The theory of surface measure [1].

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Notations.

$V_l^*(E)$	plane exterior lebesguian measure of set E.
measurable l	measurable with respect to Lebesgue's measure.
$\operatorname{Lim}_{n\to\infty} \sup_{\inf} (E_n)$	upper (lower) limit of sets $\{E_n\}$.
$E=\sum_{i=1}^{\infty} E_n = E_1+E_2+\ldots+E_n+\ldots$	sum of sets $\{E_n\}$.
$E=\prod_{i=1}^{\infty} E_i$	product (greatest common divisor) of sets $\{E_n\}$.

§ 1. Our considerations shall be carried forth in a space of 3 dimensions. This we only do in order to avoid too much complicated formulae. Generalisations of our results with concern to a space of any number „n“ of dimensions offer no difficulty.

For the sake of better understandig we beginn by paying some attention to results obtained by Messrs Gross and Jansen.

According to Gross [2]) we call „exterior surface measure“ any setfunction $\Phi(E)$ provided it satisfies the following conditions:

I. It is non negative and not identically zero nor identically $+\infty$, defined for *every* set E in the space. Vacuous sets are of measure zero.

II. E' being part of E we have always:

$$\Phi(E')\leqslant\Phi(E)$$

[1]) Thesis presented to the University of Lwów, october 1923.

[2]) Gross W: Über das Flächenmass von Punktmengen. Monatsh. für Math. und Phys XXIX Jahrg. This paper will be referred to in future as F. P.

III. Considering a sum of an at most countable number of sets $\{E_i\}_{i=1,2-\infty}$ we have:

$$\Phi(E') \leqslant \sum_{i=1}^{\infty} \Phi(E_i); \quad \sum_{i=1}^{\infty} E_i = E.$$

IV. If two sets E and E' are of positive distance, then:

$$\Phi(E+E') = \Phi(E) + \Phi(E').$$

V. $\Phi(E)$ is the lower bound of $\Phi(B)$, B being any borelian set containing E.

VI. The projection E_Θ of the set E on the plane Θ is of an exterior lebesguian plane measure not greater then $\Phi(E)$:

$$V_i^*(E_\Theta) \leqslant \Phi(E)$$

According to Caratheodory[1]) it is possible to distinguish with reference to any setfunction Ψ satisfying I—IV a class of „measurable Ψ" sets in the following manner:

A set E is, ex definitione, measurable Ψ, if for *every* set M of finite Ψ we have:

$$\Psi(M) = \Psi(M.E) + \Psi(M - M.E).$$

The exterior surface measure of a set „measurable Φ" we shall call its „Φ-area". Thus the phrase „area $\Phi(E)$" includes the measurability Φ of the set E.

Borelian sets are shown by him[1]) to be measurable Φ with respect to a n y Φ. Also theorems, well known from the theory of Lebesgue's measure e. g.: measurability of the sum, product, difference of sets measurable Φ.

§ 2. Given a set E in the space and a plane Θ. We define:

Def. A point p belonging to Θ will be said to be covered at least m-times by the projection E_Θ, if at least m points of E project themselves orthogonally to Θ in p.

Def. A point P belonging to E „projects itself at least m times on Θ", if at least $m-1$ other points of E can be found, lying together with P on the same perpendicular on Θ.

With these definitions the following theorem subsists, demonstrated by Gross[2]): *If $\Phi(E)$ is finite, then points of Θ covered*

[1]) Caratheodory: Vorlesungen über reelle Funktionen page 246–258. This book will be referred to in future as R. F.

[2]) F. P. page 174.

at least m-times by E_Θ form a set $E_{m\Theta}$, whose l-measure satisfies the relation:

$$(1) \qquad V_l^*(E) \leqslant \frac{\Phi(E)}{m}$$

Thence from results:

the pointset $E_{\infty\Theta}$ lying on Θ and covered an infinity of times by E_Θ, is of l-measure zero.

We conclude also from (1), that the lebesguian space measure of E is zero, provided $\Phi(E)$ is finite

§ 3. Above postulates do not yet define Φ uniquely. To obtain this we have to annex the following condition:

VII. *$\Phi_0(E)$ is the least of all functions, which comply with I–VI; the term „least" is to be understood thus:*

a setfunction $\Phi(E)$ being given, which fulfils I–VI, we ought to have:

$$\Phi_0(E) \leqslant \Phi(E)$$

for every E.

The existence of such least Φ_0 — this of course being unique — has been demonstrated by Gross [1]). Here we let his proof follow.

We choose a cartesian system of coordinates $x\ y\ z$ and decompose the space in cells O_ϱ, so that every point $P(x\,y\,z)$ may belong to one and only one cell O_ϱ:

$$O_\varrho = \begin{cases} x_i \leqslant x < x_{i+1} \\ y_k \leqslant y < y_{k+1} \qquad i, k, l, = 0, \pm 1, \pm 2 \ldots \\ z_l \leqslant z < z_{l+1} \end{cases}$$

the sequences $\{x_i\}$ $\{y_k\}$ $\{z_l\}$ satisfying the conditions:

$$\begin{aligned} &x_i < x_{i+1} \\ &y_k < y_{k+1} \qquad\qquad i, k, l, = 0, \pm 1, \pm 2 \\ &z_l < z_{l+1} \end{aligned}$$

$$\lim_{|i| \to \infty} |x_i| = \infty \quad \lim_{|k| \to \infty} |y_k| = \infty; \lim_{|l| \to \infty} |z_l| = \infty.$$

Such a partition of the space we shall call a space grating [2]) and denote it by the letter S.

We introduce the following notations:

[1]) F. P. page 156–159.

[2]) or space-net.

(a) E_Θ the projection of set E on the plane Θ or more generally: $E_{ikl,\ldots,\Theta}$ the projection of $E_{i,k,l,\ldots}$ on Θ.

(b) E_ϱ the set of points common to E and the cell O_ϱ:

$$E_\varrho = E \,.\, O_\varrho.$$

Consider now the set of numbers $\{V_\iota^* E_{\varrho\Theta}\}$, ϱ having been fixed, taking all possible orientations of the plane Θ. The upper bound of these numbers is defined uniquely for every ϱ. Let us call this upper bound $\sigma_\varrho(E)$. The sum:

$$\sigma(E) = \sum_{\varrho=1}^{\infty} \sigma_\varrho(E)$$

will depend of the grating only.

Let $\{S^i\}_{i=1,2\ldots\infty}$ be a sequence of gratings such that:

1º each grating S^{i+1} results from the preceding S^i by means of subdivision;

2º the maximum diameter of cells tends towards zero.

It can be easily shown, that the sequence of numbers $\sigma^i(E) = \sum_{\varrho=1}^{\infty} \sigma_\varrho^i(E) - \sigma^i(E)$ constructed for the grating S^i — represent a non decreasing sequence and that the setfunction $\Phi_0(E)$ defined as:

$$\overline{\Phi}_0(E) = \lim_{i\to\infty} \sigma^i(E)$$

possess all demanded properties, save perhaps V.

Having thus obtained $\overline{\Phi}_0(E)$, we construct without difficulties $\Phi_0(E)$ in the following manner:

$\Phi_0(E)$ is the lower bound of $\overline{\Phi}_0(B)$ for all borelian sets containing E.

$\Phi_0(E)$ satisfies all our I—VII postulates.

§ 4. Jansen was, in defining his measure, prior to Gross [1].

Let us call $X\,Y\,Z$ the planes determined by the axes $(y\,z)$ $(z\,x)$ $(x\,y)$ resp., all other notations remaining unchanged.

It can be demonstrated, that the numbers

[1] O. Jansen. Über einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt Inaugural-Dissertation. Königsberg 1907.

$$s^i = \sum_{\varrho=1}^{\infty} \sqrt{V_i^2(E_\varrho)_X + V_i^2(E_\varrho)_Y + V_i^2(E_\varrho)_Z}$$

converge to a limit, independently of the choice of the „fundamental" system of gratings $\{S^i\}_{i=1,2-\infty}$, provided that $x\,y\,z$ remain fixed.

Jansen's exterior measure is just this common limit $J(E)$:

$$J(E) = \lim_{i\to\infty} s^i(E)$$

$J(E)$ fulfils postulates I—V. Instead of VI the following property subsists:

For every E:

$$\sqrt{V_i^2 E_X + V_i^2 E_Y + V_i^2 E_Z} \leqslant J(E)$$

§ 5. We give here in advance a short summary of main results of the present paper.

In the first chapter general properties of sets measurable Φ shall be investigated. In particular with th. III we find conditions necessary and sufficient in order that such set have a finite surface [1]) measure. In pursuit we show, that each set E measurable Φ_0, its Φ_0 area being finite, may be approximated by means of closed subsets F in such a manner, that $\Phi_0(F)$ differ from $\Phi_0(E)$ as little as we please. An interesting property of sets with finite Φ is comprised by th. VI.

In the second chapter we introduce „normal" sets.

A set E is called normal, if from every family of spheres containing E an at most enumerable number of mutually seperated spheres can be extracted, their sum covering E except a set of Φ measure zero.

Parts of normal sets are too normal (th. VII). A wide class of normal sets is pointed out by th. XVII.

At last, we generalise in this chapter a well known theorem due to Gauss, concerning connection between volume and surface integrals. This theorem will be extended to domains with normal boundary.

Chapter III deals with surfaces:

$$x = \varphi(uv)$$
$$y = \Psi(uv)$$
$$z = \Xi(uv)$$

[1]) resp. jansenian.

$u \cdot v$ ranging over some plane domain; functons φ, Ψ, Ξ are supposed to satisfy Lipschitz's conditions. Such surfaces are always normal; the theorem of Gauss takes here his classical form [1]).

I.

§ 1. We make the following assumption:

1° $\Phi(E)$ is finite and the set E measurable Φ.

Until further notice our assumptions are to remain unchanged.

Therefrom ensues the following theorem:

Theorem I. [2]) *E_Θ is measurable l for every Θ.*

Proof. In order to demonstrate this we construct according with 5-th. postulate a borelian set B enclosing E, whose area is equal to the area of E:

$$B \supset E;\ \Phi(B) = \Phi(E).$$

Consequently the set $B - E$ is of area zero:

$$\Phi(B - E) = 0.$$

In virtue of 6 th. postulate we have a fortiori:

$$V_l(B - E)_\Theta = 0. \tag{1}$$

On the other side:

$$B_\Theta = E_\Theta + (B - E)_\Theta.$$

B_Θ being a set of Suslin (A) is measurable l. Therefore:

$$E_\Theta = B_\Theta - (B - E)_\Theta \cdot (B_\Theta - E_\Theta)$$

is too measurable-l (acc. to 1).

Theorem II. *The points of Θ, covered by E_Θ at least m-times, form a set measurable-l.*

Proof. Let us divide the space into a series of layers of latitude $\frac{1}{2^k}$ each, by means of planes parallel to the given plane Θ. For this purpose let us choose a pair of orthogonal axes lying in Θ and a third one z normal to Θ.

[1]) Lichtenstein has proved this theorem for surfaces with continuous tangent plane. Lichtenstein. Arch. der Math. u. Phys. (3) 27 (1918) page 31—37.

[2]) Cf. P. F. page 173

Every layer represents [1]) a set of points W_{kh}, whose x, y are arbitrary and whose z satisfy the relation:

$$\frac{h}{2^k} \leqslant z < \frac{h+1}{2^k} \qquad h = 0, \pm 1, \pm 2 \ldots \text{ [2])} \tag{2}$$

Let us suppose the set E to lie entirely on the side with non negative z. This restriction is made for the while only and there will be no difficulty to drop it.

We introduce by induction the following sets:

$$\begin{aligned} E'_{kh} &= E \,.\, W_{kh} \\ E_{kh\Theta} &= \text{projection of } E'_{kh} \text{ on } \Theta \\ E_k^0 &= E_{k0\Theta} \\ E_k^1 &= E_{k1\Theta}\ (\Theta - E_{k0\Theta})\text{ [3])} \\ E_k^h &= E_{kh\Theta} \cdot \prod_{s=0}^{s=h-1} (\Theta - E_{ks\Theta}) \end{aligned} \tag{3$_1$}$$

Generally supposing $E_k^{h_1 h_2 \ldots h_{r-1}}$ be already defined for all non negative, integer h_1 and all natural $h_2\, h_3 \ldots h_{r-1}$, we define now:

for $h_r = 1$

$$E_k^{h_1 h_2 \ldots h_r} = E_k^{h_1 h_2 \ldots h_{r-1}} . E'_{k, h_1 + h_2 + \ldots h_{r-1} + 1, \Theta} \tag{3$_2$}$$

for $h_r > 1$

$$E_k^{h_1 h_2 \ldots h_r} = E_k^{h_1 h_2 \ldots h_{r-1}} . E_{k\, h_1 + h_2 + \ldots h_{r-1} + h_r, \Theta} \cdot \prod_{s=1}^{s=h_r - 1} (E_k^{h_1 h_2 \ldots h_{r-1}} -$$

$$- E_k^{h_1 h_2 \ldots h_{r-1}} . E_{k, h_1 + h_2 + \ldots h_{r-1} + s, \Theta})$$

The sets E_{kh} are measurable-Φ, being product of two measurable sets, namely E and the borelian W_{kh}. The area of E_{kh} is finite, E_{kh} being part of E Consequently according to the precedent theorem, the sets $E_{kh\Theta}$ are measurable-l. And since the sets

[1]) By definition.

[2]) The sum $\sum\limits_{h=-\infty}^{h=\infty} W_{hk}$ is the whole space (for k constant).

[3]) Θ = the totality of points belonging to the plane Θ.

$E_k^{h_1 h_2 \ldots h_r}$ are formed from $E_{kh\Theta}$ by means of operations: sum, difference and product, their measurableness l is secured.

(It may be easily shown, that two sets $E_k^{h_1 h_2 \ldots h_r}$ and $E_k^{h'_1 h'_2 \ldots h'_r}$ have no common points, provided $(h_1\, h_2 \ldots h_r)$ be different from $(h'_1\, h'_2 \ldots h'_r)$:

$$(4) \qquad E_k^{h_1 h_2 \ldots h_r} \,.\, E_k^{h'_1 h'_2 \ldots h'_r} = 0 \text{ if } (h_1\, h_2 \ldots h_r) \neq (h'_1 h'_2 \ldots h'_r)$$

In fact we may immediately establish our supposition for $E_k^{h_1}$. Supposing its validity for $E_k^{h_1 h_2 \ldots h_{r-1}}$ we may show it to hold still for $E_k^{h_1 h_2 \ldots h_r}$ by discerning two cases (α) (β).

(α) $(h_1\, h_2 \ldots h_{r-1}) = (h'_1\, h'_2 \ldots h'_{r-1})$ that is $h_1 = h'_1$; $h_2 = h'_2, \ldots h_{r-1} = h'_{r-1}$

$$h_r \neq h'_r.$$

Let us suppose for instance $h_r < h'_r$.

In this case we have according to $(3)_2$:

$$E_k^{h_1 h_2 \ldots h_{r-1} h'_r} = E_k^{h'_1 h'_2 \ldots h'_{r-1} h'_r} =$$

$$= E_k^{h_1 h_2 \ldots h_{r-1}} \,.\, E_{k, h_1 + h_2 + \ldots h_{r-1} + h'_r, \Theta} \,.\, \prod_{s=1}^{s=h'_r - 1} \Big(E_k^{h_1 h_2 \ldots h_{r-1}} -$$

$$- E_k^{h_1 h_2 \ldots h_{r-1}} \,.\, E_{k, h_1 + h_2 + \ldots h_{r-1} + s, \Theta} \Big)$$

$$\subset E_k^{h_1 h_2 \ldots h_{r-1}} - E_k^{h_1 h_2 \ldots h_{r-1}} \,.\, E_{k, h_1 + h_2 + \ldots h_{r-1} + h_r, \Theta} \quad ,$$

Consequently the set $E_k^{h'_1 h'_2 \ldots h'_{r-1} h'_r}$ has no common points with $E_{k, h_1 + h_2 + \ldots h_{r-1} + h_r, \Theta}$, while — regarding (3) — $E^{h_1 h_2 h \ldots h_{r-1} h_r}$ belongs entirely to $E_{k, h_1 + h_2 + \ldots h_{r-1} + h_r, \Theta}$.

(β) In the second case we have already:

$$(h_1\, h_2 \ldots h_{r-1}) \neq (h'_1\, h'_2 \ldots h'_{r-1}).$$

Then sets $E_k^{h_1 h'_2 \ldots h_{r-1}}$ and $E_k^{h_1' h_2' \ldots h'_{r-1}}$ having already no common points, this remains true concerning the sets $E_k^{h_1 h_2 \ldots h_{r-1} h_r}$ and $E_k^{h'_1 h'_2 \ldots h'_{r-1} h'_r}$ belonging to $E_k^{h_1 h_2 \ldots h_{r-1}}$ resp. to $E_k^{h'_1 h'_2 \ldots h'_{r-1}}$ [1]).

We return now to the demonstration of our theorem.

Let us observe the set of those points of E, which project themselves on Θ at least m times. We denote it by E_m.

[1]) Paragraph in paranthesis won't be needed for the present; property obtained therein shall be made use of during the demonstration of th. IV.

According to our definition of E_m to every point P_1 of it, we can find at least $m-1$ other points $P_2, P_3 \dots P_m$ belonging to E and lying together with P_1 on the s a m e straight line perpendicular to Θ.

As the points $P_1\ P_2 \dots P_m$ have a positive distance from one another, for an integer k chosen great enough ($k > K$), they are contained by different layers W_{kh} (for k constant).

In consequence thereof $P_{1\Theta}$ belongs to at least m $E_{kh\Theta}$ ($k > K$).

Let us observe those $E_{kh\Theta}$, which contain $P_{1\Theta}$ and let us arrange them by increasing non negative indices h (k fixed).

$$P_{1\Theta} \,\varepsilon\, E_{k, h_1, \Theta}; \qquad P_{1\Theta} \,\varepsilon\, E_{k, h_1+h_2, \Theta} \dots; \qquad P_{1\Theta} \,\varepsilon\, E_{k, h_1+h_2+\dots h_m, \Theta}$$
$$k > K$$

while in consequence of our arrangement n o set $E_{kh\Theta}$ with an index h less then h_1, or with an index h:

$$h_1 + h_2 + \dots h_r < h < h_1 + h_2 + \dots h_r + h_{r+1} \qquad r = 1, 2, \dots m-1$$

contains $P_{1\Theta}$.

The point $P_{1\Theta}$ belonging to $E_{kh_1\Theta}$, but to neither $\{E_{ks\Theta}\}_{s=0, 1 \dots h_1-1}$, it belongs to:

$$E_{kh_1\Theta} \cdot \prod_{s=0}^{s=h_1-1} (\Theta - E_{ks\Theta}) \qquad k > K$$

and therefore according to (3) to:

$$P_{1\Theta} \,\varepsilon\, E_k^{h_1}$$

Further $P_{1\Theta}$ belonging to $E_{k, h_1+h_2, \Theta}$ but to neither $\{E_{k, h_1+s, \Theta}\}_{s=1, 2 \dots, h_2-1}$ it is contained by:

$$P_{1\Theta} \,\varepsilon\, E_k^{h_1} \cdot E_{k, h_1+h_2, \Theta} \cdot \prod_{s=1}^{s=h_2-1} (E_k^{h_1} - E_k^{h_1} \cdot E_{k, h_1+s, \Theta}) = E_k^{h_1 h_2}.$$

Continuing in the same manner we shall convince ourselves that:

$$P_{1\Theta} \,\varepsilon\, E_k^{h_1 h_2 \dots h_m} \qquad k > K$$

and all the more:

$$P_{1\Theta} \,\varepsilon \sum_{h_1 h_2 \dots h_m} E_k^{h_1 h_2 \dots h_m} = A_k^m \qquad k > K \tag{5}$$

where the sum is to be taken over all natural $h_2\ h_3 \dots h_m$ and all integer non negative h_1. This last relation holds for any k great enough, that is to say:

(6) $$P_{1\Theta} \,\varepsilon\, \lim_{k\to\infty} A_k^m$$

But since (6) concerns any point $P_{1\Theta} \,\varepsilon\, E_{m\Theta}$ we have:

(7) $$E_{m\Theta} \subset \lim_{k\to\infty} A_k^m$$

Employing the same method, we show on the other side:

(8) $$\lim_{k\to\infty} A_k^m \subset E_{m\Theta}$$

Combining (7) and (8) we finally find:

(9) $$E_{m\Theta} = \lim_{k\to\infty} A_k^m$$

and the measurability l of the set $E_{m\Theta}$ is thus demonstrated.

Remark I. If E be a borelian set, it is not necessary, in order to demonstrate th. II to assume the finiteness of $\Phi(E)$. This condition we have only used when showing measurability l of $E_{kh\Theta}$. In the case of borelian E, $E_{kh\Theta}$ and $E_k^{h_1 h_2 \dots h_r}$ are sets of Suslin (A).

Remark II. *The projection of points covering Θ exactly m times is measurable-l.*

We have found, that the sets E_m project themselves on Θ in sets measurable-l. As far as E'_m is concerned we cannot tell, whether it is measurable-Φ. Howewer if we substract from every set E_m in a certain manner (which we will describe below) sets projecting themselves on Θ into sets of l-measure zero, the remaining points of E_m will furnish a set measurable-Φ. As easily seen it comes to the same to discuss the sets E'_m or e_m consisting of points of E projecting themselves on Θ exactly m times.

Thus we establish:

Lemma I. *A set E' exists measurable-Φ, of such a kind, that:*

1^0 $$E' \subset E$$
$$V_l(E')_\Theta = 0$$

2^0 The set $D_m = (E - E') \,.\, e_m$

consists but of points projecting themselves on Θ exactly m times

3^0 D_m *is measurable-Φ.*

Proof. Let us construct for every set $e_{m\Theta}$ a borelian set b_m, being part of the former, so that:

(10) $$V_\iota e_{m\Theta} = V_\iota b_m; \quad b_m \subset e_{m\Theta}$$

and moreover borelian sets B_m consisting of those points in the space, whose projection on Θ belongs to b_m. The sets:

$$D_m = B_m . E \subset e_m$$

are measurable-Φ.

Evidently D_m consists only of points projecting themselves on Θ exactly m times. Besides that in virtue of (10).

$$V_\iota (D_{m\Theta}) = V_\iota (e_{m\Theta})$$

$$\text{The set: } E' = E - \sum_{m=1}^{\infty} E . B_m = E - \sum_{m=1}^{\infty} D_m$$

is measurable-Φ and its projection on Θ has a l-measure zero.

E' agrees with the conditions of our lemma.

We will now employ our results in order to generalise a theorem due to Prof. Banach.

To beginn with, we give some definitions.

Let — as usually — e_m denote the set of points belonging to E and projecting themselves on Θ exactly m times. Let us define the function $F_\Theta (x\, y)$ in the plane Θ in the following manner:

$$\begin{aligned} F_\Theta (x\, y) &= m && \text{in the set } e_{m\Theta}, \\ F_\Theta (x\, y) &= +\infty && \quad\text{"} \quad E_\Theta - \sum_{m=1}^{\infty} e_{m\Theta} \\ F_\Theta (x\, y) &= 0 && \text{out of } E_\Theta. \end{aligned}$$

We announce now:

Theorem III. a) The function $F_\Theta (x\, y)$ is summable-l.

$$\text{b)} \iint_\Theta F_\Theta (x\, y)\, dx dy \leqslant \Phi (E).$$

Proof. It will do to show the function $F'_\Theta (x\, y)$ equivalent to $F_\Theta (x\, y)$ to be summable l. For this purpose let us consider the borelian sets b_m and B_m defined when demonstrating the lemma. We put:

$$\begin{aligned} F'_\Theta (x\, y) &= m && \text{in the set } b_m \\ F'_\Theta (x\, y) &= 0 && \text{out of } \sum_{m=1}^{\infty} b_m . \end{aligned}$$

Clearly F' is equivalent to F. Moreover b_m being measurable-l we have:

$$\int_{\Theta}\!\!\int F'_{\Theta}(x\,y)\,dx\,dy = \sum_{m=1}^{\infty} m\,.\,V_l\,b_m$$

We have yet to show that, $\sum_{m=1}^{\infty} m\,.\,V_l\,b_m$ is finite;

According to lemma we know, that:

(11) 1^0 the sets b_m are projections of $D_m = B_m\,.\,E \subset e_m$

2^0 D_m are measurable-Φ and mutually separated.

Making use of Gross theorem mentioned in the introduction, we obtain

(12) $$V_l\,b_m \leqslant \frac{\Phi\,(D_m)}{m}.$$

On the other hand, according to $(11)_2$:

$$\sum_{m=1}^{\infty} \Phi\,(D_m) = \Phi\left(\sum_{m=1}^{\infty} D_m\right) \leqslant \Phi\,(E)$$

$$\text{since } \sum_{m=1}^{\infty} D_m \subset E$$

and finally in virtue of $(11)_1$, (12):

(13) $$\sum_{m=1}^{\infty} m\,.\,V_l\,b_m \leqslant \sum_{m=1}^{\infty} \Phi\,(D_m) \leqslant \Phi\,(E)$$ [1] q. e. d.

§ 2. We return now to denotations of th. II.

Observe the set $H = \sum_{i=1}^{\infty} e_i$, that is the set of all points, which project themselves on Θ a finite number of times. Let us arrange, on every straight line parallel to z-axis, having a non-vacuons pro-

[1]) The necessary and also *sufficient* condition that a set *measurable* J has a finite (exterior) jansenian measure is: the functions $F_1\,(x\,y)$, $F_2\,(y\,z)$, $F_3\,(z\,x)$ are summable l. F_1, F_2, F_3 are formed for planes $z = 0$, $y = 0$, $z = 0$ resp.

This may be proved in a similar way as th. III.

duct with H, the points of E by increasing z-coördinates:

(14) $$z(P_1) < z(P_2) \ldots < z(P_r) < \ldots z(P_m)$$

Let H_r denote the set of all those points of H, whose index defined by (14) is r. We state now.

Theorem IV. *In case E is a borelian set and has a finite Φ-area, H_r are measurable-Φ.*

Proof. We make the important remark, that 3-dimensional (A) sets of Suslin, having a finite exterior surface measure Φ are mesurable Φ. This may be shown in a similar manner, as that in which the l-measurability of (A) sets has been demonstrated by Messrs Sierpiński and Lusin [1]). It is only necessary to replace the words „exterior" (interior) Lebesgue's measure by „exterior" (interior) surface measure [2]).

H follows to be measurable-Φ.

We observe now the sets $S_k^{h_1 h_2 h_3 \ldots h_r}$ consisting of exactly those points of the space, whose projection on Θ is $E_k^{h_1 h_2 \ldots h_r}$ and whose z-coordinate satisfies the relation:

(15) $$0 \leqslant z < \frac{h_1 + h_2 + \ldots h_r + 1}{2^k}$$

$S_k^{h_1 h_2 \ldots h_r}$ is an 3-dimensional A-set. We define now:

$$M_k^{h_1 h_2 \ldots h_r} = H \,.\, S_k^{h_1 h_2 \ldots h_r}.$$

$M_k^{h_1 h_2 \ldots h_r}$ are measurable-Φ being A-sets of a finite Φ.

Let us now remark, that:

$$M_k^{h_1 h_2 \ldots h_r} \,.\, M_k^{h'_1 h'_2 \ldots h'_r} = 0 \qquad \text{when } (h_1\, h_2 \ldots h_r) \neq (h'_1\, h'_2 \ldots h'_r)$$

because they rest upon $E_k^{h_1 h_2 \ldots h_r}$ resp. $E_k^{h'_1 h'_2 \ldots h'_r}$, those being without common points (Th. II relation (4). Let us designe by:

$$M_{k_r} = \sum_{h_1 h_2 \ldots h_r} M_k^{h_1 h_2 \ldots h_r}$$

where the summing is to be taken over all non negative integer

[1]) M. Lusin et W. Sierpiński: Sur quelques proprietées des ensembles (A). Bull. de l'Acad. d. Sciences de Cracovie. Avril 1918.

[2]) The interior surface measure may be object of definition, since the exterior surface measure is „regular" in the sense of Caratheodory R. F. p. 258.

h_1 and all natural $h_2 h_3 \dots h_r$. We state now, that:

$$\text{(16)} \qquad \operatorname*{Lim\,inf}_{k\to\infty} M_{kr} = \sum_{s=1}^{s=r} H_s .$$

In order to show (16) we observe, that if the point P belongs to H without belonging to $\sum_{s=1}^{s=r} H_s$:

$$\text{(17)} \qquad P \,\varepsilon \sum_{s=r+1}^{\infty} H_s$$

then points $P_1\, P_2 \dots P_r$ lying together with P on the same perpendicular to Θ have according to the definition of H_s z-coordinates smaller then $z(P)$:

$$z(P_1) < z(P_2) \dots < z(P_r) < z(P).$$

If we choose k great enough $(k > K)$ we achieve, that the points $\{P_s\}_{s=1,2\dots r}$ and P belong to distinct layers W_{kh}:

$$h_s > 0 \text{ for } s = 2, 3 \dots r; \qquad P_s \,\varepsilon\, W_{k,\, h_1+h_2+\dots h_s}; \qquad s = 1, 2, \dots r$$

$$P \,\varepsilon\, W_{k,\, h_1+h_2+\dots h_r+\omega} \qquad \omega > 0 \qquad k > K$$

that means:

$$P_{1\Theta} = P_{2\Theta} = \dots P_{r\Theta} \,\varepsilon\, E_k^{h_1 h_2 \dots h_r} \qquad k > K$$

and that:

$$P_s \,\varepsilon\, M_k^{h_1 h_2 \dots h_s} \qquad s = 1, 2, \dots r \qquad k > K$$

P does not belong to $M_k^{h_1 h_2 \dots h_r}$, because $M_k^{h_1 h_2 \dots h_r}$ contains no points. but those complying with the relation (15), while we have:

$$z(P) \geqslant \frac{h_1 + h_2 + \dots h_r + \omega}{2^k} \geqslant \frac{h_1 + h_2 + \dots h_r + 1}{2^k}$$

The more P belongs to neither $M_k^{h'_1 h'_2 \dots h'_r}$ $- (h'_1\, h'_2 \dots h'_r) \neq (h_1\, h_2 \dots h_r)$ since $M_k^{h'_1 h'_2 \dots h'_r}$ projects itself on $E_k^{h'_1 h'_2 \dots h'_r}$, which does not contain P_Θ [1]) (th. II. rel. 4). Thus

$$P \, non \,\varepsilon \sum_{h_1 h_2 \dots h_r} M_k^{h_1 h_2 \dots h_r} \qquad k > K$$

consequently:

$$P \, non \,\varepsilon \operatorname*{Lim}_{k\to\infty} M_{kr}$$

[1]) $P_\Theta = P_{1\Theta} = P_{2\Theta} \dots = P_{r\Theta} \,\varepsilon\, E_k^{h_1 h_2 \dots h_r}$.

provided that P agrees with (17). Or the same result in another way:

$$\text{(18)} \qquad \operatorname*{Lim\,inf}_{k\to\infty} M_{kr} \subset \sum_{s=1}^{r} H_s .$$

In a similar mode we show, that:

$$\text{(19)} \qquad \operatorname*{Lim\,inf}_{k\to\infty} M_{kr} \supset \sum_{s=1}^{r} H_s$$

Comparing (18) and (19):

$$\operatorname*{Lim\,inf}_{k\to\infty} M_{kr} = \sum_{s=1}^{r} H_s .$$

Consequently H_r are measurable-Φ.

Remark I. *Every set E of finite Φ-area may be represented as the sum of:*

1⁰ *a set E' measurable Φ, projecting itself on Θ in a set of l-measure zero*

2⁰ *a sequence of sets H_r measurable-Φ, each of them projecting itself on Θ exactly once.*

To prove this, it will suffice to construct (according to 5^{th} postulate) a borelian set B, so that:

$$B \subset E; \qquad \Phi(B) = \Phi(E).$$

Then H_i of theor. IV, taken with respect to B agree with the conditions.

Remark II. *For a borelian E the sets $H_r \ldots e_m$ are measurable-Φ.*

H_r being measurable-Φ, it remains to prove the measurability Φ of e_m. This results from the remark, that e_m is a 3-dimensional (A) set of finite Φ.

We shall now prove — with respect to Gross Φ_0 measure — the following theorem:

Theorem V. *Any set E measurable Φ_0, of finite $\Phi_0(E)$, may be, with any accuracy wanted approximated by means of closed sets contained by E.*

We shall demonstrate at first two lemmas:

Lemma II. Let D be a set of finite Φ-area, projecting itself on Θ exactly once. Let us choose a system of coordinates like in th. II

Now consider the function $f(x\,y)$ — defined in D_Θ — whose geometrical image is the set D, that is:

$$f(P_\Theta) = z(P)$$

We shall demonstrate, that:

$$f(x\,y) \text{ is measurable-}l.$$

Proof. We have to evince, that for every number α the set F_α of those points belonging to Θ', which comply with:

$$z = f(x\,y) > \alpha$$

is measurable-l.

This takes actually place, since F_α can be represented in this manner:

$$F_\alpha = (D \,.\, R_\alpha)_\Theta$$

where R_α denotes the set of those point in the space, whereat $z > \alpha$. Now $D \,.\, R_\alpha$ being product of two measurable-Φ sets, is itself measurable-Φ and its projection measurable l.

Lemma III. *If the set D complies with conditions of the preceding lemma, it may be represented as the sum of a closed set F and complementary set $D - F$, so that:*

$$0 \leqslant V_i (D - F)_\Theta = V_i D_\Theta - V_i F_\Theta < \varepsilon$$

where ε is any positive number.

The proof is immediate by means of the following well known theorem [1]): Every function measurable l, finite and defined in a set D_Θ measurable l, is continuous save a set, whose l-measure can be made as small as wanted.

Proof of th. V. Since for every set E of a finite Φ-area a borelian set can be found belonging entirely to E, whose Φ-area is equal te $\Phi(E)$ it will be sufficient to prove th. V with respect to a borelian E.

In this case $\Phi_0(E) = \overline{\Phi}_0(F)$ and we make use of the manner in which the Gross' measure $\overline{\Phi}_0$ has been constructed. Let us choose a space grating such that:

$$\sigma(E) = \sum_{\varrho=1}^{\infty} \sigma_\varrho(E)$$

$$0 \leqslant \overline{\Phi}_0(E) - \sigma(E) < \varepsilon \qquad \varepsilon > 0 \tag{20}$$

1) R. F. page 410. th. 7.

We mention that $\sigma(E)$, $\sigma_\varrho(E)$, E_ϱ were defined in introduction. We choose a plane Θ_ϱ, so that:

$$0 \leqslant \sigma_\varrho(E) - V_l(E_\varrho)_{\Theta_\varrho} < \frac{\varepsilon}{2^\varrho}. \tag{21}$$

According to th. IV rem. 1. E_ϱ may be obtained by summing resp:

(α) the sets $\{E_{\varrho\nu}\}_{\nu=1,2\ldots\infty}$ measurable -Φ, each of them projecting itself on Θ exactly once

(β) a complementary set, its projection on Θ_ϱ being of l-measure zero:

$$E_\varrho = \sum_{\nu=1}^{\infty} E_{\varrho\nu} + \left(E - \sum_{\nu=1}^{\infty} E_{\varrho\nu}\right) \tag{22}$$

$$V_l\left(E_\varrho - \sum_{\nu=1}^{\infty} E_{\varrho\nu}\right)_{\Theta_\varrho} = 0.$$

Applying lemma III to every $E_{\varrho\nu}$ we shall find closed sets $F_{\varrho\nu}$ and a positive integer ν_ϱ, such that:

$$F_{\varrho\nu} \subset E_{\varrho\nu}$$

$$0 \leqslant V_l\left(\sum_{\nu=1}^{\infty} E_{\varrho\nu}\right)_{\Theta_\varrho} - V_l\left(\sum_{s=1}^{\nu_\varrho} F_{\varrho s}\right)_{\Theta_\varrho} < \frac{\varepsilon}{2^\varrho} \quad \text{lemma III!}$$

and in virtue of (22):

$$0 \leqslant V_l(E_\varrho)_{\Theta_\varrho} - V_l\left(\sum_{s=1}^{\nu_\varrho} F_{\varrho s}\right)_{\Theta_\varrho} < \frac{\varepsilon}{2^\varrho}. \tag{23}$$

From (23) we conclude according to (21)

$$0 \leqslant \sigma_\varrho(E) - V_l\left(\sum_{s=1}^{\nu_\varrho} F_{\varrho s}\right)_{\Theta_\varrho} < \frac{2\varepsilon}{2^\varrho}$$

(24)

$$\text{where } \sum_{s=1}^{\nu_\varrho} F_{\varrho s} \subset E_\varrho.$$

Let us now choose a positive integer r great enough, so that:

$$0 \leqslant \sigma(E) - \sum_{\varrho=1}^{r} \sigma_\varrho(E) < \varepsilon \text{ [1]}. \tag{25}$$

[1]) This is possible according to (20).

We have then in view of (24). (25):

$$0 \leqslant \sigma(E) - \sum_{\varrho=1}^{r} V_\iota \Big(\sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big)_{\Theta_\varrho} < 3\varepsilon. \tag{26}$$

With account of (20), (26) and relations between sets and their projections we may write:

$$\Phi_0 \Big(\sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big) \geqslant V_l \Big(\sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big)_{\Theta_\varrho}$$

(27):

$$0 \leqslant \overline{\Phi}_0 (E) - \sum_{\varrho=1}^{r} \overline{\Phi}_0 \Big(\sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big) \leqslant \overline{\Phi}_0 (E) - \sum_{\varrho=1}^{r} V_\iota \Big(\sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big)_{\Theta_\varrho} < 4\varepsilon$$

Since the sets $\Big\{ \sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big\}_{\varrho=1,2,\ldots,r}$ have no mutually common points, we may interchange in (27) the signs „Φ“ and „Σ“ and we find finally:

$$0 \leqslant \overline{\Phi}_0 (E) - \overline{\Phi}_0 \Big(\sum_{\varrho=1}^{r} \sum_{s=1}^{\nu_\varrho} F_{\varrho_s} \Big) < 4\,\varepsilon$$

$\sum_{\varrho=1}^{r} \sum_{s=1}^{\nu_\varrho} F_{\varrho_s}$ being a closed set. q. e. d.

§ 3. One more theorem, standing rather aside!

Let E be a set in the space and let a, b, c be three numbers such that:

$$a^2 + b^2 + c^2 = 1.$$

Let $E_{\theta(abc)}$ denote the projection of E on the plane Θ, whose normal may have the direction cosines $a\ b\ c$.

Using these notations we see, that the exterior l-measure of the set $E_{\theta(abc)}$ is a function of a, b, c:

$$V_l^* E_{\nu(abc)} = \varphi (abc).$$

We ask: of which kind is this function? *We can, provided* $\Phi (E)$ *is finite, state*:

Theorem IV. $\varphi (abc)$ *is an upper-semicontinuous function.*

Proof. We have to demonstrate that for each sequence $\{a_n b_n c_n\}_{n=1,2,\ldots,\infty}$ tending towards $a_0\, b_0\, c_0$ we have:

$$\limsup_{n\to\infty} \varphi (a_n\, b_n\, c_n) \leqslant \varphi (a_0\, b_0\, c_0)$$

On the plane $\Theta(a_0 b_0 c_0)$ we enclose the set $E_{\Theta(a_0 b_0 c_0)}$ by any 2-dimensional domain G. Let $G(abc)$ be the 2-dimensional domain situated on $\Theta(abc)$ and constructed thus:

we take the collection of straights, perpendicular to $\Theta(a_0 b_0 c_0)$ and passing through G. Points common to this family of straights and to the plane $\Theta(abc)$ form — by definition — the domain $G(abc)$.

We make the following important remark. To every point P of E a positive number $\varepsilon(P)$ can be associated in such a manner, that the point P belongs to $G(abc)$, provided that:

$$|a-a_0|+|b-b_0|+|c-c_0|<\varepsilon(P).$$

Let now E^m be the set of points P situated in E, whereat:

$$P_{\Theta(a_i b_i c_i)}\,\varepsilon\, G(a_i b_i c_i)\ ^{1)}$$

$$\text{for } i=m, m+1, \ldots \infty$$

With regard to our remark and to the definition of E^m we have:

$$(\alpha)\qquad (E^m)_{\Theta(a_i b_i c_i)} < G(a_i b_i c_i) \qquad i=m, m+1, \ldots \infty$$

$$(28)\quad (\beta)\qquad E^m \subset E^{m+1}; \qquad \operatorname*{Lim}_{m\to\infty} E^m = E$$

$$(\gamma)\qquad \lim_{m\to\infty} \Phi(E^m) = \Phi(E).$$

From (28) γ we infer:

$$(29)\qquad \lim_{m\to\infty} \Phi(E-E^m)=0$$

provided that E^m be the product of E and a set measurable -Φ. In order to show this consider the family of straights perpendicular to $\Theta(a_i b_i c_i)$ and passing through $G(a_i b_i c_i)$. This family forms a 3-dimensional domain: $W(a_i b_i c_i)$.

It is easily percevied that:

$$E^m = E.\prod_{i=m}^{\infty} W(a_i b_i c_i)$$

and $\prod\limits_{i=m}^{\infty} W(a_i b_i c_i)$ is measurable Φ being a G_δ.

Proceeding from the identity:

$$E=E^m+(E-E^m)$$

1) $P_{\Theta(abc)}$ the projection of P on the plane $\Theta(abc)$.

we can write:

$$\varphi(a_n b_n c_n) = V_i^* E_{\Theta(a_n b_n c_n)} \leqslant V_i^* E^n_{\Theta(a_n b_n c_n)} + V_i^* (E - E^n)_{\Theta(a_n b_n c_n)}$$

Making use of (28) (α) and of $\Phi(M) \geqslant V_i^* M_\Theta$ — for every set M — we can estimate $\varphi(a_n b_n c_n)$ thus:

$$\varphi(a_n b_n c_n) \leqslant \Phi(E - E^n) + V_i\, G(a_n b_n c_n).$$

Passing to the limit, we find according to (29):

(30) $$\limsup_{n\to\infty} \varphi(a_n b_n c_n) \leqslant \lim_{n\to\infty} V_i\, G(a_n b_n c_n) = V_i\, G$$

(30) remaining valid for every domain G containing $E_{\Theta(a_0 b_0 c_0)}$ we have finally:

$$\limsup_{n\to\infty} \varphi(a_n b_n c_n) \leqslant V_i^* E_{\Theta(a_0 b_0 c_0)} = \varphi(a_0 b_0 c_0).$$

II.

§ 1. **Definition I.** *A family R of spheres $\{K\}$ will be said to cover the set E*, if to every point P belonging to E we can associate a sequence $\{K_i\}_{i=1,2\ldots\infty}$ contained by R, such that:

1° P is the centre of the corresponding $K_i(P)$ for $i = 1, 2\ldots\infty$ and that:

2° the limit of the radii of K_i is zero, when $i \to \infty$.

Definition II. A set E shall be called *normal (with respect to the set function Φ)*, if from e v e r y family covering E we may extract a sequence $\{K_i\}_{i=1,2\ldots\infty}$ endowed with the two following properties:

(α) $$K_i \cdot K_{i'} = 0 \qquad i \neq i'$$

(β) $\sum_{i=1}^{\infty} K_i$ covers E excepting a set of area -Φ zero.

We shall now prove the following theorem:

Theorem VII. *A set E being supposed normal and of finite Φ area, every measurable -Φ subset of E is normal with respect to Φ_0.*

Proof. Every set normal with respect to a Φ complying with I—VI postulates, being also normal with respect to Gross' Φ_0- measure, it will be sufficient to discuss sets normal with respect to Φ_0.

Let E' be the subset of E, which we intend to show to be normal. Of course, it will do to demonstrate, that from every family $R_{E'}$ covering E' a finite number of mutually seperated spheres $\{K_i\}_{i=1,2\ldots n}$ can be extracted, covering a part F of E with $\Phi_0(F)$ as near to $\Phi_0(E)$ as we like.

Let — according to th V — F denote a closed subset of E' of such a kind, that:

$$\Phi_0(E-F) < \varepsilon.$$

We complete the family $R_{E'}$ in the following manner to a certain family R_E covering E: for every point P of $E-E'$ we choose a sequence of spheres $\{K_i\}_{i=1,2..\infty}$ — P their centre — having no common points with F (this is possible in virtue of the closedness of F) in such a way, that the radii of K_i tend to zero. We annex all those K_i to $R_{E'}$ and obtain thus a family R_E covering E.

The set E being on supposition normal we may extract from R_E an at most enumerable number of spheres K_n mutually separated such, that the sum $\sum_{i=1}^{\infty} K_i$ covers E save a set of Gross' measure zero.

Those of the extracted K_n which do not belong to $R_E - R_{E'}$, cover F except a set of Φ_0 measure zero.

A similar reasoning has been applied by Prof. Banach on another purpose.

Remark 1. The property of E to be normal with respect to Φ_0 is equivalent to the existence of a positive number K of such a kind, that:

for every subset E' of E a finite number of mutually seperated spheres $\{K_i\}_{i=1,2\ldots n}$ can be extracted from the given family R_E (covering E), so that:

$$\sum_{i=1}^{n} \Phi_0(E.K_i) \geqslant \varkappa . \Phi_0(E') \qquad \varkappa > 0.$$

From theorems V and VII results also the following theorem VIII, which shall be announced without demonstration.

Theorem VIII [1]. *Let A be a normal set, having a finite area $\Phi(A)$. Let us denote by R_E a family of spheres covering the set $E \subset A$; E measurable Φ.*

Then we may choose a sequence of spheres $\{K_i\}_{i=1,2..\infty}$ belonging to R_E, so that:

[1] Cf. R. F. § 291.

$$\Phi\left[\left(\sum_{i=1}^{\infty} K_i\right)(A-E)\right] < \varepsilon. \qquad \varepsilon > 0$$

$$K_i \,.\, K_k = 0 \qquad i \neq k$$

$$\Phi\left[\left(\sum_{i=1}^{\infty} K_i\right).E\right] = \Phi(F)$$

ε being as small as we please.

§ 2. Let E be a set of finite $\Phi(E)$ area, P any point of the space. Let $W(P,a)$ denote the product of E and a sphere, whose centre be P and its radius a. $W(P,a)$ is measurable -Φ and moreover:

$$\Phi[W(P,a)] \leqslant \Phi(E).$$

Let E_1 denote the set of those points P of E, for which:

(1) $$\Phi[W(P,a)] = 0$$

provided that $a \leqslant a(P)$.

The set E_1 is of zero measure Φ. Indeed, according to Lindelöf's theorem [1]) we may find an at most countable number of points belonging (all) to E_1 so, that:

(2) $$E_1 \subset \sum_{i=1}^{\infty} W(P_i, a_i)$$

$W(P_i, a_i)$ complying with (1). The second member (2) is thus of Φ- area zero and so is E_1.

The numbers:

$$\Phi[W(P,a)] = \varphi(P,a)$$
$$V_i[W(P,a)]_\Theta = \psi(P,a)$$

are functions of P and a. In order to show some properties of those functions we observe, that definitions of functions „measurable Φ" and „summable Φ" can be transferred from the theory of Lebesgue's integrals. All theorems of importance will, as easily established remain precerved.

Theorem IX. *For every E of finite Φ- area, the corresponding $\varphi(Pa)$ and $\psi(Pa)$ are:*

[1]) Cf. R. F. § 60, page 46.

1° *continuous from the right side for fixed* P
2° *upper semi-continuous in* P *for constant* a.

Proof. Let us commence with $\varphi(Pa)$ and the first part of our theorem. We choose a point P constant and consider a decreasing sequence $\{a_n\}$ tending to a:

(4) $$a_{n+1} < a_n; \qquad n = 1, 2 \ldots \infty; \qquad \lim a_n = a.$$

We have to prove, that:

$$\lim \varphi(Pa_n) = \varphi(Pa).$$

We have:

$$\lim_{n\to\infty} \varphi(Pa_n) = \lim_{n\to\infty} \Phi[W(P, a_n)]$$

$$\operatorname*{Lim}_{n\to\infty} W(P, a_n) = W(P, a)$$

as easily verified.

Consequently the Φ- areas of sets $\{W(P, a_n)\}$ tend towards the area of $W(P, a)$:

$$\lim_{n\to\infty} \varphi(Pa_n) = \varphi(Pa).$$

In order to demonstrate the second part of our theorem — for $\varphi(Pa)$ — we shall show, that the set E_α of points, whereat:

$\varphi(Pa) \geqslant \alpha$ is a closed set, when P varies (a constant).

In fact let P be a limit point of E_α and let $\{P_n\}_{n=1,2\ldots\infty}$ be a sequence of points of E_α tending towards P. For numbers $\{\Delta_n\}_{n=1,2\ldots\infty}$ properly chosen we achieve:

$$W(P_n a) \subset W(P, a + \Delta_n); \qquad \lim_{n\to\infty} \Delta_n = 0.$$

In each point P_n we have:

$$\alpha \leqslant \varphi(P_n a) = \Phi[W(P_n, a)] \leqslant \Phi[W(P, a + \Delta_n)].$$

Passing to the limit we obtain:

$$\alpha \leqslant \Phi[W(P, a)] = \varphi(Pa).$$

Let us now in a similar way proceed with $\psi(Pa)$. At first concerning continuity from the right side, P remaining constant. Let the sequence of numbers $\{a_n\}$ be chosen as above (rel. 4). Further let $\Omega(P, a_n)$ denote the set of those points of $W(P, a_n)$, which:

1° do not belong to $W(Pa)$:

$$\Omega(P, a_n) \subset W(P, a_n) - W(P, a) \qquad \text{while}$$

2° $[\Omega(P,a_n)]_\Theta$ belongs entirely to $[W(P,a)]_\Theta$

$$[\Omega(P,a_n)]_\Theta \subset [W(P,a)]_\Theta.$$

We may of course write:

(5) $W(P,a_n) = W(P,a) + \Omega(P,a_n) + [W(P,a_n) - W(P,a) - \Omega(P,a_n)].$

We immediately perceive, that:

(6) $$\lim_{n\to\infty} V_i[W(P,a_n) - W(P,a) - \Omega(P,a_n)]_\Theta = 0$$

We will now show, that:

$$\lim_{n\to\infty} V_i[\Omega(P,a_n)]_\Theta = 0.$$

For this purpose let us observe the relations:

(7_1) $\Omega(P,a_n) \supset \Omega(P,a_{n+1});\qquad [\Omega(P,a_n)]_\Theta > [\Omega(P,a_{n+1})]_\Theta$

(7_2) $$\operatorname{Lim}_{n\to\infty} \Omega(P,a_n) = 0.$$

Yet we cannot conclude, that $\operatorname{Lim}_{n\to\infty} [\Omega(P,a_n)]_\Theta = 0.$

In any case each point of $\operatorname{Lim}_{n\to\infty} [\Omega(P,a_n)]_\Theta$ is the projection of an infinity number of points (belonging to E) and therefore according to (7):

(8) $$\lim_{n\to\infty} V_i[\Omega(P,a_n)]_\Theta = V_i\{\operatorname{Lim}_{n\to\infty}[\Omega(P,a_n)]_\Theta = 0\}.$$

According to (5) we have:

(9) $$\begin{aligned}\Psi(P,a) \leqslant \Psi(P,a_n) \leqslant \Psi(P,a_n) + V_i[\Omega(P,a_n)]_\Theta +\\ + V_i[W(P,a_n) - W(P,a) - \Omega(P,a_n)]_\Theta\end{aligned}$$

and finally with respect (6) and (8) we conclude from 9:

$$\Psi(P,a) = \lim_{n\to\infty} \Psi(P,a_n).$$

The upper semi-continuity of $\Psi(P,a)$ — a being fixed — can be demonstrated in the same manner as the upper semi-continuity of $\varphi(P,a)$.

Thus is our theorem entirely evinced.

Remark: the function $\dfrac{\Psi(Pa)}{\varphi(Pa)}$ *is continuous from the right side for fixed* P, *and measurable* Φ *for constant* a.

In the preceding we have seen, that the ratio $\dfrac{1}{\varphi(P_a)}$ is in $E-E_1$ defined for *every* $a>0$, E_1 being part of E of zero measure Φ.

Let us now examine — in $E - E_1$ — the ratio $\frac{\Psi(Pa)}{\varphi(Pa)}$.

Let $D^*(P)$ and $D_*(P)$ denote the largest and least resp. limit, which may be obtained from $\frac{\Psi(Pa)}{\varphi(Pa)}$, when $a \to 0$" P remaining fixed. We announce:

Theorem X. *If E is measurable Φ, $\Phi(E)$ being different of zero and finite, $D^*(P)$ and $D_*(P)$ are summable -Φ.*

It is sufficient to prove, that D^* and D_* are measurable -Φ, since their summability ensues as a consequence of their boundedness.

The demonstration of this theorem may be accomplished by means of the same classical method, which is made use of when demonstrating the measurability l of „derivatives" of absolutely continuous setfunctions [1]), since $\frac{\Psi}{\varphi}$ is *measurable* Φ for constant a, and continuous from the right side for fixed P.

$D^*(D_*)$ we shall call the upper (lower) ratio of enlargement of the set E on the plane Θ.

In order to obtain further results, we suppose E normal.

Theorem XI. Let:

1° *E be a normal set, of finite Φ_0 area*

2° *E' a measurable subset of E, such that its projection E'_Θ be of zero measure -l,*

then in „almost every" [2]) point P of E' is the upper ratio of enlargement D^*. calculated with respect to E, equal to zero.

Proof. $D^*(P)$ being a function measurable -Φ (th. X), the set E'_α of those points of E' whereat:

$$D^*(P) > \alpha > 0 \qquad P \varepsilon E' \tag{10}$$

is surely measurable -Φ. Let us suppose:

$$\Phi(E'_\alpha) = \beta > 0. \tag{11}$$

This supposition will lead to a contradiction.

[1]) absolutely continuous with respect to l-measure. Cf. R. F. § 436. th. II. page 482 - 484.

[2]) that is except a set of Φ measure zero.

E'_α being part of a normal set, is itself normal.

Let us for every point of E'_α choose a sequence of sets $\{W(P, a_n)\}_{n=1..\infty}$ [1]) in such a manner, that:

$$\frac{\Phi(P, a_n)}{\varphi(P, a_n)} > D^*(P) - \varepsilon(P) > \alpha; \qquad \lim_{n\to\infty} a_n = 0 \tag{12}$$

$\varepsilon(P)$ being chosen small enough.

E'_α being normal, we can extract [2]) (acc. to th. VIII) a sequence $\{W(P_i, a_i)\}_{i=1,2\ldots\infty}$ so that:

$$W(P_i, a_i) . W(P_k, a_k) = 0 \qquad i \neq k$$

$$\Phi\left[\sum_{i=1}^{\infty} W(P_i, a_i) - E'_\alpha . \sum_{i=1}^{\infty} W(P_i, a_i)\right] < \eta \tag{13}$$

$$\Phi\left[E'_\alpha - E'_\alpha . \sum_{i=1}^{\infty} W(P_i, a_i)\right] = 0$$

η being chosen as small as wanted.

According to (12) we have:

$$\sum_{i=1}^{\infty} V_l[W(P_i a_i)]_\Theta > \alpha . \sum_{i=1}^{\infty} \Phi\, W[(P_i a_i)] = \alpha . \Phi\left(\sum_{i=1}^{\infty} W(P_i a_i)\right) \tag{14}$$

We observe that

$$\sum_{i=1}^{\infty} V_l[W(P_i, a_i)]_\Theta \leq \sum_{i=1}^{\infty} V_l[\underset{\mathrm{I}}{\underset{\uparrow}{W(P_i, a_i) . E'_\alpha}})]_\Theta + \tag{15}$$
$$+ \sum_{i=1}^{\infty} V_l[\underset{\mathrm{II}}{\underset{\uparrow}{W(P_i, a_i) . (E - E'_\alpha)}}]_\Theta .$$

The first part (I) in the right member in (15):

$$(\mathrm{I}) = 0 \tag{16}$$

as each $W(P_i, a_i) . E'_\alpha$ projects itself on Θ in a set of l-measure zero. On the other hand according to $(13)_2$:

[1]) $W(P, a)$ are calculated with respect to E.

[2]) from $\{W(P, a)\}$ complying with (12).

$$(17)\qquad \begin{aligned}(\mathrm{II}) &\leqslant \sum_{i=1}^{\infty} \Phi[W(P_i, a_i).(E-E'_\alpha)] = \sum_{i=1}^{\infty} \Phi[W(P_i, a_i)] - \\ &- \sum_{i=1}^{\infty} \Phi.[W(P_i, a_i).E'_\alpha] < \eta.\end{aligned}$$

Combining (14), (15), (16), (17) we may write:

$$(18)\qquad \alpha . \Phi\left(\sum_{i=1}^{\infty} W(P_i a_i)\right) < \eta.$$

But the relation $(13)_2$ can be written in the following manner:

$$(19)\qquad \Phi\left(\sum_{i=1}^{\infty} W(P_i a_i)\right) = \Phi(E'_\alpha) + \vartheta\,\eta \qquad |\vartheta| \leqslant 1$$

and we obtain finally regarding (11), (12), (18):

$$\alpha.(\beta + \vartheta\,\eta) < \eta.$$

But this is impossible, as η tends to zero and as $\alpha.\beta \neq 0$.

Theorem XII. Let:

1° *E be normal of a finite -Φ_0 area*

2° *E' be a measurable -Φ_0 subset of E*

3° *E' project itself on Θ exactly once,*

then:

$$V_t\, E'_\Theta = \int_{E'} D^* d\sigma = \int_{E'} D_* \, d\sigma$$

the symbol $\int_{E'}$ denoting a surface integral, taken in the set E'. We mention that D^* and D_* are constructed with respect to the set E.

Proof. As the particulars of the demonstration would but slightly differ from the fashion, in which an „absolutely continuous and additive" setfunction is shown to be reproduced by the integral of its „field derivatives", we confine ourselves to state, that the property of our set to be normal is to be used instead of Vitalis „covering theorem".

Theorem XIII. *Let E be a normal set of finite Φ-area — then:*

$$D_* = D^* = D$$

save a set of Φ-area zero we have.

Proof. Indeed, let us represent E as the sum of:

1° sets $\{Q_\nu\}_{\nu=1,2\ldots\infty}$ measurable Φ, projecting themselves on Θ exactly once:

2° the complementary set, measurable Φ, whose projection:

$$\left(E - \sum_{\nu=1}^{\infty} Q_\nu\right)_\Theta$$ is of l-measure zero.

In the set $E - \sum_{\nu=1}^{\infty} Q_\nu$ we have with respect to th. XI:

$$D^* = D_* = 0$$

neglecting a set of Φ_0 measure zero.

In every one of the sets Q_ν (acc to th. XII):

$$V_l\, Q_{\nu\Theta} = \int_{Q_\nu} D^*\, d\sigma = \int_{Q_\nu} D_*\, d\sigma \tag{20}$$

From (20) results regarding $D^* \geqslant D_*$:

$$D^*(P) = D_*(P); \qquad P \varepsilon Q_\nu;\ \nu = 1, 2 \ldots \infty$$

save a set of Φ_0 measure zero.

* * *

§ 3. Let G be any domain, E its boundary. Let us divide the points of the boundary into three categories:

The I[st] consists of those points P of the boundary, whereat the perpendicular on Θ through P contains a segment AB with its centre in P, such that (AP) contains points of G only and (PB) but points exterior to G.

To the II[nd] category belong all those points, where either:

(α) $\langle AP) + (PB \rangle$ belongs entirely to G

or

(β) $\langle AB) + (PB \rangle$ belongs entirely to the „exterior“ of G.

The III[rd] contains all remaining points of E.

Theorem XIV. Let:

1° *G be a bounded domain.*

2° *its boundary E be normal, of a finite Φ_0 area*

3° *all points of II category belonging to E project themselves on Θ in a set of l-measure zero.*

4° *the function $F(xyz)$ defined in $G+E$ be for almost every pair (xy) absolutely continuous with respect to z.*

5° *the partial derivative $\frac{\partial F}{\partial z}(xyz)$ be summable L in $G+E$* [1]).

6° *$F(xyz)$ be summable Φ on the boundary E.*

then

$$\iiint_G \frac{\partial F}{\partial z}\,dx\,dy\,dz = \int_E \dot{D}\,.\,F\,d\sigma.$$

The dot over D indicates, that we have yet to furnish D with the sign $+$ or $-$ in the following manner:

$$\dot{D}(P) = \pm D(P)\ ^{2})$$

according as the point P be preceded by an uneven or even number of points belonging to the boundary E and lying together with P on the same perpendicular on Θ.

Proof. Since the lebesguian space measure of E is zero, we have:

$$\iiint_G \frac{\partial F}{\partial z}\,dx\,dy\,dz = \iiint_{G+E} \frac{\partial F}{\partial z}\,dx\,dy\,dz. \tag{1}$$

According to Fubini's theorem [3]):

$$\iiint_{G+E} \frac{\partial F}{\partial z}\,dx\,dy\,dz = \iint_{E_\Theta} \varphi(xy)\,dx\,dy \tag{2}$$

where

$\varphi(xy) = \int \frac{\partial F}{\partial z}\,dz$ for almost every point xy belonging to E_Θ.

Let us now observe the following points set situated on $\Theta(xy)$:

(α) The set $(E-H)_\Theta$.

We call to the readers mind the definitions of H, H_r, e_m introduced, when

[1]) that is summable with respect to the 3-dimensional Lebesgue's measure. We choose the system of coördinates like in th. II.

[2]) $D(P)$ — the „ratio of enlargement" with respect to the plane Θ.

[3]) Cf. R. F. § 544, page 621.

demonstrating th. IV. The knowledge of H_r, e_m and their properties shall be used in some lines below. H, H_r, e_m are constructed with respect to the boundary E.

(β) Points of $\Theta(xy)$, whereat $F(x\,y\,z)$ is not absolutely continuous with respect to z.

(γ) Points of $\Theta(x\,y)$, whereat $\int \frac{\partial F}{\partial z}\,dz$ does not exist.

(δ) The projection on Θ of those points belonging to E, which are of IInd category.

The sum $\alpha+\beta+\gamma+\delta$ has the l-measure zero. This follows from our preceding theorems and partly from suppositions of th. XIV.

Consequently a borelian set e can be found in the plane $\Theta(x\,y)$ containing $\alpha+\beta+\gamma+\delta$ and contained by E_Θ, whose l-measure is also zero:

$$\alpha+\beta+\gamma+\delta \subset e \subset E_\Theta$$
$$V_l\,(e)=0.$$

Consider those points of E, which project themselves on e. They form a set $\mathfrak{e}$ measurable Φ:

$$(\mathfrak{e})_\Theta = e; \qquad V_l\,(e) = V_l\,(\mathfrak{e})_\Theta = 0.$$

Therefore all conditions of th. XI are satisfied and we conclude:

$$D(P)=0 \qquad \text{for } P\,\varepsilon\,\mathfrak{e} \tag{3}$$

excepting a set, whose Φ_0-measure is zero.

From (3) we obtain thus:

$$\int\!\!\int_e \varphi(x\,y)\,dx\,dy = \int_{\mathfrak{e}} F\,.\,D\,d\,\sigma = 0$$

and it remains to prove:

$$\int\!\!\int_{E_\Theta - e} \varphi\,(x\,y)\,dx\,dy = \int_{E-\mathfrak{e}} F\,.\,D\,d\,\sigma.$$

The set $E-\mathfrak{e}$ can be decomposed into sets $(E-\mathfrak{e})\,.\,e_r\,.\,H_m\,.$ — — $m=1,2\ldots r$; $r=1,2\ldots\infty$.

$(E-\mathfrak{e})\,.\,e_r\,.\,H_m$ are measurable Φ — as immediately results from th. IV rem. I, — and the correspondence:

$$(E-\mathfrak{e})\,.\,e_r\,.\,H_m \longleftrightarrow [(E-\mathfrak{e})\,.\,e_r\,.\,H_m]_\Theta \qquad r \text{ and } m \text{ fixed}$$

is one to one.

Moreover for constant r all sets:

$$(E-\mathfrak{e}).e_r.H_m \qquad m-1,2,\ldots r$$

have the same projection ω_r:

$$\omega_r=[(E-\mathfrak{e}).e_r.H_m]_\Theta \qquad m=1,2,\ldots r$$

Besides that:

(4) $$\sum_{r=1}^{\infty}\omega_r=E_\Theta-e.$$

Hence we can further decompose $\iint\limits_{E_\Theta-e}\varphi(xy)\,dx\,dy$:

(5) $$\iint\limits_{E_\Theta-e}\varphi(xy)\,dx\,dy=\sum_{r=1}^{\infty}\iint\limits_{\omega_r}\varphi(xy)\,dx\,dy.$$

The function $F(xyz)$ being summable -Φ in E (cond. 6⁰), it is also summable in every one of the sets $(E-\mathfrak{e}).e_r.H_m$. Besides that, the correspondence $(E-\mathfrak{e}).e_r.H_m$ to $[(E-\mathfrak{e}).e_r.H_m]_\Theta$ being unifold, F [1]) turns over into a function $f_{rm}(xy)$ defined on ω_r. We have for every (xy) belonging to ω_r:

(6) $$\varphi(xy)=\int\frac{\partial F}{\partial z}dz=f_{r2}-f_{r1}+f_{r4}-f_{r3}+\ldots f_{rr}-f_{r,r-1}$$

as no point of ω_r belongs to $\alpha+\beta+\gamma+\delta$.

We state now that:

Every function $f_{rm}(xy)$ is summable -l.

At first $f_{rm}(xy)$ is measurable -l. Indeed the point set, whereat:

$$f_{rm}(xy)>\alpha$$

can be regarded at the projection of those points of $(E-\mathfrak{e}).e_r.H_m$ whereat:

$$F(xyz)>\alpha.$$

This last set being measurable -Φ, the projection is measurable l. To prove the summability we observe, that the lebesguian integral:

(7) $$\iint\limits_{\omega_r}|f_{rm}(xy)|\,dx\,dy\leq\int\limits_E|F|.D\,d\sigma.$$

1) regarded in $(E-\mathfrak{e})\,e_r.H_m$ only

The integral $\int\limits_{E} |F| . D d\sigma$ is finite, the function D being less then 1. The last inequality (7) is an immediate consequence of th. XII [1]).

The said theorem enables us to state still:

$$(8) \qquad \int\int\limits_{\omega_r} f_{rm}(xy)\, dz\, dy = \int\limits_{(E-e).E_m e_r} F . D\, d\sigma. \text{ }^{1})$$

According to (6), (8) we can write:

$$(9) \qquad \int\int\limits_{\omega_r} \varphi(xy)\, dx\, dy = \int\int\limits_{\omega_r} f_{r2}\, dx\, dy - \int\int\limits_{\omega_r} f_{r1}\, dx\, dy + \ldots$$

$$+ \ldots \int\int\limits_{\omega_r} f_{rr}\, dx\, dy - \int\int\limits_{\omega_r} f_{rr-1}\, dx\, dy = \sum_{m=1}^{r} \int\limits_{(E-e).E_m.e_r} (-1)^r . D . F d\sigma = \int\limits_{(E-e)e_r} D . F d\sigma$$

summing up (9) upon r we obtain with help of (4), (5):

$$\int\int\limits_{(E-e)_\Theta} \varphi(x\, y)\, dx\, dy = \int\limits_{E-e} F . D\, d\sigma. \qquad \text{q. e. d.}$$

III.

§ 1. All results hitherto obtained hold still for Jansen's exterior measure, with the restriction only, that theorems dealing with relations between sets and their plane projections are not of general validity. They may be announced only with concern to planes, drawn through axes $x\,y\,z$ of coördinates used in order to construct Jansen's measure. If in particular the jansenian measure of a given set is independent of the system of coördinates chosen, relating theorems can be generalised for any plane.

It is an advantage of Jansen's measure to be finite provided that Φ is finite. This is a consequence of the following relation:

$$(1) \qquad J(E) \leqslant 3\, \Phi(E).$$

[1]) The following theorem (whose particular cases are relations 7 and 8) results from th. XII: M be a normal set of finite Φ_0 area projecting itself on Θ exactly once. Moreover F be a function summable Φ_0 defined on M. Then F regarded as a function f defined on M_Θ is summable l and we have:

$$\int\int\limits_{M_\Theta} f(xy)\, dx\, dy = \int\limits_{M} F . D\, d\sigma.$$

From (1) we may conclude: every set measurable Φ, its Φ-measure being finite, is also measurable J.

We shall now demonstrate a theorem referring to Jansen's measure, which will represent a generalisation of the classical formula $\iint \sqrt{EG-F}\,du\,dv$ for regular surfaces. Let:

1° *A be a set measurable -l, on the plane of variables uv; $V_\iota A$ finite. $\overline{A}$ its unifold and continuous image in the xyz space.*

Remark I: we shall denote in general by $\overline{M}$ the image of the set M belonging A, obtained by means of the above mentioned correspondence.

2° *the exterior jansenian measure $J(\overline{A})$ be finite.*

3° $J(\overline{M}) = \Psi(M) \qquad M \subset A.$

be a setfunction absolutely continuous in A [1]).

Before we pass over to infer anything from our suppositions, we define the function $\Delta(Pr)$.

Taking any point P on the plane uv as centre, we describe a circle $w(Pr)$ of the radius r (its centre P). With $\Delta(Pr)$ we shall denote the ratio:

$$\Delta(Pr) = \frac{\sqrt{V_\iota^2\,\overline{[A\,.\,w(Pr)]}_x + V_\iota^2\,\overline{[A\,.\,w(Pr)]}_y + V_\iota^2\,\overline{[A\,.\,w(Pr)]}_z}}{r^2\pi}$$

$\overline{[A\,.\,w(Pr)]}_x$ being the projection of $\overline{[A\,.\,w(Pr)]}$ on the plane $x=0$ etc.

We announce now:

Theorem XV. (α) *at almost every point $P(uv)$ the ratio:*

$$\Delta(P) = \lim \Delta(P_r) \qquad \text{exists.}$$

$$(\beta) \qquad \iint_M \Delta(P)\,du\,dv = J(\overline{M})$$

for every measurable -l subset M of A.

Proof. The setfunction $\Psi(M)$ being absolutely continuous and additive we have:

$$\iint_M \Delta_\Psi\,du\,dv = \Psi(M) = J(\overline{M})$$

for every M measurable l. Δ_Ψ is the „derivative" of the setfunction $\Psi(M)$.

[1]) absolutely continuous with respect to l-measure.

On the other side let us consider the setfunction:

$$\Omega(M) = \sqrt{V_i^2(\overline{M})_X + V_i^2(\overline{M})_Y + V_i^2(\overline{M})_Z}.$$

For any two sets M_1 and M_2 the following relation takes place:

(3) $$\Omega(M_1 + M_2) \leqslant \Omega(M_1) + \Omega(M_2)$$

$\Omega(M)$ is also an absolutely continuous set function since we always have:

(4) $$\Omega(M) \leqslant \Psi(M) = J(\overline{M}).$$

Denoting by Δ_Ω^+ resp. Δ_Ω^- the upper resp. lower „derivative“ of $\Omega(M)$ we conclude using (3) and the absolute continuity of $\Omega(M)$:

(5) $$\Omega(M) \leqslant \iint_M \Delta_\Omega^- \, du\, dv \leqslant \iint_M \Delta_\Omega^+ \, du\, dv$$ [1]

M measurable -l.

It ensues from (4):

(6) $$\Delta_\Omega^+(P) \leqslant \Delta_\Psi(P) \qquad \text{almost everywhere in } A.$$

(5) and (6) give:

(7) $$\Omega(M) \leqslant \iint_M \Delta_\Omega^- \, du\, dv \leqslant \iint_M \Delta_\Omega^+ \, du\, dv \leqslant \iint_M \Delta_\Psi \, du\, dv = J(\overline{M}).$$

We construct now in the space xyz a grating dense enough, in order that:

(8) $$\sum_{q=1}^{\infty} \sqrt{V_i^2(\overline{M}_q)_X + V_i^2(\overline{M}_q)_Y + V_i^2(\overline{M}_q)_Z} \geqslant J(\overline{M}) - \varepsilon$$

$\overline{M}_q$ designing as usually the set of points common to $\overline{M}$ and the q-th cell of our grating.

Let us remark, that the sets M_q — corresponding to $\overline{M}_q$ on the plane uv — are measurable l, M being measurable l. Moreover $\{M_q\}_{q=1,2,..\infty}$ have no mutually common points:

$$M_q \cdot M_{q'} = 0 \qquad q \neq q'$$

(9) $$\sum_{q=1}^{\infty} M_q = M.$$

[1] For the proof of rel. 5 compare R. F. § 437, § 438.

Applying (7), (8) to the sets $\{M_q\}_{q=1,2\ldots\infty}$ we have thus (acc. to 9):

$$J(\overline{M})-\varepsilon \leqslant \sum_{q=1}^{\infty} \Omega(M_q) \leqslant \sum_{q=1}^{\infty} \iint_{M_q} \Delta_{\overline{\Omega}}\, du\, dv \leqslant \sum_{q=1}^{\infty} \iint_{M_q} \Delta_{\Omega}^{+}\, du\, dv =$$

$$= \sum_{q=1}^{\infty} J(\overline{M}_q) = J(\overline{M})$$

and when ε tends towards zero we obtain:

$$J(\overline{M}) = \iint_{M} \Delta_{\overline{\Omega}}\, du\, dv = \iint_{M} \Delta_{\Omega}^{+}\, du\, dv$$

so that:

$$\Delta_{\overline{\Omega}} = \Delta_{\Omega}^{+} = \Delta$$

almost everywhere in A.

§ 2. We suppose now A to be a bounded domain and $\overline{A}$ a surface defined by means of 3 functions:

$$x = \varphi(u\,v) \qquad y = \Psi(u\,v) \qquad z = \Xi(u\,v) \tag{1}$$

each of them being à condition de Lipschitz, that means, that to every point $(u_0\, v_0)$ in the domain A exists a number $r(u_0\, v_0)$ such that:

$$\left| \frac{\varphi(u_0+h,\ v_0+k)-\varphi(u_0\, v_0)}{\sqrt{h_2+k_2}} \right| \leqslant \mu; \qquad \sqrt{h_2+k_2} \leqslant r(u_0 v_0) \tag{2}$$

and similarly with respect to the remaining functions.

Surfaces $\overline{A}$ of this kind have been investigated by Rademacher.

We assume also the correspondence $\overline{A}$ to A given by 1) to be one to one.

Let us now apply our results obtained beforehand.

Theorem XVI. *The surface $\overline{A}$ has a finite jansenian measure.*

Proof. If M be any measurable $-l$ part of A and $\overline{M}$ its image on the surface $\overline{A}$, we have:

$$V_\iota(\overline{M})_X \leqslant 4\mu \,.\, V_\iota M; \quad V_\iota(\overline{M})_Y \leqslant 4\mu \,.\, V_\iota M; \quad V_\iota(\overline{M})_Z \leqslant 4\mu \,.\, V_\iota M. \tag{3}$$

Dividing the space xyz into cells O_ν we obtain:

$$\overline{A} = \sum_{\nu=1}^{\infty} \overline{A}_\nu \text{ [1]}.$$

$$A_\nu \,.\, A_{\nu'} = 0 \qquad\qquad \nu \neq \nu' \tag{4}$$

[1] $A_\nu = A.\, O_\nu$.

According to (3):

$$\sqrt{V_\iota^2(\overline{A}_\nu)_x + V_\iota^2(\overline{A}_\nu)_y + V_\iota^2(\overline{A}_\nu)_z} \leqslant 12\mu \,.\, V_\iota A_\nu. \tag{5}$$

Summing (5) according to ν we obtain regarding (4):

$$\sum_{\nu=1}^{\infty} \sqrt{V_\iota^2(\overline{A}_\nu)_x + \ldots + \ldots} \leqslant 12\mu \,.\, V_\iota A$$

and finally passing to the limit:

$$J(\overline{A}) \leqslant 12\mu \,.\, V_\iota A.$$

Remark: In the same manner may be shown, that for every measurable -l subset M of A the following inequality subsists:

$$J(\overline{M}) \leqslant 12\mu \,.\, V_\iota M. \tag{6}$$

By means of (1), (6) we convince ourselves, that the function $\Psi(M) = J(\overline{M})$ introduced in the preceding theorem is absolutely continuous and $J(\overline{A})$ finite. Hence all conditions of said theorem are satisfied:

We conclude:

$$J(\overline{M}) = \iint_M \Delta(uv)\, du\, dv \qquad M \subset A.$$

According to Rademacher it is possible to calculate the function $\Delta(uv)$. Let us introduce the following notations:

$$I_1 = \begin{vmatrix} \frac{\partial \varphi}{\partial u}, & \frac{\partial \Psi}{\partial u} \\ \frac{\partial \varphi}{\partial v}, & \frac{\partial \Psi}{\partial v} \end{vmatrix} \qquad I_2 = \begin{vmatrix} \frac{\partial \Psi}{\partial u}, & \frac{\partial \Xi}{\partial u} \\ \frac{\partial \Psi}{\partial v}, & \frac{\partial \Xi}{\partial v} \end{vmatrix} \qquad I_3 = \begin{vmatrix} \frac{\partial \Xi}{\partial u}, & \frac{\partial \varphi}{\partial u} \\ \frac{\partial \Xi}{\partial v}, & \frac{\partial \varphi}{\partial v} \end{vmatrix} \tag{7}$$

It has been shown by Rademacher [1]), that these 3 jacobians exist simultaneously at almost every point of A and their value gives the „ratio of enlargement" of the set A with respect to the plane considered. For instance $|I_1|$ is the „ratio of enlargement" with respect to the plane xy:

[1]) Rademacher: Ueber totale Differenzierbarkeit. Part I, 1919. Part II, 1920.

$$\lim_{r\to 0}\frac{V_\iota\,\overline{(w(Pr)\,.\,A)}_x}{r^2\pi}=|I_1|$$

at almost every point P belonging to A [1]).

Thence from and from (7) results, that almost everywhere in A:

$$\Delta(u\,v)=\sqrt{I_1^2+I_2^2+I_3^2}$$

$$J(\overline{M})=\iint\limits_{M}\sqrt{I_1^2+I_2^2+I_3^2}\,du\,dv$$

$\iint\limits_{M}\sqrt{I_1^2+I_2^2+I_3^2}\,du\,dv$ *being independent of the choice of axes* xyz *the same property subsists for* $J(\overline{M})$. *We conclude that:*

(7₁) $$\Phi_0(\overline{M})\leqslant J(\overline{M})\ ^{2)}.$$

We will now demonstrate:

Theorem XVII. $\overline{A}$ *is normal* (with respect to J measure and consequently also with respect to Φ_0).

The proof of this theorem we shall accomplish in a few steps. I step. *Let* M *be a set measurable* l, *belonging to* A. *Suppose:*

$$|I_1|>0$$

throughout M.

Denoting by M' a set, whose l-measure is zero and by $K(Pr)$ a sphere with its centre in $\overline{P}$, its radius being r, we can announce:

To every point $\overline{P}$ *of* $\overline{M-M'}$ *a number* $\varrho(P)$ *can be assiociated in such a way, that:*

$$\frac{V_\iota\,[K(\overline{Pr})\,.\,\overline{M}]_z}{r^2\pi}>\frac{|I_1|}{18\mu^2}\ ^{3)}$$

provided that $r\leqslant\varrho(P)$.

Indeed save a set M', whose $V_\iota M'=0$ the following relation subsists:

$$\lim_{r_1\to 0}\frac{V_\iota\,\overline{[w(Pr_1)\,.\,M]}_z}{r_1^2\pi}=|I_1|;\qquad P\,\varepsilon\,M-M'$$

[1]) The ratio of enlargement $|I_1|$ of A with respect to the plane xy i different from the ratio of enlargement D_z of $\overline{A}$ on the plane xy. For the definition of $|I_1|$ we use circles, for that of D_z spheres. Vide th XVIII.

[2]) I shall prove in a additional note, that $J(\overline{A})=\Phi_0(\overline{A})$.

[3]) The number μ occuring in rel (2).

Hence we conclude:

(8) $$\frac{V_\iota[w(Pr_1).M]_z}{r_1^2\pi} > \frac{|I_1|}{2} > 0; \qquad P \varepsilon M - M'$$

for $r_1 \leqslant \varrho_1(P)$.

Be now $\overline{P}$ a point of $\overline{M - M'}$. Consider any sphere $K(\overline{Pr})$, its radius $r \leqslant 3\mu.\varrho_1(P)$.

The image of the circle $w\left(P, \frac{r}{3\mu}\right)$ is situated entirely within the sphere $K(\overline{Pr})$.

Consequently:

(9) $$\frac{V_\iota[\overline{M}.K(\overline{P}r)]_z}{r^2\pi} \geqslant \frac{V_\iota\left[w\overline{\left(P.\frac{r}{3\mu}\right).M}\right]_z}{9\mu^2\left(\frac{r}{3\mu}\right)^2\pi}.$$

We may write, since $\frac{r}{3\mu} \leqslant \varrho_1(P)$ according to (8), (9)

$$\frac{V_\iota[\overline{M}.K(\overline{P}r)]_z}{r^2\pi} \geqslant \frac{|I_1|}{18\mu^2}$$

Thus $\varrho(P) = 3\mu.\varrho_1(P)$ satisfies the desired conditions.

II step. *Let R be a family of spheres $\{K(\overline{P}r)\}$ covering the surface $\overline{A}$ and M a measurable l subset of A, whereat:*

(10) $$|I_1| > \beta; \qquad |I_2| > \beta; \qquad |I_3| > \beta$$

β being > 0.

Besides that suppose that $\overline{M}$ belongs entirely to a given domain G.

We can then extract from R a finite number of spheres $\{K_i\}_{i=1,2,\dots\infty}$ in such a manner, that:

$$K_i.K_{i'} = 0; \qquad \text{if } i \neq i'; \qquad i = 1, 2, \dots n$$

$$K_i \subset G.$$

$$\frac{\beta}{20\mu^2}\sqrt{V_\iota^2(\overline{M})_x + V_\iota^2(\overline{M})_y + V_\iota^2(\overline{M})_z} \leqslant$$

$$\leqslant \sum_{i=1}^{n}\sqrt{V_\iota^2[K_i.(\overline{M})]_x + V_\iota^2[K_i.(\overline{M})]_y + V_\iota^2[K_i.\overline{M}]_z}$$

Proof of II step We remove from the family R:

1^0 all spheres, which do not entirely belong to G.

2^0 those spheres $K(\overline{P}r)$, where $r \geqslant \varrho(P)$.

3^0 spheres, whose centres $\overline{P}$ belong to $\overline{M}'$ [1]).

Thus we obtain a family (of spheres) R_G.

Let us construct a family „f" of circles covering $(\overline{M})_x$ by projecting the family R_G. According to a Vitali's theorem we can find in f an at most enumerable number of circles $\{(K_i)_x\}_{i=1,2\dots\infty}$, so that:

$$(K_i)_x . (K_{i'})_x = 0 \qquad i \neq i'; \qquad i, i' = 1, 2, \dots \infty$$

$$\sum_{i=1}^{\infty} V_t (K_i)_x \sum_{i=1}^{\infty} V_t [(K_i)_x . (\overline{M})_x] = V_t \, \overline{M}_x .$$

Hence in virtue of 1^{st} step and (10):

$$\sum_{i=1}^{\infty} V_t \, [K_i . \overline{M}]_x \geqslant \frac{\beta}{18\mu^2} . V_t (\overline{M})_x .$$

But since the circles $(K_i)_x$ are mutually separated, the spheres $\{K_i\}$ are separated. For a n_1 properly chosen we have:

$$K_i . K_i' = 0; \qquad i \neq i'; \qquad i, i' = 1, 2 \dots n_1$$

$$\sum_{i=1}^{n_1} V_t [K_i . \overline{M}]_x \geqslant \frac{\beta}{20\mu^2} V_t (\overline{M})_x . \tag{11}$$

In the same manner we convince onrselves, that we are able to extract from R a finite number of spheres $\{K_i\}_{i=n_1+1, n_1+2 \dots n_2-1, n}$ belonging to G, having no common points neither mutually, nor with the group $\{K_i\}_{i=1,2,\dots n_1}$ so that:

$$K_i . K_{i'} = 0; \qquad i \neq i'; \qquad i, i' = 1, 2 \dots n_2$$

$$\sum_{i=1}^{n_2} V_t \, [K_i . \overline{M}]_Y \geqslant \frac{\beta}{20\mu^2} . V_t (\overline{M})_Y . \tag{12}$$

At last we add a third group $\{K_i\}_{i=n_2+1, \dots n_3-1, n_3}$ situated entirely within G, so that:

$$K_i . K_{i'} = 0; \qquad i \neq i'; \qquad i, i' = 1, 2 \dots n_3$$

$$\sum_{i=1}^{n_3} V_t [K_i \overline{M}]_z \geqslant \frac{\beta}{20\mu^2} . V_t (\overline{M})_z . \tag{13}$$

[1]) M' was defined in step I.

Summarising (11), (12), (13) we have a fortiori:

$$\left.\begin{aligned}\sum_{i=1}^{n_3} V_i[K_i.\overline{M}]_X &\geqslant \frac{\beta}{20\mu^2}\cdot V_i(\overline{M})_X\\ \sum_{i=1}^{n_3} V_i[K_i.\overline{M}]_Y &\geqslant \frac{\beta}{20\mu^2}\cdot V_i(\overline{M})_Y\\ \sum_{i=1}^{n_3} V_i[K_i.\overline{M}]_Z &\geqslant \frac{\beta}{20\mu^2}\cdot V_i(\overline{M})_Z\end{aligned}\right\}\begin{aligned}&K_i.K_{i'}=0.\\ &i,i'=1,2\dots n_3\end{aligned}$$

Consequently:

$$\frac{\beta}{20\mu^2}\sqrt{V_i^2(\overline{M})_X+V_i^2(\overline{M})_Y+V_i^2(\overline{M})_Z}\geqslant$$

$$\leqslant\sum_{i=1}^{n_3}\sqrt{V_i^2[K_i.\overline{M}]_X+V_i^2[K_i.\overline{M}]_Y+V_i^2[K_i.\overline{M}]_Z}$$

q. e. d.

III step. The conditions remain unchanged (as in II$^{\text{nd}}$ step).

We can then extract from the given family R a finite number of spheres in such a way that:

$$K_i.K_{i'}=0;\qquad i,i'=1,2\dots n;\qquad i\neq i'$$

$$\sum_{i=1}^{n} J(\overline{M}.K_i)\geqslant\frac{\beta}{40\mu^2}J(\overline{M}).$$

Proof of III$^{\text{rd}}$ step. We shall denote by $\overline{A}_{x=c}$ the set of points common to $\overline{A}$ and the plane $x=c$ and similarly $\overline{A}_{y=c}$, $\overline{A}_{z=c}$.

We make the following simple remark: for an at most enumerable number of planes $x=c$, $y=c$, $z=c$ resp we have:

$$J(\overline{A}_{x=c})>0;\qquad J(\overline{A}_{y=c})>0;\qquad J(\overline{A}_{z=c})>0\ \text{ resp.}$$

In fact let we regard for instance those planes parallel to the $z=0$ plane where

$$J(\overline{A}_{z=c})>m>0.$$

The number — N_m — of such planes is finite, namely:

$$N_m\leqslant\frac{J(\overline{A})}{m}.$$

Making use of this property we construct a space grating dense enough, in order that:

$$(14)\qquad \sum_{q=1}^{\infty}\sqrt{V_\iota^2(\bar{M}_q)_x + V_\iota^2(\bar{M}_q)_y + V_\iota^2(\bar{M}_q)_z} > \frac{1}{2}J(\bar{M})$$

and that the walls of the cells 0_γ contain only subsets of $\bar{A}$ of J-measure zero. (That is possible according to what has been just shown).

Denoting by m_q the subset of $\bar{M}_q$ which is situated in the interior of the q-th cell, we obtain instead of (14):

$$(14_1)\qquad \sum_{q=1}^{\infty}\sqrt{V_\iota^2(\bar{m}_q)_x + V_\iota^2(\bar{m}_q)_y + V_\iota^2(\bar{m}_q)_z} > \frac{1}{2}J(\bar{M}).$$

We extract now from the given family R [1]) — for q constant — a finite number of spheres $\{K_i^q\}_{i=1,2\ldots n_q}$ in such a way that they lie entirely within the interior of q-th cell and that:

$$K_i^q \,.\, K_{i'}^q = 0 \qquad i \neq i'; \qquad i, i' = 1, 2\ldots n_q$$

$$(15)\qquad \sum_{i=1}^{n_q}\sqrt{V_\iota^2(\bar{M}\,.\,K_i^q)_x + V_\iota^2(\bar{M}\,.\,K_i^q)_y + V_\iota^2(\bar{M}\,.\,K_i^q)_z} =$$

$$= \sum_{i=1}^{n_q}\sqrt{V_\iota^2(\bar{m}_q . K_i^q)_x + V_\iota^2(\bar{m}_q . K_i^q)_y + V_\iota^2(\bar{m}_q . K_i^q)_z} \geqslant$$

$$\geqslant \frac{\beta}{20\mu^2}\sqrt{V_\iota^2(\bar{m}_q)_x + V_\iota^2(\bar{m}_q)_y + V_\iota^2(\bar{m}_q)_z}$$

Summing (15) according to q and taking $(14)_1$ into account we obtain:

$$\sum_{q=1}^{\infty}\sum_{i=1}^{n_q} J(\bar{M}\,.\,K_i^q) \geqslant \sum_{q=1}^{\infty}\sum_{i=1}^{n_q}\sqrt{V_\iota^2(\bar{M}.K_i^q)_x + V_\iota^2(\bar{M}.K_i^q)_y + V_\iota^2(\bar{M}.K_i^q)_z}$$

$$\geqslant \frac{\beta}{40\mu^2} J(\bar{M}).$$

The groups $\{K_i^q\}_{i=1,2\ldots n_q}$ and $\{K_i^{q'}\}_{i=1,2\ldots n'_q}$ are without common points for $q \neq q'$, as they are contained by the interior of different cells.

The sum of spheres $\{K_i^q\}_{\substack{q=1,2\ldots\infty\\ i=1,2\ldots n_q}}$ fulfils III step.

What we have proved in the 2nd and 3rd step shows according to theorem VII remark 1, that every set M satisfying (10) is normal.

[1]) according with 2nd step.

Also following sets are in a similar way shown normal:

The set of points where:	$\lvert I_1 \rvert > \beta > 0;$	$\lvert I_2' \rvert > \beta;$	$\lvert I_3 \rvert = 0$
„ „	$\lvert I_1 \rvert > \beta;$	$\lvert I_2 \rvert = 0;$	$\lvert I_3 \rvert > \beta$
„ „	$\lvert I_1 \rvert = 0;$	$\lvert I_2 \rvert > \beta;$	$\lvert I_3 \rvert > \beta$
„ „	$\lvert I_1 \rvert > \beta;$	$\lvert I_2 \rvert = 0;$	$\lvert I_3 \rvert = 0$
„ „	$\lvert I_1 \rvert = 0;$	$\lvert I_2 \rvert > \beta;$	$\lvert I_3 \rvert = 0$
„ „	$\lvert I_1 \rvert = 0;$	$\lvert I_2 \rvert = 0:$	$\lvert I_3 \rvert > \beta$

And since the surface $\bar{A}$ [1]) is sum but of such sets complying with one of the conditions either (10) or (10′) it is, too, itself normal.

The surface $\bar{A}$ being normal we can apply th. XIII and we conclude, that the ratio $D_z(P)$ [2]) exists in $\bar{A}$ excepting a set of J measure zero.

Theorem XVIII.

$$D_z(P) = \frac{|I_1|}{\sqrt{I_1^2 + I_2^2 + I_3^2}}$$

almost everywhere in A.

Proof. At first we have to prove that D_z defined originally on $\bar{A}$ is also measurable -l as a function of uv defined in A.

This is an immediate consequence of the fashion in which D_z has been obtained — that is to say by passing to limit with functions upper-semicontinuous in $\bar{A}$. Yet, functions semicontinuous in $\bar{A}$ are functions of the same kind in A.

We denote B_1 resp B_2 the set of points belonging to A, wherein:

(11)$_1$
$$0 \leqslant D_z . \sqrt{I_1^2 + I_2^2 + I_3^2} < |I_1|$$
resp.

(11)$_2$
$$0 \leqslant |I_1| < D_z \sqrt{I_1^2 + I_2^2 + I_3^2}$$

The sets B_1, B_2 have the lebesguian l-measure zero. We shall demonstrate this in an indirect way.

Let us suppose the set B_1 to have a positive l-measure. $\bar{B}_1$ is measurable J and of positive J measure, because:

$$J(\bar{B}_1) = \iint_{B_1} \sqrt{I_1^2 + I_2^2 + I_3^2}\, du\, dv \geqslant \iint_{B_1} |I_1|\, du\, dv.$$

[1]) Save a set of J measure zero.

[2]) $D_z(P)$ is the ratio of enlargement at the point P of the surface $\bar{A}$ on the plane $z = 0$. The measure used in order to define D_z is the jansenian measure.

The integral $\iint_{B_1} |I_1|\, du\, dv$ is taken in the set B_1, which is of positive l-measure in virtue of our supposition and $|I_1| > 0$ in B_1 in virtue of $(11)_1$.

Hence:

$$J(\overline{B}_1) > 0$$

The projection $(\overline{B_1})_Z$ on the plane Z can be considered as the image of the set B_1 obtained by means of functions:

$$x = \varphi(uv) \qquad y = \Psi(uv)$$

$|I_1|$ is the „ratio of enlargement" referring to this correspondence. That is, we have almost everywhere in B_1:

$$\lim_{r\to 0} \frac{V_\iota \overline{[B_1 . w(Pr)]_Z}}{V_\iota [B_1 . w(Pr)]} = |I_1(P)|$$

at almost every point P belonging to B_1.

And since $|I_1|$ is positive in the set B_1 and as B_1 is of positive l-measure, the image $(\overline{B_1})_Z$ ought to posses a positive l-measure. In virtue of theor. IV a set $\overline{C_1}$ can be found belonging entirely to $\overline{B_1}$, projecting itself on Z exactly o n c e and such that:

$$V_\iota(\overline{B}_1)_Z = V_\iota(\overline{C}_1)_Z > 0. \qquad C_1 \subset B_1$$

Consequently $J(\overline{C_1})$ being not less then $V_\iota(\overline{C_1})_Z$ is too positive. Moreover, also the l-measure of the set $C_1 \subset B_1$ is positive. Indeed, if it were zero, we had also:

$$J(\overline{C_1}) = \iint_{C_1} \sqrt{I_1^2 + I_2^2 + I_3^2}\, du\, dv = 0.$$

Taking into account, that the correspondence $C_1 \longleftrightarrow \overline{(C_1)}_Z$ is one to one we obtain:

$$V_\iota \overline{(C_1)}_Z = \iint_{C_1} |I_1|\, du\, dv.$$

On the other side $\overline{C_1}$ being normal we conclude according to th XII:

$$\int_{\overline{C}_1} D_Z\, d\sigma = V_\iota \overline{(C_1)}_Z.$$

But we can turn the surface integral $\iint\limits_{\overline{C}_1} D_z d\sigma$ into a lebesguian integral:

$$\int\limits_{\overline{C}_1} D_z d\sigma = \iint\limits_{C_1} D_z \sqrt{I_1^2 + I_2^2 + I_3^2}\, du\, dv.$$

Finally we obtain:

$$\begin{cases} \iint\limits_{C_1} D_z \sqrt{I_1^2 + I_2^2 + I_3^2} = \iint\limits_{C_1} |I_1|\, du\, dv; \quad C_1 \subset B_1 \\ V_\iota(C_1) > 0; \quad |I_1| > 0. \end{cases}$$

But this is impossible, being in contradiction with $(11)_1$.

B_2 is in similar mode shown to be of l measure zero.

§ 3. With reference to the surface $\overline{A}$ take the conditions of th XIV a very much simplified shaps. We prove:

Lemma. *Given in the space a closed jordanian surface S.*

We will show, that in every point P of this surface, whereat the tangent plane exists, exists also the inward direction of the normal, that is to say:

Two points A and B can be found on the normal such, that the segment $< AB >$ contains P in its interior and that one of the segments $< AP)$, $(PB >$ lies within, the other without S.

Proof. At first, we will give the definition of tangent plane. We shall say, the tangent plane Θ exists at the point P, if Θ satisfies three following conditions α, β, γ:

(α) Θ passes through the point P.

(β) $\{P_n\}_{n=1,2\ldots\infty}$ be any sequence of points belonging to S and tending towards P. Then the angle between PP_n and the perpendicular to Θ tends to $\pm \frac{D}{2}$.

In order to explain (γ) more easily we employ the following notations; Consider any closed neighbourhood of P, (the neighbourhood taken only with respect to the surface S) small enough, that it can be represented as an image $\overline{K}$ of a circle K, lying in the plane uv — the correspondence K to $\overline{K}$ being one to one.

Let us now project $\overline{K}$ on the plane Θ. This projections determines a continuous correspondence „Ψ" between K and the set $(\overline{K})_\Theta$.

By the condition (β) we are ascertained that, if we choose $\overline{K}$ sufficiently small, the relation:

$$\Psi(p) = P \qquad\qquad p \varepsilon K$$

holds for exactly one point p.

Any closed curve c lying within K and containing p in its interior transforms itself into a curve $\Psi(c)$, which does not pass through P. The „order" of P with respect to $\Phi(c)$ is thus uniquely defined. It may easily be shown, that this order is independent of the choice of c (c containing h in its interior).

We shall call the common value of this order the „index" at the point P.

(γ) declares: the index at the point P ought to be an odd number.

We pass now to the proof of the announced lemma.

At first we observe, that according to (β) we can find on the normal through P at all events a segment AB, containg P in its interior so, that no other point of $< AB >$ belongs to S. It remains to show, that o n e of the points A, B lies within S, the o t h e r without S. Let us suppose, that it be not so: that, for instance both A and B lie inside of S. In this case, they can be joined together by a broken line L, which does not cut S. $L + < AB >$ from a closed polygon W having with S but o n e common point: P.

Let us denote by c the circumference of the circle K. The order of P with respect to $\Psi(c)$ is an odd number (cond γ). The set $(S - \overline{K}) + \overline{c}$ has a positive distance from the polygon W. We can therefore construct a polyhedron Π', whose vertices lie on $(S - \overline{K}) + \overline{c}$ in such a way, that it has still a positive distance from W.

We mark out those vertices of Π', which belong to $\overline{c}$. Let us arrange them in a cyclic mode: $P_1, P_2, \ldots P_n, P_1$. We complete now the o p e n polyhedron Π' to a c l o s e d one Π approximating the surface S by adding the triangles PP_1P_2, $PP_2P_3 \ldots PP_{n-1}P_n$, PP_nP_1. Projecting the broken line $P_1P_2, \ldots P_nP_1$ on Θ we obtain the c l o s e d polygon $\Psi(P_1)$ $\Psi(P_2)$ $\Psi(P_n)$ $\Psi(P_1)$ approximating the curve $\Psi(c)$. The order of P with respect to $\Psi(P_1)$ $\Psi P_2) \ldots$ $\Psi(P_n)$ $\Psi(P_1)$ is equal to the order of P with respect to $\Psi(c)$, if we choose the triangular faces of Π small enough. Then for e v e r y point Q lying on Θ s u f f i c i e n t l y near to P the following property subsists:

Q is covered by an odd number of triangles: $P\,\Psi(P_i)\,\Psi(P_{i+1})$ [1].

We deplace now the polygon W — by means of the translation parallel to the vector PQ — so that P arrives Q. We can choose the translation in such a manner, that W' have in its new position still no common points with the polyhedron Π'. It will suffice to take the vector PQ in its absolute value less then the distance of W from Π'.

W „penetrates" in its new position the closed polyhedron Π, that means:

(α) no vertex of the polygon W belongs to Π.

(β) no edge of the polyhedron Π_1 has common points with W.

We observe now, that if the triangle $P\,\Psi(P_i)\,\Psi(P_{i+1})$, covers the point Q, corresponding triangle $P\,P_i\,P_{i+1}$ is „penetrated by W and inversely, so that the closed polygon W penetrates the closed polyhedron Π an odd number of times. But this is impossible in virtue of a well known theorem of Brouwer [2]).

We have yet to investigate the manner, in which our definition of the tangent plane is related to the classical one. In this concern the following theorem can easily be demonstrated:

Suppose the neighbourhood of the point P be represented by means of 3 functions:

$$x = \varphi(uv);\quad y = \Psi(uv);\quad z = \Xi(uv);\quad \sqrt{(u-u_0)^2+(v-v_0)^2} < r(P)$$

the values $u_0\,v_0$ corresponding to P.

Then the tangent place at P exists, if all three functions φ, Ψ, X posses at $u_0\,v_0$ total differentials and if the rank of the matrix:

$$\left\|\begin{matrix} \frac{\partial\varphi}{\partial u}, & \frac{\partial\Psi}{\partial u}, & \frac{\partial\Xi}{\partial u} \\ \frac{\partial\Psi}{\partial v}, & \frac{\partial\Psi}{\partial v}, & \frac{\partial\Xi}{\partial v} \end{matrix}\right\| \text{ is } 2.$$

Therefore if around any point P of S the $\varphi(uv)$, $\Psi(uv)$, $\Xi(uv)$ are à condition de Lipschitz, then excepting a set of surface measure 0 exists on S an inward direction of the normal.

[1]) we suppose also, that Q is situated on neither segment $P\,\Psi(P_1)$, $P\,\Psi(P_2)$... $P\,\Psi(P_n)$.

[2]) Brouwer: Beweis des n-dimensionalen Jordanschen Satzes Math. Annalen t. 71, (1912).

Theorem XIX. *Given a closed jordanian surface S of such a kind, that around any point P the representing functions φ, Ψ, Ξ are im „kleinen“ à condition de Lipschitz. Then denoting by G the interior of S we have:*

$$\iiint_G \frac{\partial F}{\partial z} dx\, dy\, dz = \int_S \cos(n z) F d\sigma$$

F being a function complying with conditions (4), (5), (6) of th. XIV, and n the direction of the inward normal.

Proof. In order to prove this theorem, we have to demonstrate that:

1^0 the condition 3^0 of th. XIV is satisfied.

2^0 $\cos(n z) = \dot{D}$ almost every where.

In fact, consider any point on S, whereat the tangent plane exists and $|I_1| \neq 0$. This means, that the tangent plane through P is not parallel to z-axis, therefore the normal not parallel to the plane $z = 0$.

Let us suppose, that is possible to find a sequent APB with its centre in P, parallel to the z-axis and lying — except the point P — entirely within the domain G.

This supposition will lead us to a contradiction.

Let for instance $< AP)$ lie on the same side of the tangent plane, as the outward normal and $< BP)$ on that of the inward. We choose on the outward normal a point A' such, that the segment $< A' P)$ may belong entirely to the exterior of S. Let us now take on $(PA>$ points $\{A_n\}_{n=1,2\ldots\infty}$ tending towards P and on $(PA'>$ points $\{A_n'\}_{n=1,2\ldots\infty}$ tending also towards P. Since (according to our supposition) A_n belongs to G and since A_n' lies in the exterior of S, the segment $A_n A_n'$ ought to have a point A_n'' common with S. The points $\{A_n''\}_{n=1,2\ldots\infty}$ have P as limit point. All points $\{A_n''\}_{n=1,2\ldots\infty}$ lie in the interior of acute angle APA'. In the interior of the acute angle APA' exists thus an infinite number of points belonging to S and tending towards P.

But the impossibility of this last property is an immediate consequence of the properties of the tangent plane.

In a similar way can be shown, that $< AP) + (PB>$ does not entirely belong to the exterior of S.

Hence the points P of S, whereat the segment $< AB>$ lies within G (on the outside of S) belong to the set of those points where $|I_1| = 0$. But this last set has on the plane $z = 0$ a projection of l-measure zero. The condition 3^0 of th. XIV is thus demonstrated.

We have yet to show that almost everywhere in S:

$$\dot{D} = \cos(nz).$$

From th. XVIII results at all events:

$$|\dot{D}| = |D| = |\cos(nz)| \qquad \text{almost everywhere.}$$

It remains only to prove, that the equation holds still with sign. This it will suffice to prove in those points only, where $D \gtrless 0$. If e. g. $D(P) > 0$, then $\dot{D} = D$ and the point P is the upper end of a segment QP parallel to z-axis and belonging entirely to G. Then $\cos(nz)$ must be also positive in P. This is easily seen, the angle between $\vec{PQ}$ and the inward normal being acute.