Concerning the relation between separability and the proposition that every uncountable point set has a limit point.

By

R. L. Moore (Austin, Texas, U. S. A.).

In this paper I will establish two theorems relating to the theory of abstract sets.

Theorem 1. *In order that every subclass of a given class D of Fréchet should be separable it is necessary and sufficient that every uncountable subclass of that class D should have a limit point* [1]).

[1]) A class L of Fréchet is a set of elements such that (1) if P is an element of L and $P_1, P_2, P_3, \ldots$ is a countable sequence of elements of L then the statement that P is the sequential limit of the sequence $P_1, P_2, P_3, \ldots$ has a definite meaning and the question whether this statement is true or false has a determinate answer as soon as the element P and the sequence in question are themselves determined, (2) if the element P is the sequential limit of the sequence $P_1, P_2, P_3, \ldots$ and $n_1, n_2, n_3, \ldots$ is an infinite sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$ then P is also the sequential limit of the sequence $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$ (3) if P is an element of L, P is the sequential limit of the sequence $P, P, P, \ldots$ whose elements all coincide with the element P. Cf. M. Fréchet, Palermo Rend. Circ. Mat., 22, 1906 (1—72). A class D is a class L in which with every pair of elements A and B there is associated a number (A, B) such that (1) $(A, B) = (B, A) \geqq 0$, (2) $(A, B) = 0$ if, and only if, $A = B$, (3) if A, B and C are any three elements then $(A, B) + (B, C) \geqq (A, C)$, (4) the element P is the sequential limit of the sequence of elements $P_1, P_2, P_3, \ldots$ if and only if (P_n, P) approaches zero as a limit as n approaches infinity. In the present paper the elements of a class L will be called points. A point P is said to be a *limit point* of the set of points M if and only if P is the sequential limit of a sequence of distinct points each of which belongs to M. A set of points M is said to be *separable* if it contains a countable subset K such that every point of M either belongs to K or is a limit point of K. Cf. Fréchet, loc. cit.

Proof.[1]. Let D_s denote a class D every uncountable subclass of which has a limit point and let K denote a set of points belonging to it. Let the points of K be arranged in a well ordered sequence β. Suppose n is any positive integer. Let P_1 denote the first point of β. If there exists any point of K at a distance greater than $1/n$ from P_1 let P_{2n} denote the first such point in β. If there exists any point of K at a distance greater than $1/n$ both from P_1 and from P_{2n} let P_{3n} denote the first such point in β. Continue this process thus obtaining a well ordered sequence α_n such that (1) the first, second and third elements of α_n are P_1, P_{2n} and P_{3n} respectively, (2) if γ is a subsequence of α_n such that every element which, in α_n, precedes an element of γ belongs to γ then either γ is identical with α_n or there is an element of β which is at a distance greater than $1/n$ from every element of γ, in which case the first such element in β is the first element in α_n which follows all the elements of γ. Let K_n denote the point set consisting of all the elements of α_n. If K_n is uncountable then, by hypothesis, it has a limit point X. But, since every two points of K_n are at a distance apart greater than $1/n$, it is easy to see that this is impossible in a space D. It follows that, for each n, K_n is countable. Hence the point set $K_1 + K_2 + K_3 + \ldots$ is countable. But clearly every point of K is at a distance less than or equal to $1/n$ from some point of K_n. It follows that every point of K either belongs to $K_1 + K_2 + K_3 + \ldots$ or is a limit point of it. Thus the sufficiency of the condition in question has been established. That it is necessary is evident.

Theorem 2. *If D_s is a separable class D than every uncountable subclass of D_s contains a point of condensation*[2].

Proof. Suppose, on the contrary, that D_s contains an uncountable point set H which contains no point of condensation of itself. For each point P of H and each positive integer m let I_{P_m} denote the

[1] This Summer I proposed to my class in The Theory of Sets the problem of proving that in a class D of Fréchet every compact set is separable. Mr. W. L. Ayres obtained a proof. I have since found that a proof is given in Hausdorff's *Grundzüge der Mengenlehre* (1914). These proofs and the proof here given for Theorem 1 are closely related.

[2] A point P is said to be interior to a point set M if P belongs to M and M contains at least one point, distinct from P, of every point set which has P as a limit point. The point P is said to be a point of condensation of the point set M if every point set which has P in its interior contains uncountably many points of M.

set of all those points of D_S which are at a distance from P less than, or equal to, $1/m$. Then for each point P of H there exists a positive integer m_P such that the set of all those points of H which belong to $I_{P_{m_P}}$ is countable. Since H is uncountable it easily follows that there exists an integer n and an uncountable subset K of the point set H such that, for every point P which belongs to K, $m_P = n$. For the point set K and this particular integer n construct a well ordered sequence α_n in the manner indicated in the above proof of Theorem 1 and let K_n denote the set of all those points of K that belong to α_n. Each point of K belongs to I_{P_n} for some point P of K_n. But K is uncountable and, for each P, I_{P_n} is countable. It follows that K_n is uncountable. Since every two points of K_n are at a distance apart greater than $1/n$ therefore there exists a positive integer k such that if X and Y are distinct points of K_n then I_{X_k} and I_{Y_k} have no point in common. Hence, since K_n is uncountable, the set G of all point sets I_{X_k}, for all points X of K_n, is an uncountable collection of mutually exclusive point sets. Since D_S is separable it contains a countable point set L such that every point of D_S either belongs to L or is a limit point of it. Hence every point set of the collection G contains a point of L. But this is clearly impossible. The truth of Theorem 2 is therefore established.

As a consequence of the above theorems we have the following.

Theorem 3. *Every subclass of a separable class D is itself separable.*

Theorem 4. *In order that every uncountable subclass of a given class D should contain a point of condensation of itself it is necessary and sufficient that every uncountable subclass of D should have a limit point.*

Sierpiński [1]) has shown that there exists an uncountable class S which is separable (and therefore in which every uncountable point set contains a limit point of itself) but in which no uncountable point set has a point of condensation. In the same paper he has also shown that there exists an uncountable class S which is not separable but in which every uncountable point set contains a point of condensation of itself. It is easy to see that there exists a class

[1]) W. Sierpiński, *Sur l'équivalence de trois propriétés des ensembles abstraits* Fund. Math., vol. 2 (1921), pp. 179—188. A class S is a class L in which every derived set is closed.

S in which every uncountable point set *has* a limit point but in which not every one *contains* a limit point of itself.

Sierpiński [1]) has proved that, for a space S, (1) the proposition that every well ordered descending sequence of distinct closed point sets is countable is equivalent to the proposition that every uncountable point set contains a point of condensation of itself and (2) the proposition that every ascending sequence of closed distinct point sets is countable is equivalent to the proposition that every point set is separable. From these results and Theorems 1 and 4 above it follows that the following proposition holds true.

Theorem 5. *In order that every ascending sequence of distinct closed subsets of a given class D should be countable it is necessary and sufficient that every descending one should be.*

[1]) Loc. cit.

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