where h is canonical isomorphism and $g(s^{-1}r\otimes a) = r\otimes a$. Hence $\ker f_s = \ker(hgf_s) = \{a \in A \mid sa = 0\} \subseteq T(A)$, and evidently, $\bigcup_{s\in S} \ker f_s = T(A)$. Thus

$$T(A) = \varinjlim(\ker f_s) \cong \ker f \cong \mathrm{Tor}_1(K, A)\,.$$

3.4. Proposition. *For every module A there exists an S-module.*

Proof. Write $A_s = R'\otimes A$ and consider the map $\varphi\colon A\to A_s$ ($\varphi(a) = 1\otimes a$, $a \in A$). Since A_s is S-divisible, therefore $A' = A_s/T(A_s)$ is S-torsion free and S-divisible.

If ψ_A is the combined map $A\xrightarrow{\varphi}A_s\xrightarrow{g}A'$, where g is the natural homomorphism, then $\psi_A(a) = 0$ implies that $\varphi(a) \in T(A_s)$, i.e., $\sigma\varphi(a) = 0$ for some $\sigma \in S$. Hence by 3.3 $\sigma a \in T(A)$, and it is clear that $\ker\psi_A = T(A)$.

Now every element $x \in A'$ can be written as $g(s^{-1}\otimes a)$, so that $sx = \psi_A(a)$. This completes the proof.

It follows that, *if A is S-torsion module, then its S-module is zero.*

3.5. Proposition. *If $0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$ is an exact sequence of modules, then the sequence*

$$0\to A'\to B'\to C'\to 0$$

is exact, where (ψ_A, A'), (ψ_B, B'), and (ψ_C, C') are S-modules of A, B, and C respectively.

Proof. The exactness of the given sequence implies the exactness of

$$0\to A_s\xrightarrow{f^*}B_s\xrightarrow{g^*}C_s\to 0 \qquad (f^* = 1_{R'}\otimes f, \text{ and } g^* \text{ similarly}).$$

Since A_s is S-divisible, therefore by 2.1 we obtain the exact sequence $0\to A'\xrightarrow{\overline{f^*}}B'\xrightarrow{\overline{g^*}}C'\to 0$.

In view of 3.5 and the remark at the end of 3.4 it follows that $A' \cong \big(A/T(A)\big)'$ for any module A. Furthermore, it is evident that A' is a covariant exact functor of A.

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A note on the Levitzki radical of a semiring

by

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Abstract. In this paper the authors prove that the Levitzki radical of an arbitrary semiring S is necessarily a k-ideal of S. A preliminary lemma states that if I is a locally nilpotent ideal of a semiring S then the closure of I is also a locally nilpotent ideal of S. These results strengthen certain of those obtained by E. Barbut [1].

1. A set S with two binary operations $+$ and $\cdot$ is called a *semiring* if $(R, +)$ is a commutative semigroup with zero, $(R, \cdot)$ is a semigroup, and both the left and right distributive laws hold for multiplication over addition. It is also required that $0\cdot x = x\cdot 0 = 0$ for all $x \in S$. A non-empty subsemiring I is called a *right ideal of* S if for all $x \in I$ and $r \in S$, $xr \in I$. Left ideals and (two-sided) ideals are defined in a similar manner. An ideal I of S is called a *k-ideal of* S [5] if $x+y \in I$ and $y \in I$ implies $x \in I$ for each $x, y \in S$.

E. Barbut [1] defined the Levitzki radical of a semiring and could prove many results concerning this radical providing the Levitzki radical is a k-ideal. In this note we prove that this radical is necessarily a k-ideal which strengthens many of Barbut's results.

2. If I is an ideal of the semiring S, the quotient semiring S/I is the one defined by S. Bourne [2] where for $a, b \in S$,

$$a \equiv b(\mathrm{mod}\, I) \text{ iff there exists } i_1, i_2 \in I \text{ such that } a+i_1 = b+i_2\,.$$

Definition 1. A semiring S is called *locally nilpotent* if every finite subset F of S generates a nilpotent subsemiring of S, or equivalently, if for each finite subset F of S there exists a positive integer N_F such that every product of N_F elements from F is zero.

Definition 2. [1] The *Levitzki radical* $L(S)$ of a semiring S is the sum of all locally nilpotent ideals of S.

E. Barbut [1] has shown that $L(S)$ is a locally nilpotent ideal of S which contains every locally nilpotent right or left ideal of S.

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DEFINITION 3. [3] If I is an ideal of a semiring S then the *closure of* I, I^*, is

$$I^* = \{x \in S: x+i \in I \text{ for some } i \in I\}.$$

It is known that if I is an ideal of S then I^* is an ideal of S. One can also see from the definition that $I \subseteq I^*$, $(I^*)^* = I^*$ and that I is a k-ideal of S if and only if $I = I^*$.

LEMMA 1. *If I is a locally nilpotent ideal of the semiring S then I^* is also a locally nilpotent ideal of S.*

Proof. We must show that for every finite subset F of I^* we can find a positive integer p such that any product of p elements from F is zero. We do so by induction on the number, n, of elements in F which are in I^*-I.

Any finite subset F of I^* which has $n = 0$ elements from I^*-I is a subset of I which is locally nilpotent. Now assume for induction that for any finite subset of I^* with $n = k$ elements from I^*-I that there exists a positive integer p such that any product of p elements from this subset is zero.

Let F be any finite subset of I^* with $n = k+1$ elements from I^*-I. Choose $x_1 \in F$ such that $x_1 \in I^*-I$ so that by definition of I^* $x_1+a = b$ for some $a, b \in I$. Then $F_1 = \{F-\{x_1\}\} \cup \{a, b\}$ is a finite subset of I^* with only $n = k$ elements from I^*-I. By induction, there exists a positive integer p_0 such that any product of p_0 elements from F_1 is zero.

Let $F_2 = F_1 \cup \{x_1\} = F \cup \{a, b\}$. We show that any product of p_0 elements from F_2 is zero by induction on the number, m, of times x_1 occurs in the product.

Consider any product of p_0 elements from F_2. If x_1 occurs $m = 0$ times the product is entirely of elements from F_1 and thus must be zero. Now assume that any product of p_0 elements from F_2 in which x_1 occurs $m = k$ times is zero. Any product of p_0 elements from F_2 in which x_1 occurs $m = k+1$ times can be written $A \cdot x_1 \cdot B$ where A and B are products of elements from F_2. But $x_1+a = b$ for some $a, b \in I$ so

$$A \cdot x_1 \cdot B + A \cdot a \cdot B = A \cdot b \cdot B.$$

However, $A \cdot a \cdot B$ and $A \cdot b \cdot B$ are products of p_0 elements from F_2 in which x_1 occurs $m = k$ times and as a result must be zero. Consequently, $A \cdot x_1 \cdot B$ is zero.

By induction, then, any product of p_0 elements from F_2 is zero and since $F \subseteq F_2$, any product of p_0 elements from F is zero. Thus we have completed our original induction and we conclude that I^* is locally nilpotent.

THEOREM 1. *For any semiring S, the Levitzki radical $L(S)$ of S is a k-ideal of S.*

Proof. $L(S)$ is a locally nilpotent ideal of S ([4], p. 26) so by Lemma 1, $L(S)^*$ is locally nilpotent. However $L(S)$ contains all locally nilpotent ideals of S. Thus $L(S)^* \subseteq L(S) \subseteq L(S)^*$ and $L(S)$ is a k-ideal of S.

Theorem 1 renders E. Barbut's Lemma 11 [1] unnecessary and we may improve his Lemma 6, Theorem, and Corollary all in [1] as follows.

LEMMA. *In a semiring R, $L(R/L(R)) = 0$.*

THEOREM. *If R is a semiring which satisfies the ascending chain condition on left and right annihilators then any nil subsemiring of R is nilpotent.*

COROLLARY. *If R is a semiring satisfying the ascending chain condition on left and right k-ideals then any nil subsemiring of R is nilpotent.*

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