTS. Hence every irreducibly connected space is WLOTS-wise connected.

In order to proceed we will need the following simple result on connected expansions, which has a number of important applications.

THEOREM 7. *Let (X, τ) be connected, $V \in \tau$, $p \in \partial V$, and let $\tau' = \tau(V \cup \{p\})$. If p is a cluster point of $V \cap C$ for each component C of $X - \{p\}$, then (X, τ') is connected.*

Proof. Since $C \cap (V \cup \{p\}) = C \cap V$, $\tau'|C = \tau|C$. Since p is a τ-cluster point of $C \cap V$ it is a τ'-cluster point of C. Hence X is τ'-connected.

THEOREM 8. *Let* (X, τ) *be a* WLOTS-*wise connected space. If* X *is essentially connected then it is irreducibly connected.*

Proof. Let $A \subseteq X$ be a WLOTS having x and y as endpoints, and let C be any connected subset of X containing x and y. By Theorem 1 we may assume that $X = A \cup C$. Suppose $p \in A - C$, and let $[x, p)$ and $(p, y]$ be the components of $A - \{p\}$. Since A is connected, p is a cluster point of each. Since $(p, y] \in \tau|A$, there is a $V \in \tau$ with $V \cap A = (p, y]$.

Let $\tau' = \tau(V \cup \{p\})$. Then by Theorem 7, (X, τ') is connected, but A is τ'-disconnected, contradicting the essential connectedness of X. Hence $A \subseteq C$. Then A is a subspace minimum with respect to being connected and containing x and y. Hence X is irreducibly connected.

Not every essentially connected space is irreducibly connected, however. Let X be the disjoint sum of two copies of the real interval $[0, 2]$ with the two copies of $[0, 1)$ identified. This is essentially connected but $\bigcap \{X - [0, i): i \in [0, 1)\}$ is disconnected. By modifying the neighborhoods of the 1's a Hausdorff example may be obtained. Note, however, that finite intersections of connected sets are connected, so this example does not clarify the position of Theorem 3.

To characterize essential connectedness in WLOTS-wise connected spaces we need only combine Theorem 8 with a characterization of essential connectedness in irreducibly connected spaces. For this we need the following definitions pertaining to a connected subset C of X. A *branch of* C is a component of $X - C$ which clusters in C. A point $p \in C$ is a *branch point of* C if it is a cluster point of some branch of C.

THEOREM 9. *Let* (X, τ) *be irreducibly connected. Then* X *is essentially connected if and only if the branches of each segment* $[x, y]$ *of* X *form a closure-preserving family.*

Proof. It is shown in [3] that each $[x, y]$ is closed, that each component of $X - [x, y]$ is a branch, and that each branch of $[x, y]$ has a unique cluster point in $[x, y]$. Let $\mathcal{B}$ be a family of branches of $[x, y]$ and suppose $z \in X$ is a cluster point of $\bigcup \mathcal{B}$. Suppose $z \in B$, where B is a branch of $[x, y]$ clustering to $p \in [x, y]$. Let $\mathcal{B}_x$ consist of those members of $\mathcal{B}$ clustering in $[x, p]$, $\mathcal{B}_y$ of those clustering in $[p, y]$. Either $\bigcup \mathcal{B}_x$ or $\bigcup \mathcal{B}_y$ clusters to z; hence without loss of generality let $\mathcal{B}_x = \mathcal{B}$. Then $[x, p] \cup [\bigcup \mathcal{B}] \cup \{z\}$ is connected, hence contains $[p, z]$. Thus $(p, z]$ is contained in some member of $\mathcal{B}$; that is, $B \in \mathcal{B}$. Therefore we may assume that $z \in [x, y]$, and again without loss of generality we assume that the members of $\mathcal{B}$ cluster in $[x, z]$.

Let $\tau' = \tau(\{z\} \cup (X - [x, z]))$. If z is not a cluster point of some member of $\mathcal{B}$, then $X' = [x, z] \cup (\bigcup \mathcal{B})$ is τ'-connected by Theorem 7, while $[x, z]$ is τ'-disconnected. This contradicts the essential connectedness of X' and hence of X. Therefore the branches of $[x, y]$ form a closure preserving family.

For the converse, let $\tau' \supset \tau$ disconnect some τ-connected subset $C \subseteq X$. If $a, b \in C$ are separated, then $[a, b] \subseteq C$ is τ'-disconnected. Let V, W be a $\tau'|[a, b]$ separation of $[a, b]$ with $a \in V$, $b \in W$. Then V and W are τ'-closed, since $[a, b]$ is τ-closed.

Let F be the union of V and those branches of $[a, b]$ which cluster in V, and let G be the union of W and those branches of $[a, b]$ which cluster in W. Since each branch has a unique cluster point in $[a, b]$, F and G are well-defined, disjoint, nonempty, and $X = F \cup G$. If $x \notin F$ is a τ'-cluster point of F, it is not a τ'-cluster point of V; hence it is a τ'-cluster point and therefore a τ-cluster point of the family of branches of $[a, b]$ contained in F. But the branches of $[a, b]$ are τ-closure preserving, a contradiction. Hence F is τ'-closed. A similar argument shows that G is τ'-closed, so that X is τ'-disconnected. Hence (X, τ) is essentially connected.

Note that a connected expansion of an essentially connected space is essentially connected. The class of spaces to which the Theorem of [9] applies is not closed under connected expansions, and such expansions cannot be recognized as essentially connected by a direct application of that theorem. In contrast, if our Theorems 8 and 9 identify a space as essentially connected, they will apply directly to any connected expansion.

We now show that for LOTS-wise connected spaces, and particularly for arcwise connected spaces, the condition on the branches can be replaced by a simpler condition.

THEOREM 10. *Let* (X, τ) *be a* LOTS-*wise connected space. Then* X *is essentially connected if and only if it is irreducibly connected and the branch points in each segment are discrete.*

Proof. If X is essentially connected, the branches of $[x, y]$ are closure preserving. Since each branch has a unique cluster point in $[x, y]$, the branch points of $[x, y]$ cannot cluster.

Conversely suppose the branch points of $[x, y]$ are discrete. If the union of a family $\mathcal{B}$ of branches cluster to a point p not in the closure of one of them, then the proof of Theorem 9 shows $p \in [x, y]$; without loss of generality, let $\mathcal{B}$ be a family of branches of $[x, p]$ each clustering in $[x, p)$. Then the branch points associated with members of $\mathcal{B}$ lie in $[x, q]$ with $q < p$, for they are absent from some order interval about p. Then $[x, p] \cap ([x, q] \cup [\bigcup \mathcal{B}] \cup \{p\}) = [x, q] \cup \{p\}$ is disconnected, contradicting the irreducible connectedness of X. Hence the branches of $[x, y]$ are closure preserving, and by Theorem 9, X is essentially connected.

§ 4. Maximal and strong connectedness. We have already observed that maximal connectedness is a special case of essential connectedness. We now identify a class of spaces in which the two concepts are equivalent.

A space and its topology are called *principal* if each point has a minimum neighborhood, or equivalently, an arbitrary intersection of open sets is open. Maximal connected principal topologies were characterized by Thomas [10]. The two-point indiscrete space is an example of an essentially connected principal space which is not maximally connected. It turns out that this is a unique anomaly.

THEOREM 11. *Apart from the indiscrete doubleton, the concepts of essential and maximal connectedness coincide for principal spaces.*

Proof. Eliminating trivial cases, we assume X is an essentially connected principal space with at least three points. Let V_p denote the minimum neighborhood of a nonisolated point p. Suppose V_p contains a distinct nonisolated point q. If $x \in V_q \subseteq V_p$, isolating q and x disconnects $\{q, x\}$ while leaving V_p connected, a contradiction. Hence $V_p = \{p, q\} = V_q$ is indiscrete. Since $X - V_p$ cannot also be open, V_p meets V_x for some $x \in X - V_p$; hence $V_p \subset V_x$. Then isolating p and q disconnects V_p but leaves V_x connected, again contradicting essential connectedness. Hence $V_p - \{p\}$ must consist of isolated points.

We now refer to Theorem 5 of [10]. Condition (i) is clear, while (ii) and (iii) follow by the same proofs found in [10], since the expansions exhibited in violation of maximal connectedness also violate essential connectedness by disconnecting an original minimum neighborhood. Hence X is maximally connected.

In certain other situations, essentially connected spaces can be recognized as strongly connected. If D is a dense subspace of (X, τ), Bourbaki [2; p. 124] defines an expansion τ' of τ to be D-maximal if it is maximal with respect to the properties that $\tau'|D = \tau|D$ and D is τ'-dense in X, and observes that every topology τ has D-maximal expansions. By [2; p. 139], if D is submaximal and X is D-maximal, then X itself is submaximal.

THEOREM 12. *Let X be an essentially connected space having a dense maximally connected subspace D. Then every D-maximal expansion of X is maximally connected.*

Proof. Since D remains dense, X remains connected, and since maximally connected spaces are submaximal, X is submaximal by the result quoted above. Then D is open and $X - D$ is closed and discrete. A connected expansion of X cannot create interior in $X - D$, and by the essential connectedness of X cannot disconnect D. By the maximal connectedness of D, its topology is not altered; hence by the D-maximality of X, the expansion is not proper. Hence X is maximally connected.

It is not known whether there exist any maximally connected Hausdorff spaces other than singletons. We are able to show however that not every connected Hausdorff space has a maximally connected expansion; that is, we exhibit a large class of connected spaces which are not strongly connected.

Observe that an important feature of our characterization of essentially connected spaces is the absence of anything resembling a simple closed curve. Since one prominent property of simple closed curves is the absence of cutpoints, we begin by studying spaces having no cutpoints. Swingle [8] defined a space to be *widely connected* if it is connected and every nonsingleton connected subspace is dense, and showed that such spaces appear as subspaces of Euclidean spaces. Evidently a widely connected T_1 space X has no cut point. For if $X - \{x\}$ is totally disconnected and (F, G) is a separation of $X - \{x\}$, then $F \cup \{x\}$ is a connected subset which is neither dense nor a singleton. Thus $X - \{x\}$ contains a component which is dense, so $X - \{x\}$ is connected.

THEOREM 13. *Let X be an essentially connected T_1 space. Then X is widely connected if and only if it has no cut points.*

Proof. We have just seen that a widely connected space has no cut points. If X is essentially connected and without cut points, let C be a proper closed connected set with more than one point. Then $X - C$ is open. Let $p \in \partial(X - C)$. Then expanding by $(X - C) \cup \{p\}$ gives a connected topology by Theorem 7, but disconnects C by isolating p. Hence X is widely connected.

The T_1 topology obtained by adjoining the empty set to an ultrafilter finer than the filter of cofinite sets is an example of a widely connected space which is maximally connected and hence essentially connected.

In a maximally connected space we can count the cut points in many situations. A space is called *quasiregular* if every nonempty open set contains the closure of some nonempty open set.

THEOREM 14. *Let X be a maximally connected space. Then every nonempty nondense open set has a cut point of X in its boundary.*

Proof. Let V be a nonempty nondense open set. If $V \cup \{p\}$ is open for each $p \in \partial V$, then $\mathrm{Cl}(V)$ is also open; hence $V \cup \{p\}$ is not open for some $p \in \partial V$. If p is not a cut point, Theorem 7 allows a proper connected expansion by $V \cup \{p\}$; hence p is a cut point.

COROLLARY 14A. (i) *If X is Hausdorff, X has infinitely many cut points.*
(ii) *If X is quasiregular, the set of cut points is dense in X.*

Proof. (i) Any finite set of points in a Hausdorff space is contained in a nondense open set. Hence the finite set cannot include all the cut points.

(ii) If every nonempty open set V contains $\mathrm{Cl}(W)$ for some nonempty open set W, then $X-\mathrm{Cl}(W)$ is a nonempty nondense open set whose boundary is contained in V. Hence V contains a cut point.

COROLLARY 14B. *No widely connected Hausdorff space is maximally connected.*

Proof. By Theorem 13, a widely connected Hausdorff space has no cut points, so cannot be maximally connected by Corollary 14A.

THEOREM 15. *No connected Hausdorff space with a dispersion point is strongly connected.*

Proof. Let X be a space with dispersion point p. Then p is the only cut point of X, and by Corollary 14A, X is not maximally connected. But in any connected expansion of X, $X-\{p\}$ remains totally disconnected, hence p remains the only cut point of X. Hence X has no maximally connected expansion.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITTSBURGH
Pittsburgh, Pennsylvania

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