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On the connection between Hausdorff measures and generalized capacity

by

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Abstract. The object of this note is to compare the methods of using Hausdorff measure and generalized capacity for measuring thinness of sets. It is shown that both methods are equivalent for subsets E of R^k: $m_h(E) = 0$ for every Hausdorff measure m_h if, and only if, $C_k(E) = 0$ for every generalized capacity C_k. Then a problem of Choquet is settled by showing (with the aid of the continuum hypothesis) that there exists a universal null subset A of R^2 which is not capacitable for some capacity which is both positive and of order A_∞ on the set K_{R^2} of compact subsets of R^2.

Introduction. For any continuous increasing function h on $[0, \infty)$ with $h(0) = 0$, Hausdorff [6] defined a Cartheadory measure, m_h; such measures are called Hausdorff measures. Frostman [5] defined capacity, C_Φ, with respect to a non negative, continuous decreasing function Φ on $(0, \infty)$ satisfying $\lim_{t\to 0+} \Phi(t) = \infty$. For background, we refer the interested reader to C. A. Rogers book [8], S. J. Taylor's paper [11] and Lennart Carleson's monograph [2] with 1049 references. However, a bit of caution is in order: The notation in the literature is quite inconsistent. For instance m_h carries a differ[illegible] meaning in each of this note, [2], [7], and [11]; in [2] our m_h is denoted [illegible] Λ_h, in [7] it is denoted by m_h only in § 5, and in [11] m_h corresponds to h^*-m. Our notation is intended to permit us to reach into the references for technical results with a minimum of verbosity.

Notation. Throughout we fix a positive integer k and restrict the discussion to subsets of R_k, Euclidean k-space. Thus (cf. [7]), m_h is defined for all subsets of R_k and C_Φ is defined for all Borel subsets of R_k.

Denote by $\mathfrak{B}$ the set of Borel subsets of R_k and define the outer capacity C_Φ^* by

$$C_\Phi^*(E) = \inf\{C_\Phi(U):\ U \text{ open},\ U \supset E\}$$
$$= \min\{C_\Phi(B):\ B \in \mathfrak{B},\ B \supset E\}$$

since $C_\Phi(B) = \inf\{C_\Phi(U):\ U \supset B\}$ when $B \in \mathfrak{B}$, C_Φ is monotone, and $\mathfrak{B}$ is closed under countable intersection.

Let $\mathcal{H}$ denote the set of subsets E of R_k which satisfy $m_h(E) = 0$ for all Hausdorff measures m_h, and let $\mathcal{F}$ denote the set of subsets G of R_k verifying $C_\Phi^*(G) = 0$ for every outer capacity C_Φ^*.

THEOREM. *If E is a subset of R_k, then $E \in \mathcal{H}$ if, and only if, $E \in \mathcal{F}$.*

It suffices to suppose henceforth that E is a subset of the open, positive square

$$S_k = \{r_i\}_{i=1}^k:\ 0 < r_i < \tfrac{1}{2}\} \quad \text{in } R_k,$$

and show that $E \in \mathcal{H} \Leftrightarrow E \in \mathcal{F}$. This reduction permits us to apply an argument of Kametani directly. We can also suppose that $h(1) = \Phi(1) = 1$.

To show that $\mathcal{H} \subset \mathcal{F}$, suppose $E \in \mathcal{H}$. Pick an admissible Φ and let $h = \Phi^{-1}$. Then [7, Theorem 2] there is a Borel set $B \supset E$ with $m_h(B) = 0$. Consequently [7, Theorem 11] $C_\Phi(B) = 0$ which implies $C_\Phi^*(E) = 0$.

In order to show that $\mathcal{F} \subset \mathcal{H}$ associate an admissible Φ with an admissible h by setting $\Phi(t) = 1 - \ln\big(h(t)\big)$, $t > 0$, and notice that $-\int_0^1 h\, d\Phi = 1$. Also notice that Theorem 57 of [8] implies that the proof of Theorem 13 in [7] is complete, so we can apply this latter result to Borel subsets B of S_k.

Thus, $C_\Phi(B) > 0$ if $m_h(B) > 0$. Hence, $m_h(E) > 0$ implies that $C_\Phi^*(E) = \min\{C_\Phi(B):\ B \in \mathcal{B},\ B \supset E\} > 0$, and the theorem is established.

Recall ([4, Lemma 1]) that $\mathcal{H}$ is characterized simply as follows; $E \in \mathcal{H}$ if, and only if, E has property (C): for each sequence $\{\varepsilon_i\}$ of positive numbers there is a sequence $\{x_i\}$ of points in R_k such that $E \subset \bigcup N(x_i, \varepsilon_i)$, where $N(x_i, \varepsilon_i)$ is the open ball in R_k with center x_i and radius ε_i.

A subset E of R^k is said to be a *universal null set* if $\mu(E) = 0$ for each Radon measure μ without point masses.

Clearly every set in $\mathcal{H}$ is a universal null set, and Sierpiński showed [10, p. 57] that $\mathcal{H}$ is a proper subset of the set $\mathcal{N}$ of universal null sets by showing that while property (C) is invariant under continuous maps the property of being a universal null set (property β) is not invariant under continuous maps. Besicovitch [1] also studied property (C).

Next we shall settle Problem 33.3 of [3]: If A is a universal null set in R^2 is A capacitable for each capacity which is both $\geqslant 0$ and of order A_∞ on K_{R^2}?, in the negative by showing (by the aid of the continuous hypothesis) that there exists a universal null subset A of R^2 and a capacity F which is both $\geqslant 0$ and of order A_∞ on K_{R^2} such that A is not f-capacitable.

It follows from some known results ([10], p. 261, line 4–9), that

(*) *The continuum hypothesis implies the existence of a universal null subset A of R^2 whose projection on a certain straight line is non-measurable in the sense of Lebesgue.*

The original proof of Theorem 33.1 of [3] gives actually the following proposition:

(**) *There exists a capacity $f \geqslant 0$ and of order A_∞ on K_{R^2} such that every subset of the plane having the non-measurable projection on a straight line is not f-capacitable.*

Propositions (*) and (**) give the desideratum.

Notice that the set A used in the preceding example does not have property (C), so we are left with the following question. Is every set in $\mathcal{H}$ capacitable for every capacity which is both $\geqslant 0$ and of order A_∞ on K_{R^n}?

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